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Algebraic Reynolds stress modeling of turbulence subject to rapid homogeneous and non-homogeneous compression or expansion

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A recently developed explicit algebraic Reynolds stress model (EARSIM) by Grigoriev *et al.* [“A realizable explicit algebraic Reynolds stress model for compressible turbulent flow with significant mean dilatation,” *Phys. Fluids* **25**(10), 105112 (2013)] and the related differential Reynolds stress model (DRSM) are used to investigate the influence of homogeneous shear and compression on the evolution of turbulence in the limit of rapid distortion theory (RDT). The DRSM predictions of the turbulence kinetic energy evolution are in reasonable agreement with RDT while the evolution of diagonal components of anisotropy correctly captures the essential features, which is not the case for standard compressible extensions of DRSMs. The EARSIM is shown to give a realizable anisotropy tensor and a correct trend of the growth of turbulence kinetic energy K , which saturates at a power law growth versus compression ratio, as well as retaining a normalized strain in the RDT regime. In contrast, an eddy-viscosity model results in a rapid exponential growth of K and excludes both realizability and high magnitude of the strain rate. We illustrate the importance of using a proper algebraic treatment of EARSIM in systems with high values of dilatation and vorticity but low shear. A homogeneously compressed and rotating gas cloud with cylindrical symmetry, related to astrophysical flows and swirling supercritical flows, was investigated too. We also outline the extension of DRSM and EARSIM to include the effect of non-homogeneous density coupled with “local mean acceleration” which can be important for, e.g., stratified flows or flows with heat release. A fixed-point analysis of direct numerical simulation data of combustion in a wall-jet flow demonstrates that our model gives quantitatively correct predictions of both streamwise and cross-stream components of turbulent density flux as well as their influence on the anisotropies. In summary, we believe that our approach, based on a proper formulation of the rapid pressure-strain correlation and accounting for the coupling with turbulent density flux, can be an important element in CFD tools for compressible flows. © 2016 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4941352>]

I. INTRODUCTION

Modeling approaches to turbulence have evolved significantly over the last half-a-century. However, even in recent works, most attention has been paid to the effects of shear and rotation while the influence of mean dilatation on the development of turbulence has attracted less consideration. One of the exceptions is given by Mahesh *et al.*¹ who used rapid distortion theory (RDT) to carefully examine the influence of homogeneous compression on turbulence that initially has been subjected to a certain amount of shear. We are unaware of the application of Direct Numerical Simulation

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(DNS), Large Eddy Simulation (LES), or Reynolds averaged Navier Stokes (RANS) equations to a similar problem. In fact, to our knowledge the first known attempt to consider the effect of significant mean dilatation on turbulent flows in the RANS framework is Grigoriev *et al.*² In particular, the pressure-strain rate model was generalized for flows with non-zero mean dilatation. The resulting model preserves realizability in the limit of large mean dilatation where standard model extensions for compressible flows will become unrealizable. Simple and useful physical examples appropriate for comparison with the model are not easily identified and for this reason we considered two simple flow cases to show the realizability of the model which makes it advantageous over previous compressible models like Wallin and Johansson.³ In the work of Grigoriev *et al.*,⁴ we extended the investigation to account for the influence of the density gradient, coupled with the pressure gradient and gravity, on the flow. We confirmed that taking into account density fluxes does not deteriorate the realizability of the model and described the trends of the influence of the model parameters on the turbulence (although only one of the two possible calibration branches has been considered there).

In this paper, we formulate the explicit algebraic Reynolds stress model (EARSM) for compressible two-dimensional mean flows and then illustrate the convergence of the corresponding differential Reynolds stress model (DRSM) to EARSM in the case of homogeneously sheared and compressed flow. This is important since EARSM represents an asymptotic state with approximately constant anisotropies while DRSM captures transient processes. Then, this case is employed for comparison of DRSM with RDT, Mahesh *et al.*,¹ to confirm the close behaviour of turbulence kinetic energies and similar trends of diagonal anisotropies. The behaviour of the EARSM in the same setup is then compared with DRSM and RDT results revealing realizability of the model and consistent evolution of turbulence kinetic energy. On the contrary, an eddy-viscosity model (EVM) reveals unphysical trends both in the evolution of anisotropies and turbulence kinetic energy. To emphasize the importance of the self-consistency relation within the EARSM formulation when using a fully compressible model, we introduce a new astrophysically related case of axisymmetrically rotating and compressed flow with zero shear. Then, we discuss the influence of density gradient of the flow through the emerging turbulent density flux as a result of coupling with “local mean acceleration.” The behaviour of turbulence in adiabatically compressed or expanded density-stratified flow with forcing has been analyzed as a first step to assess the impact of the rapid phenomena of detonation and deflagration on turbulence evolution. Finally, we have performed a fixed-point analysis of a DNS study of combustion in a wall-jet flow and confirmed that even with a simple calibration an extended EARSM accounting for the coupling with turbulent density flux yields a reasonable agreement with DNS data.

The initial evolution of turbulence at rapid deformation rates of the mean flow can readily be described using rapid distortion theory where the non-linear terms in the Navier-Stokes equations are neglected, see the work of Hamlington and Ihme.⁵ The expected objection to the employment of turbulence models in RDT limit is that in real cases the non-linear effects cannot be neglected and the strain-rate normalized with the turbulence time scale would be of order unity. Apparently, this is true for incompressible sheared flow which gradually converges to its asymptotic state. But strongly compressible flow processes such as cylinder flows (e.g., in internal combustion engine) and nozzle flows (e.g., in ramjets and scramjets) are typically of very fast and abrupt nature. In both steady and unsteady cases, compressible flows achieve their asymptotic states in a short interim (measured rather by compression ratio ρ/ρ_0 not by time itself) while remaining in a high-strain regime. For this reason, the investigation of strongly compressible turbulence in RDT limit is meaningful, and we can rely on the data, Mahesh *et al.*,¹ for assessing the performance of our EARSM and DRSM in conditions when rapid distortion theory is applicable.

We hope that this work can attract attention to the application of differential and explicit algebraic turbulence models not only to engineering flows with strong dilatation such as gas cycles and supercritical flow phenomena in which density can change significantly near pseudo-critical point resulting in large dilatations even under slight changes in temperature but also to astrophysical flows subject to rotation and compression. Additionally, certain stages of the processes of deflagration and detonation can be approximately considered as a nearly homogeneous compression (with possible shear) and the developed models can be used to assess the turbulence behaviour in such cases. This can be of importance since turbulence essentially effects the initiation and evolution of combustion phenomena.

II. EARSM FOR COMPRESSIBLE FLOW

In the work of Grigoriev *et al.*,² we developed an EARSM for compressible turbulent flows taking into account strong dilatation of the flow. This was achieved by formulating a self-consistent model for the rapid pressure-strain correlation $\Pi_{ij}^{(r)}$ in the form formally identical to the $\Pi_{ij}^{(r)}$ in incompressible case. In a two-dimensional case with $\partial_z = U_z \equiv 0$, the model for Reynolds stress anisotropy tensor $\mathbf{a} = \mathbf{R}/(\bar{\rho} K) - \frac{2}{3} \delta^{3D}$ is described by the relations

$$\begin{aligned} \mathbf{a} &= -\frac{6}{5} \frac{1}{N^2 - 2II_{\Omega}} \left(N \mathbf{S}^{2D} + \mathbf{S}^{2D} \mathbf{\Omega} - \mathbf{\Omega} \mathbf{S}^{2D} \right) - \frac{3}{5} \frac{\mathcal{D}}{N} \left(\delta^{2D} - \frac{2}{3} \delta^{3D} \right), \quad \mathbf{S}^{2D} = \begin{pmatrix} \mathcal{J} & \sigma & 0 \\ \sigma & -\mathcal{J} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{\Omega} &= \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{D} = \tau (\partial_x U_x + \partial_y U_y), \quad \mathcal{J} = \frac{\tau}{2} (\partial_x U_x - \partial_y U_y), \quad \sigma, \omega = \frac{\tau}{2} (\partial_y U_x \pm \partial_x U_y), \\ \mathbf{S}^{2D} &= \mathbf{S} - \frac{\mathcal{D}}{2} \left(\delta^{2D} - \frac{2}{3} \delta^{3D} \right), \quad \frac{\mathcal{P}}{\varepsilon} = -\frac{2}{3} \mathcal{D} + \frac{6}{5} \frac{N}{N^2 - 2II_{\Omega}} \left(II_S - \frac{II_{\Omega}}{3} \mathcal{D}^2 N^{-2} \right), \end{aligned} \quad (1)$$

where δ^{2D} and δ^{3D} are two-dimensional and three-dimensional Kronecker tensors, respectively, $\mathbf{S}^{2D}$, $\mathbf{S}$, and $\mathbf{\Omega}$ are two-dimensional and three-dimensional traceless strain-tensors and vorticity-tensor, respectively (by calling a tensor T_{ij} two-dimensional we assume $T_{i3} \equiv T_{3j} \equiv 0$). $\tau = K/\varepsilon$ is the turbulence time scale and $\mathcal{P}/\varepsilon \equiv -\text{tr}(\mathbf{a}\mathbf{S}) - 2/3 \mathcal{D}$ is the turbulence production-to-dissipation ratio (K , ε , and $\varepsilon = \rho \epsilon$ are turbulence kinetic energy per unit mass, turbulence dissipation per unit mass and per unit volume, respectively). Finally, the parameter $N \equiv c'_1 - 9/4 \text{tr}(\mathbf{a}\mathbf{S})$ is determined from the solution to a quartic equation which depends on a model constant $c'_1 = 9/4(c_1 - 1)$ (we use only standard value of $c_1 = 1.8$), the dilatation $\mathcal{D}$, and invariants of the flow $II_S = \text{tr}(\mathbf{S}^2) = 2(\sigma^2 + \mathcal{J}^2) + \mathcal{D}^2/6$ and $II_{\Omega} = \text{tr}(\mathbf{\Omega}^2) = -2\omega^2$,

$$N^4 - c'_1 N^3 - \left(2II_{\Omega} + \frac{27}{10} II_S \right) N^2 + 2c'_1 II_{\Omega} N + \frac{9}{10} II_{\Omega} \mathcal{D}^2 = 0. \quad (2)$$

Note that in incompressible flow ($\mathcal{D} \equiv 0$) N is just proportional to the production-to-dissipation ratio whereas in an expanding flow ($\mathcal{D} > 0$) it can significantly exceed $\mathcal{P}/\varepsilon$ and in compressed flow ($\mathcal{D} < 0$) to be much lower.

In the work of Grigoriev *et al.*,² we investigated the behaviour of the model in two steady-state setups. The first setup was an expanding homogeneously sheared and strained plane flow with large $\mathcal{D} > 0$ arising due to spatial acceleration of the flow (created, e.g., by a favourable pressure gradient or by heating). Applying the fixed-point analysis to the case we have shown that at any values of shear $\sigma \equiv \omega$ and dilatation $\mathcal{D} > 0$ the model given by (1) remains realizable. The second case was a quasi-one-dimensional nozzle flow expanding from subsonic to supersonic speed. We modeled the evolution of turbulence in the nozzle making the somewhat artificial assumption that the RDT regime is preserved, i.e., influence of the turbulence on the flow is not important, and viscous effects can be neglected. Again, it was illustrated that the model given by Grigoriev *et al.*² is self-consistent and realizable and, hence, is suitable for being applied in more complex geometries and using more realistic techniques, e.g., time marching technique (Anderson⁶).

In the present paper, we first extend the application of the models to unsteady sheared and compressed or expanding homogeneous flows in the RDT regime. Though the setup is to some extent artificial with the mean flow being prescribed and driven by some forcing (which is not necessarily given explicitly), we will be able to make decisive conclusions about the performance of our EARSM and its advantages over EVM. We are also going to analyze the trends of the transient period predicted by the DRSM version of the EARSM developed and to compare results with RDT theory. The two generic flow configurations are formulated below in Sections III and IV. In Section V, we will also apply a generalized version of our model, accounting for the effect of variable density, to a wall-bounded non-homogeneous flow with heat release and will compare the results with DNS data.

III. PLANE HOMOGENEOUS COMPRESSION AND SHEAR

We begin with the consideration of rapid homogeneous one-dimensional shear and compression in a plane geometry. We concentrate mostly on compression processes here but some attention will be paid to the case of expansion too, especially in Subsection V A. The evolution of the mean quantities depends on the parameters s_0 and d_0 responsible for shear and dilatation ($d_0 < 0$ in the case of compression and we will also use $s_0 < 0$ to keep the ratio s_0/d_0 positive), respectively, and is prescribed as

$$U_x = \left(1 + d_0 t\right)^{-1} \left(d_0 x + s_0 y\right), \quad U_{y,z} = 0, \quad \rho = \left(1 + d_0 t\right)^{-1} \rho_0, \quad P = \left(1 + d_0 t\right)^{-\gamma} P_0 \rightarrow$$

$$\rightarrow \mathcal{D} = \tau d_0 (1 + d_0 t)^{-1}, \quad \mathcal{J} = \frac{1}{2} \tau d_0 (1 + d_0 t)^{-1}, \quad \sigma = \omega = \frac{1}{2} \tau s_0 (1 + d_0 t)^{-1}. \quad (3)$$

These relations identically fulfill the following equations of motion of a homogeneous flow (in general case — a flow with low level of turbulence kinetic energy):

$$\rho D_t U_i + \partial_i P = \rho g_i, \quad (g_i \equiv 0), \quad \partial_t \rho + \partial_k (\rho U_k) \equiv 0, \quad D_t P + \gamma P \partial_k U_k = (\gamma - 1) Q, \quad (Q \equiv 0). \quad (4)$$

We have to keep in mind that in a general case g_i represents gravity or hydrodynamic forcing which can be essential for sustaining the kinematics of the flow. Similarly, in general, the pressure does not have to follow the adiabatic law, when P/ρ^γ is passively advected with the mean flow, and can change due to external heating Q_{ext} or heating due to chemical reactions Q_{ch} . To complete the picture, we must note that four components of the viscous stress tensor are non-zero under the kinematics given by (3), namely, the shear component $\tau_{xy} = \mu s_0 (1 + d_0 t)^{-1}$ and diagonal components $\tau_{xx} = (\mu_v + 4/3 \mu) d_0 (1 + d_0 t)^{-1}$, $\tau_{yy} = \tau_{zz} = (\mu_v - 2/3 \mu) d_0 (1 + d_0 t)^{-1}$, where μ_v is the dynamic viscosity. Thus, some external influence is required to set the fluid in motion according to (3) although the divergence of τ_{ij} is zero. Since d_0 is negative, the flow will be compressed in time until $-d_0 t = 1$ when the density will become singular. The evolution will be plotted vs $-d_0 t$ and ρ/ρ_0 which are related as depicted in Fig. 1 (left column) illustrating the strong non-linearity.

Although the EARSM given by Grigoriev *et al.*² is self-consistent, it is important to ensure that the basis of the model — the corresponding DRSM — performs reasonably when applied to rapidly distorted flows in capturing the unsteady transient process. This is the object of Subsection III A.

A. Convergence of DRSM towards EARSM

To assess how much information one can extract from comparison of DRSM with the rather limited RDT data of Mahesh *et al.*,¹ we first look at the convergence of DRSM to EARSM in the

FIG. 1. Left — compression ratio $\rho/\rho_0 \equiv (1 + d_0 t)^{-1}$ vs nondimensionalized time $-d_0 t$. Right — evolution of the turbulence Reynolds number Re_t versus “incompressible time scale” $|s_0| t$. Solid lines: $s_0/d_0 = 1000, 100, 50, 30$. Dashed lines: $s_0/d_0 = 20, 15, 10, 5$. Dotted-dashed lines — $s_0/d_0 = 4, 3, 2, 1$. Dotted line — $s_0/d_0 = 0.2$. Arrows point in the direction of increasing $|d_0|$. Thick solid line almost coincides with the asymptotics $s_0/d_0 \rightarrow \infty$.

case of homogeneous shear and compression. The DRSM with neglected non-linear transport terms and with $\partial_t a_{ij} = D_t a_{ij}$ (both due to homogeneity in space) is described by the equation

$$\begin{aligned} \partial_t a_{ij} = \tau^{-1} & \left(-(c_1 - 1 + \Psi/\varepsilon - a_{lm} S_{ml}) a_{ij} - \frac{8}{15} S_{ij} + \frac{4}{9} (a_{ik} \Omega_{kj} - \Omega_{ik} a_{kj}) \right. \\ & \left. + (1 - c_p) \left[\frac{\Psi_{ij}}{\varepsilon} - \frac{2}{3} \frac{\Psi}{\varepsilon} \delta_{ij} \right] \right), \end{aligned} \quad (5)$$

where in the general case $\Psi_{ij} = -\overline{\rho' u'_i (D_t U_j - g_j)} - \overline{\rho' u'_j (D_t U_i - g_i)}$ can be important when density varies and “local mean acceleration” is non-zero. EARSMS given by (1) follows from DRSM-equation (5) after applying the weak-equilibrium assumption ($\partial_t a_{ij} \equiv 0$) and putting $\Psi_{ij} \equiv 0$. Note that when $\Psi_{ij} \neq 0$ the parameter N has to be defined as $N = c'_1 + \frac{9}{4} (-\text{tr}(\mathbf{a}\mathbf{S}) + \Psi/\varepsilon)$ and Equation (2) will give only an approximate solution for N -equation (see the work of Grigoriev *et al.*⁴) while the right-hand side of (1) has to be supplemented with terms proportional to $\frac{9}{4} (1 - c_p) \left(\Psi_{ij}/\varepsilon - \frac{2}{3} \Psi/\varepsilon \delta_{ij} \right)$. In our study, both the EARSMS and DRSM are complemented by a $K - \omega$ model Grigoriev *et al.*² ($\varepsilon = C_\mu \bar{\omega} K$, where we use $\bar{\omega}$ for the “turbulence frequency” to avoid confusion with rotation rate ω),

$$\begin{aligned} \partial_t K = & - \left(a_{lm} S_{ml} + \frac{2}{3} \mathcal{D} - \frac{\Psi}{\varepsilon} + 1 \right) \varepsilon, \partial_t \bar{\omega} = -C_\mu (C_{\varepsilon_1} - 1) \left(a_{lm} S_{ml} + \frac{2}{3} \mathcal{D} \right) \bar{\omega}^2 - C_\mu (C_{\varepsilon_2} - 1) \bar{\omega}^2 + \\ & + \tau^{-1} \left[\frac{2}{3} (C_{\varepsilon_1} - 1) + \frac{1}{3} - n(\gamma - 1) \right] \mathcal{D} \bar{\omega} + C_\mu (C_{\varepsilon_b} - 1) \frac{\Psi}{\varepsilon} \bar{\omega}^2. \end{aligned} \quad (6)$$

We may expect that after the initial transient the asymptotic state of DRSM will be approximately given by EARSMS. Formally, this is true when (a) $\tau_{DRSM}(t) \rightarrow \tau_{EARSMS}(t)$, $\tau \sim \bar{\omega}^{-1}$, in the course of evolution or $\max |\tau \partial_j U_i|$ achieves asymptotically high values in both DRSM and EARSMS; (b) the asymptotic state is stationary or $\partial_t a_{ij}$ becomes much less than the driving terms on the right-hand side of (5). However, though DRSM growth rates of K and ω converge to the asymptotic growth rates given by EARSMS, the quantities themselves may be different due to initial transient effects.

Here, we will follow the framework of Mahesh *et al.*¹ and assume that the initial state of turbulence has the same anisotropy as in the RDT after applying an amount of initial shear $\beta_0 \equiv -s_0 t_0$ to isotropic turbulence (during the interval $t \in [-t_0, 0]$; s_0 is chosen negative for convenience, which leads to positive a_{12}). Assuming the RDT limit with the dimensionless “total strain-rate” $S^* = \sqrt{2(IIS + \mathcal{D}^2/3)} \gtrsim 10$ (which means that kinematic time scale is much smaller than the turbulence time scale), we plot the evolution of the anisotropies a_{11} and a_{22} versus $-d_0 t$ in the upper row of Fig. 2 while the evolution of logarithm of the turbulence kinetic energy K (chosen instead of K which grows too fast) versus the compression ratio ρ/ρ_0 is plotted in the lower row of the figure. The left column corresponds to the pure compression process with shear-to-compression ratio $s_0/d_0 = 0$ and the right column to the process with $s_0/d_0 = 2$. Green arrows indicate the changes in DRSM evolution under variation of initial conditions with increasing β_0 . Here, and later in the paper, we do not provide all the data in the figures not to make the presentation excessively cumbersome. The EARSMS solution is, naturally, independent of the initial anisotropy and immediately approaches the weak equilibrium asymptotic solution with constant anisotropy and rapidly varying turbulence scales K and $\bar{\omega}$. Apparently, anisotropies computed with DRSM do really converge to the ones given by EARSMS. However, starting DRSM with $a_{ij} = 0$ ($\beta_0 = 0$) we find that convergence is slow and only achieved when $-d_0 t \rightarrow 0.95$ which corresponds to $\rho/\rho_0 \rightarrow 20$ and is in the close vicinity of the density singularity. In general, we find that the convergence improves significantly at higher β_0 (simply because of an initial condition closer to equilibrium) and at higher s_0/d_0 .

Fig. 2 (lower row) shows that in both DRSM and EARSMS, K at the initial stage grows slower than exponentially versus ρ/ρ_0 but faster than linearly. The corresponding $\bar{\omega}$ grows slower than linearly and the strain-rate $S^* \sim \tau \partial_j U_i$ increases during the compression. Analyzing Equations (6) it is straightforward to show that when S^* is high enough, K and ω evolve according to a power law $\sim (1 + d_0 t)^{-p_K}$ and $\sim (1 + d_0 t)^{-p_\omega}$, respectively, where p_K depends exclusively on s_0/d_0 while p_ω changes with γ and the other model parameters in the equation for $\bar{\omega}$ in (6). We assume that $\gamma = 1.4$, $C_\mu = 0.09$, $C_{\varepsilon_1} = 1.56$, $C_{\varepsilon_2} = 1.83$, $C_{\varepsilon_b} = 1.0$, and $n = 2/3$ remain fixed and all the results presented are for this particular set. At higher shear-to-compression ratio ($s_0/d_0 \gtrsim 7$ for the given

FIG. 2. Behaviour of DRSM and EARSIM in rapid distortion theory limit. Upper row — anisotropy tensor plotted vs $-d_0 t$: red a_{11} and blue a_{22} — DRSM, black lines — the corresponding asymptotic EARSIM values. Lower row — $\ln K$ plotted vs ρ/ρ_0 : red — DRSM, black — EARSIM. Left column — pure dilatation case with $s_0/d_0 = 0$, right column — shear+dilatation with $s_0/d_0 = 2.0$. Coloured solid, dashed, dashed-dotted, and dotted lines — DRSM with $\beta_0 = 0, 1, 2, 3$, respectively. Arrows point in the direction of increasing β_0 .

set of parameters), $\bar{\omega}$ starts to grow a bit faster than linearly but quickly stabilizes at a linear growth. This aspect will be discussed in more detail in Subsection III C.

At $s_0/d_0 = 0$ DRSM growth rates of both K and $\bar{\omega}$ increase with increasing β_0 but at least while $\beta_0 \leq 3$ the growth rates are limited by that of EARSIM. However, the reverse trend is seen when s_0/d_0 rises above ~ 2 (EARSIM values and DRSM values with $\beta_0 = 3, 2, 1$ one by one sink below the values of K and $\bar{\omega}$ given by DRSM with $\beta_0 = 0$). A somewhat different situation would occur if s_0 changes sign at $t = 0$ so that $s_0 < 0$ for $t < 0$ and $s_0 > 0$ for $t > 0$. Initial value of a_{12} would still be positive, but the positive s_0 would asymptotically drive a_{12} to become negative in a DRSM context. Hence, the shear production would initially become negative delaying the development of K . EARSIM cannot capture such transitions and predicts the development of K independent of the sign of s_0 . If we would increase the absolute value of s_0/d_0 keeping it negative, the previously described trend would hold but with one exception: EARSIM would always predict larger values of K and $\bar{\omega}$ than DRSM as long as $\beta_0 > 0$.

The turbulence Reynolds number can be estimated as $Re_t \sim \nu^{-1} K/\bar{\omega}$ ($\nu = \mu/\rho$ decreases with compression since growth of $\mu \sim T^n \sim \rho^{n(\gamma-1)}$, $n \approx 2/3$, and $\gamma \approx 1.4$ for an ideal diatomic gas is suppressed by the growth of ρ). While in this paper we use $-d_0 t$ and ρ/ρ_0 as nondimensional time variables, in case of incompressible shear flows $|s_0| t$ is usually employed. Adopting very high magnitude of s_0/d_0 , we arrive at a formally incompressible case and we can observe that Re_t grows during the evolution. Plotting Re_t versus $|s_0| t$ at different $d_0 < 0$, we find that the growth rate of Re_t increases with higher $|d_0|$ as shown in Fig. 1 (right column) for DRSM. This implies that adding compression to a sheared flow ensures that turbulent flow will remain turbulent.

B. RDT verification of DRSM

Now, we will turn our attention to how well DRSM captures the initial transients given by RDT. On one hand, we can regard the RDT data by Mahesh *et al.*¹ as fairly limited because it tracks the evolution of all the quantities only up to $\rho/\rho_0 = 4$ (or $-d_0 t \rightarrow 0.75$) implying that it is impossible to make definite conclusions about the asymptotic state. But on the other hand, the fourfold compression is quite a radical process and we can consider the data in the work of Mahesh *et al.*¹ to be sufficient for understanding if the DRSM which constitutes a basis for Grigoriev *et al.*² is consistent with the RDT solution. Note that there are many studies dedicated to the comparison of the behaviour of turbulence in incompressible sheared flow using RDT theory with its behaviour in DRSM and EARS. Since these studies indicate that the predictions of the anisotropy tensor by DRSM are rather poor (see, e.g., Johansson and Hallbäck⁷), we take the initial values of anisotropies (at $t = 0$) directly from the RDT results.

We assume that the turbulence has accumulated a certain amount of shear in advance and proceeds to evolve to a highly compressed state. The left column of Fig. 3 shows (both vs compression ratio, upper row, and vs time, lower row) the evolution of the diagonal anisotropies for accumulated shear $\beta_0 = 3$ (there are no data for diagonal components of a_{ij} at another β_0 's in Mahesh *et al.*¹). Though it may seem that our DRSM does not capture the behaviour of the anisotropies very closely, it must be stressed that the essential trends are captured correctly even using the rapid pressure-strain correlation model that is linear in a_{ij} , S_{ij} , and Ω_{ij} , Grigoriev *et al.*² This is in sharp contrast to the standard compressible extensions of pressure-strain models (e.g., Wallin and Johansson³) which exhibit completely wrong trends of the $a_{\alpha\alpha}$ evolution. The prediction of a_{12} (at $\beta_0 = 1, 2, 3$) is not quite as convincing which again confirms the weakness of DRSM for capturing effects of homogeneous shear.

FIG. 3. Behaviour of diagonal anisotropies and turbulence kinetic energy under homogeneous compression and shear at $s_0/d_0 = 0.1$. Left column — diagonal anisotropies at $\beta_0 = 3$: red — a_{11} , blue — a_{22} , green — a_{33} . Right column — K : red — $\beta_0 = 3$, blue — $\beta_0 = 2$, green — $\beta_0 = 1$, black — $\beta_0 = 0$; purple dotted line — EARS. Dashed lines — RDT (from Mahesh *et al.*¹), solid lines — DRSM. Upper row — ρ/ρ_0 on abscissa, lower row — $-d_0 t$ on abscissa.

The right column of Fig. 3 demonstrates the behaviour of turbulence kinetic energy K for $\beta_0 = 0, 1, 2, 3$ (K is scaled by the value of turbulence kinetic energy at $t = 0$, at the moment when compression starts). Since shear at $s_0/d_0 = 0.1$ is almost negligible (in the work of Mahesh *et al.*¹ it served only to create initial anisotropic states), we can conclude that the production term is only due to dilatation (or, more precisely, due to strain $\mathcal{J} = \mathcal{D}/2$ and dilatation $\mathcal{D}$). Surprisingly, for all available β_0 , there is fairly good quantitative agreement between the behaviour of K predicted by DRSM and K given by RDT (for $\beta_0 = 0$ it is a bit worse than for the other cases). In contrast, the old pressure-strain rate model Wallin and Johansson³ would underestimate K not less than by one third. It means that the model for the rapid pressure-strain correlation, Grigoriev *et al.*², is very good for capturing compressibility effects and can be used in further investigations and extensions of RANS models for strongly compressible flows. We stress that the simplicity of the model, which does not employ non-linear terms (see the work of Sjögren and Johansson⁸) to achieve better predictions for turbulence quantities in particular flow cases, makes it a rather universal tool.

Finally, the purple dotted line in Fig. 3 represents K given by EARSM. Interestingly, it almost coincides with K provided by RDT which starts at $\beta_0 = 3$. At the moment, we cannot say conclusively if it is just a coincidence or if an anisotropic state with $\beta_0 = 3$ really is a “saturated” state which is energetically close to the asymptotic EARSM state (though with different anisotropies).

C. The analysis of EARSM and EVM

Now we turn to the characterization of the behaviour of the turbulence predicted by our EARSM. Below we will separate the turbulence production into “shear” and “dilatational” components according to

$$\left(\frac{\mathcal{P}}{\varepsilon}\right)_{\text{shear}} = -2\beta_1\sigma^2, \quad \left(\frac{\mathcal{P}}{\varepsilon}\right)_{\text{dil}} = -2\beta_1\mathcal{J}^2 + \frac{\mathcal{D}^2}{5}N^{-1} - \frac{2}{3}\mathcal{D}, \quad \beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}, \quad (7)$$

which is different from the separation into “incompressible” and “dilatational” components adopted in the work of Grigoriev *et al.*² where the importance of $\mathcal{J}$ and $\mathcal{D}$ exchanges during the evolution while in the current setup the quantities are just proportional (and shear also is important). Fig. 4 shows the evolution of the components of production, anisotropies, K and S^* . The last quantity is plotted to check if we are in the RDT regime ($S^* \gtrsim 10$). Unlike the previous results where we assumed $S^* \rightarrow \infty$ (so that numerical constants, e.g., c'_1 , become completely unimportant), we will assume more moderate strain-rates in the following discussion. The cases plotted comprise the magnitudes of “shear-to-compression” ratio (only the ratio is physically important in the RDT limit when plotting the quantities against the compression coefficient ρ/ρ_0) $s_0/d_0 = 0.3, 1.0$, and 2.0 . In all three cases, the model remains realizable and stays in the RDT regime. Moreover, S^* and components of $\mathcal{P}/\varepsilon$ grow while the anisotropy tensor a_{ij} remains almost constant and a transient to some other state such as equilibrium does not occur. It means that in the RDT limit a compression of turbulence accompanied by a limited amount of shear is distinguished by an *asymptotic conservation of anisotropies* (as opposed to cases with higher shear-to-dilatation ratio, see below). Starting from a low-strain condition, the turbulence eventually would reach the RDT limit with corresponding values of a_{ij} . We note that K grows faster than linearly (and $\bar{\omega}$, albeit not shown, slower than linearly) with respect to ρ/ρ_0 .

The initial magnitude of total production-to-dissipation ratio gradually increases following $|s_0/d_0|$. The development of $\mathcal{P}/\varepsilon$ for the different curves closely follows each other. By the moment $\rho/\rho_0 = 4$, all the curves with $|s_0/d_0| \lesssim 3$ nearly collapse. This is remarkable since the plotted cases with $s_0/d_0 = 0.3, 1.0, 2.0$ cover a very wide range of turbulence regimes and is the consequence of the fact that $\mathcal{P}/\varepsilon$ has smaller growth rate at higher $|s_0/d_0|$ — the same is true for S^* , although K and $\bar{\omega}$ always grow faster with increasing $|s_0/d_0|$. Moreover, if the $\mathcal{P}/\varepsilon$ -curves chosen are characterized by $|s_0/d_0| \leq 1$ they remain rather close under subsequent evolution, but otherwise the curves diverge largely and after leaving the interval $\rho/\rho_0 \in (0; 4)$ all the curves change order with respect to each other (a higher becomes a lower). At $|s_0/d_0| \approx 7$, the productions by shear and dilatation stay constant during the evolution at about 30 and 10 (for the chosen magnitude of initial strain), respectively, as

FIG. 4. Upper left — $\mathcal{P}/\varepsilon$: red — “shear” part, blue — “dilatational” part, black — total. Upper right — anisotropy tensor: a_{11} — red, a_{33} — green, a_{12} — magenta. Lower left — total strain. Lower right — K : red — EARS, green — EVM. Dashed, thin solid, thick solid and dashed-dotted lines correspond to the cases with $s_0/d_0 = 0.3, 1.0, 2.0, 40$, respectively. Arrows point in the direction of increasing s_0/d_0 . Note that the data presented is incomplete to make the presentation more clear.

well as a_{ij} and S^* . Note that a_{11} reacts more slowly to the change in s_0/d_0 than other components of the anisotropy tensor and its change is partially compensated by the sum $a_{22} + a_{33}$.

When $|s_0/d_0|$ exceeds ~ 7 a transient stage emerges during which the anisotropies vary substantially while the turbulence regime becomes altered. Thereafter, an equilibration of the kinematic gradients and turbulence time scale τ is reached and S^* approximately approaches a constant less than ~ 40 . Importantly, the final state depends only on the magnitude of shear-to-compression ratio while increasing the magnitude of initial strain we only shift the transition process to higher ρ/ρ_0 . Now $\bar{\omega}$ demonstrates a linear growth with respect to ρ/ρ_0 while K tends to exponential growth at the initial stage of evolution, which later slows down and saturates at a more moderate power law growth. At $s_0/d_0 \geq 15$, total strain S^* falls below ~ 10 and the flow is not rapidly distorted anymore under these extreme conditions. Indeed, large magnitude of s_0/d_0 formally corresponds to an incompressible problem in which case the described equilibrium, characterized by a converged moderate value of total strain, is rapidly (i.e., at low ρ/ρ_0) achieved. Further increasing $|s_0/d_0|$ we approach an asymptotic state with $S^* \sim 5$ and $\mathcal{P}/\varepsilon$ close to 1.5 while the anisotropies converge exactly to $a_{11} = 0.3$, $a_{22} = -a_{12} = -0.3$, $a_{33} = 0$. A case with $s_0/d_0 = 40$, illustrating the transient, is shown in Fig. 4 (dashed-dotted lines). Finally, we stress that the magnitude of s_0/d_0 characterizing the described transition, as well as the quantities characterizing the asymptotic state, is very sensitive to the change of parameters in (6). For example, increasing γ we decrease the critical s_0/d_0 (at $\gamma \approx 2.6$ total strain S^* stays constant even at $s_0 = 0$), increase the growth rate of $\bar{\omega}$ (although p_ω remains unity) while the exponent p_K for the turbulence kinetic energy growth becomes lower and the anisotropies vary more substantially during the transition stage.

An interesting question is whether an EVM is able to capture anything of this complex dynamics. As is known, an EVM with

$$a_{ij} = -2 C_\mu S_{ij}, \quad \frac{\mathcal{P}}{\varepsilon} = -\frac{2}{3} \mathcal{D} + 2 C_\mu II_S, \quad \varepsilon = C_\mu \bar{\omega} K, \quad C_\mu = 0.09 \quad (8)$$

can be used, in principle, even when the realizability conditions

$$-2/3 \leq \lambda_{1,2,3}, \quad \lambda_{1,2} = \frac{1}{2} \left[-a_{33} \pm \sqrt{4a_{12}^2 + (a_{22} - a_{11})^2} \right], \quad \lambda_3 = a_{33} \quad (9)$$

are not satisfied. One can call it “standard $K - \omega$ model” instead of “EVM” to avoid the issues connected with the turbulence structure. EVM is basically unjustified for the analysis of RDT because $\mathcal{P}/\varepsilon$ in (8) is not proportional to $d_0 f(s_0/d_0)$ (f is some function) as it is in EARS (1) but has a term proportional to $d_0^2 f((s_0/d_0)^2)$. That means that even in the case of asymptotically high strain the predicted by EVM behaviour of turbulence quantities plotted vs ρ/ρ_0 will not depend exclusively on s_0/d_0 but also on the absolute value of d_0 which is not consistent with RDT. For example, giving d_0 the same magnitude as we used when plotting the EARS curves in Fig. 4, EVM will severely overpredict the growth of turbulence kinetic energy as shown by green lines in the lower right of Fig. 4 (the decrease of τ is greatly overpredicted too).

The initial growth rate of turbulence using EVM is hence completely arbitrary and different numerical values of d_0 would unphysically give different curves. But even if, at some s_0/d_0 , d_0 is chosen so that EVM has the initial growth rate of K smaller than that in EARS, it shows excessive growth rates at large compression ratios (regardless of the choice of d_0). Moreover, the growth rate of K in EVM increases drastically with increased $|s_0/d_0|$, a trend that is unphysical. Thus, EVM cannot, in principle, be calibrated to provide us with consistent results even at some specific d_0 .

To conclude, EVM possesses an intriguing property — specifically, independently of initial S^* (or, equivalently, d_0) EVM always converges to an asymptotic state which depends only on s_0/d_0 and is characterized by constant a_{ij} , S^* , and components of $\mathcal{P}/\varepsilon$ while ω grows linearly and K grows much faster than linearly but slower than exponentially, i.e., according to a power law but with significantly larger exponent than in the EARS or DRSM.⁹ Moreover, in this highly compressed asymptotic state, the exponent depends only on s_0/d_0 , similar to the algebraic models, but the pre-exponential factor depends on d_0 as well as on the initial strain.

D. Old model and quartic vs cubic N -equation

The so-called consistency condition given by (2) will ensure that the model prediction of $\mathcal{P}/\varepsilon$ is consistent with that used within the EARS solution procedure. The quartic equation is unpractical and different approximations are preferred, which will be discussed below.

But first, we would like to point out the importance of the more physically correct modeling of the pressure-strain term for compressible flows introduced in the work of Grigoriev *et al.*² Both DRSM and EARS versions of the previous compressible model Wallin and Johansson³ with $(c_1 - 1) \rightarrow (c_1 - 1 - 2/3 \mathcal{D})$ in the case of compression give realizable results with reduced diagonal anisotropies and increased shear anisotropy. But the growth of kinetic energy becomes essentially underpredicted at moderate values of s_0/d_0 (when the ratio is increased to extreme magnitudes, i.e., in formally incompressible case, it becomes overpredicted), whereas the preceding analysis indicates that the model of Grigoriev *et al.*² with $c'_1 = \frac{9}{4}(c_1 - 1)$ gives correct behaviour of K . Subsequently, EARS by Wallin and Johansson³ becomes unrealizable in the case of expansion, unless s_0/d_0 is rather large (N in this case has to be computed using formula C.1 from Grigoriev *et al.*² but both signs have to be minus). The same is true for the corresponding DRSM, but note that when the isotropic initial state $\beta_0 = 0$ is chosen the model becomes only marginally unrealizable (and only after a substantial expansion $\rho/\rho_0 \sim 0.1$) and increasing β_0 we need higher s_0/d_0 to suppress the unrealizability (K falls slower than in the correct DRSM).

The last term in quartic equation (2) is obviously unimportant in cases with low dilatation (free shear flows) or in cases with high dilatation but weak rotation (quasi-one-dimensional nozzle flow). Neglecting this term will reduce the consistency condition to a cubic (if $II_\Omega \neq 0$) equation for N . One may suggest that always when both compressibility and rotation are strong the term

FIG. 5. $s_0/d_0 = 1.25$. Left: green lines — N , black lines — P/ε . Right — anisotropy tensor: a_{11} — red, a_{22} — blue, a_{33} — green, black horizontal line — $\lambda = -2/3$. Thin lines — original case (3), thick lines — (3) with $\sigma \equiv 0$. Dashed lines — cubic equation, solid lines — quartic equation.

produces a significant effect. However, in the case of homogeneously sheared and compressed flow, the hypothesis is wrong. Fig. 5 (thin lines) illustrates the high strain-rate case with $s_0/d_0 = 1.25$ and shows that N following from (2) is $\sim 5\%$ larger than N following from the corresponding cubic equation. The use of the cubic equation increases the diagonal components of anisotropy tensor by $\sim 5\%$ and decreases a_{12} by $\sim 5\%$. However, the components of production-to dissipation ratio, K and $\bar{\omega}$ show only minor differences and gain less than 2% in magnitude (until $\rho/\rho_0 = 4$) after neglecting the last term in (2) ($S^* \sim \omega^{-1}$ and slightly decreases). Hence, we can conclude that in this case only the anisotropies are more or less susceptible to the approximation of the N -equation.

The above discussion emphasizes the robustness of the model Wallin and Johansson³ based on the cubic equation (but amending the definition of c'_1 according to Grigoriev *et al.*²). Hence, a question arises if it is really necessary to complicate the model by introducing quartic equation (2). If we would switch-off shear in the case considered and take sufficiently high $|s_0/d_0|$, we would find that the use of (2) becomes crucial both for the prediction of all the quantities (at $|s_0/d_0| \gtrsim 0.7$) and for realizability (at $|s_0/d_0| \gtrsim 1.2$). Thick lines in Fig. 5 highlight this fact for $s_0/d_0 = 1.25$ showing that the model with the approximate cubic equation becomes unrealizable during the evolution ($\lambda_3 \equiv a_{33}$ is the lowest eigenvalue), while K is overpredicted by $\sim 10\%$. With higher $|s_0/d_0|$ the total strain S^* grows but N and K decline. Note that switching off shear by artificially setting $\sigma = 0$ we change the sign of a_{12} (which remains non-zero due to the interaction of strain $\mathcal{J}$ and rotation ω) while a_{11} and a_{33} decrease, and a_{22} increases to become positive if $|s_0/d_0| \gtrsim 1.5$.

Thus, we can draw the conclusion that strong shear dramatically decreases the impact of dilatation-rotation coupling in the last term of (2), while at low shear the term is crucial. In Section IV, we will clarify the conditions when this interaction of rotation and dilatation is physical and when it is artifact of the general two-dimensional model (1) which, nevertheless, has to be taken into account when considering complex flows.

IV. INTERACTION OF COMPRESSION AND ROTATION

In principle, case (3) can be generalized by taking $U_x \sim (d_x x + s_x y)$, $U_y \sim (d_y y + s_y x)$, $d_0 = d_x + d_y$ and assuming, if needed, that we work in rotating frame of reference with ω_{ext} (a non-zero forcing is required then to establish the mean motion). If $s_x \approx -s_y$ then the shear is close to zero $\sigma = \tau/2 (s_x + s_y) \approx 0$ while effective rotation $\omega = \tau/2 (s_x - s_y) + 13/4 \omega_{ext}$ (see Wallin and Johansson³) can be significant and, in addition, $\mathcal{J} = \tau/2 (d_x - d_y)$ becomes a free parameter. Such a generalization can be considered to represent situations which are physically meaningful and achievable. As was shown in Section III D, at small shear $\sigma \rightarrow 0$, the combined influence of rotation ω and dilatation $\mathcal{D}$, represented by the last term in consistency relation (2), can be significant and the fourth order algebraic equation really has to be used. In this section, we aim to clarify the conditions

when this “interaction” of compression and rotation is physical and essential as opposed to situations when the effect is negligible or absent.

A. Spinned-up solid body rotation with compression

We start this section by introducing a homogeneously compressed two-dimensional gas cloud subject to solid-body rotation with cylindrical symmetry ($U_z \equiv 0$, $\partial_\theta \equiv 0$). The mean equations describing the system are

$$\begin{aligned} \partial_t \rho + r^{-1} \partial_r (r U_r \rho) &= 0, \quad \partial_t U_\theta + r^{-1} U_r \partial_r (r U_\theta) = 0, \quad \partial_t U_r + U_r \partial_r U_r - U_\theta^2/r = -\rho^{-1} \partial_r P + g_r, \\ \partial_t P + U_r \partial_r P + \gamma P r^{-1} \partial_r (r U_r) &= (\gamma - 1) Q. \end{aligned} \quad (10)$$

The set of Equations (10) can be satisfied by taking

$$\begin{aligned} \rho &= \left(1 + d_0 t\right)^{-1} \rho_0, \quad U_r = d_r r \left(1 + d_0 t\right)^{-1}, \quad U_\theta = -\frac{s_0}{2} r \left(1 + d_0 t\right)^{-1}, \quad d_0 = 2 d_r \rightarrow \\ &\rightarrow \sigma = \mathcal{J} = 0, \quad \omega = \frac{\tau}{2} s_0 (1 + d_0 t)^{-1}, \quad \mathcal{D} = \tau d_0 (1 + d_0 t)^{-1} \end{aligned} \quad (11)$$

and

$$-\rho^{-1} \partial_r P + g_r = [D_t \mathbf{U}]_r \equiv -r (d_r^2 + s_0^2/4) (1 + d_0 t)^{-2}. \quad (12)$$

Equation (12) gives nontrivial results only when the body force $g_r \neq 0$ or/and the volume heating $Q \neq 0$ while the continuity equation and equation for the U_θ -component are satisfied only due to $U_r \sim r$, $U_\theta \sim r$, $d_0 = 2 d_r$ similarly to the case described by relations (3). Observe that here we use parameter s_0 to quantify the rotation though the shear is zero. In rectilinear coordinates, the velocity components are $U_x = (d_r x + s_0/2 y) (1 + d_0 t)^{-1}$, $U_y = (d_r y - s_0/2 x) (1 + d_0 t)^{-1}$, which illustrates why ω in (11) looks identical to that in (3).

A fluid element in such a gas cloud having an initial radial position r_0 evolves as $r = r_0 \sqrt{1 + d_0 t}$ according to (11). Thus, we confirm the conservation of mass of an arbitrarily thin circular strip of the gas matter since $\rho(t) (r_2(t)^2 - r_1(t)^2)$ does not depend on time. Moreover, the angular momentum per unit mass of such a strip is $U_\theta r = s_0 r^2 (1 + d_0 t)^{-1} = s_0 r_0^2$ and remains constant during the evolution. Consequently, the state of the gas cloud can be characterized as *spinned-up solid body rotation with compression*.

It has to be stressed that the full consideration of the case in an astrophysical context may involve magnetohydrodynamical, radiation, and general relativity effects. Indeed, one can show that the diagonal components of the stress tensor are non-zero $\tau_{rr} = \tau_{\theta\theta} = (\mu_v + 1/3 \mu) d_0 (1 + d_0 t)^{-1}$, $\tau_{zz} = (\mu_v - 2/3 \mu) d_0 (1 + d_0 t)^{-1}$. This indicates that we need some mechanism for producing strain in the gas cloud on the boundaries of the domain we are interested in. This mechanism we cannot explicitly identify at the moment. For example, strong stresses on the equator of a neutron star are produced by a poloidal magnetic field, which can fracture the crust (Wood and Hollerbach¹⁰). For this particular study, we are adding an artificial mass force g_r .

The solution to the basic EARSME-equation

$$N \mathbf{a} = -\frac{6}{5} \mathbf{S} + \mathbf{a} \mathbf{\Omega} - \mathbf{\Omega} \mathbf{a} \quad (13)$$

may not include the rotation tensor not only in the absence of rotation ($\mathbf{\Omega} \equiv 0$). Particularly, if $a_{11} \equiv a_{22}$, $a_{12} \equiv a_{23} \equiv a_{31} \equiv 0$ due to geometry of the problem (axis z is in the direction of rotation), $\mathbf{\Omega}$ drops out of Equation (13). Hence, the consistency relation reduces to a quadratic equation $N^2 - c'_1 N - \frac{27}{10} II_S = 0$, which describes the axisymmetric case presented above along with rotation-free regimes like quasi-one dimensional nozzle flow, Grigoriev *et al.*² An extended three-dimensional case of shearless rotating flow with radial compression and additional compression along z -direction $\partial_z U_z = d_z$ ($II_S = 2/3 \tau^2 (d_r - d_z)^2$ then) is also represented by the quadratic equation and is independent of ω .

Considering an opposite case with high rotation rate and arbitrary strain-tensor S_{ij} , we would expect a strong dependence on ω . Certainly, this is true if ω is comparable to the components of S_{ij} . But if $|\omega|$ exceeds the eigenvalues of S_{ij} by a factor ~ 5 , only axisymmetric components of the anisotropy tensor will survive while the other components proportional to $1/\omega$ become negligible and serve only to equate the non-axisymmetric (in respect to z -axis) part of S_{ij} ,

$$a_{ij} \rightarrow \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = -\frac{6}{5} N^{-1} S_{33} \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

Hence, the case of axisymmetrically rotating and compressed flow represents two opposite flow situations on an equal footing. In Subsection IV B, we will formally show how the described effects allow us to simplify consistency relation (2) in two-dimensional flow.

B. Simplification strategies for the consistency relation

Equation (2) for a general two-dimensional flow can be rewritten in two following forms:

$$\left(N^2 - c'_1 N - \frac{9}{20} \mathcal{D}^2 \right) - \frac{27}{5} \frac{\sigma^2 + \mathcal{J}^2}{N^2 - 2 II_\Omega} N^2 = 0, \quad \left(N^2 - c'_1 N - \frac{27}{10} II_S \right) - \frac{54}{5} \frac{\sigma^2 + \mathcal{J}^2}{N^2 - 2 II_\Omega} II_\Omega = 0. \quad (15)$$

Both forms show that quartic equation (2) is factorized into a quadratic equation $N^2 - c'_1 N - \frac{9}{20} \mathcal{D}^2 = 0$ when $\sigma = \mathcal{J} \equiv 0$ (note that the EARSMS approximation is still valid due to $\mathcal{D} \neq 0$). Hence, the parameter N for setup (11) is independent of ω and its formal dependence on $II_\Omega \mathcal{D}^2$ is just an artifact of the general approach. Hence, turbulence predicted by our EARSMS indeed demonstrates an evolution independent of rotation in the case of axisymmetric compression and rotation.

Similar to the results in Fig. 5 for the behaviour of (3) with $\sigma \equiv 0$, we find that quartic equation (2) in case (11) (or, identically, the quadratic equation following after factorization) gives us realizable results with asymptotic values of anisotropies $a_{11} = a_{22} = 0.3$, $a_{33} = -0.6$, $a_{12} = 0$ (independent of ω) while the cubic equation leads to unrealizable anisotropies even at lower rotation-to-dilatation ratio $s_0/d_0 \approx 0.31$. The last fact is the consequence of the decline of $\mathcal{J} = 1/2 \mathcal{D}$ to $\mathcal{J} = 0$ which reduces the physical N and in addition makes the rotation-dilatation term in (2) much more important. The first form of Equation (15) confirms that decrease in $|\mathcal{J}|$ and increase in $|II_\Omega|$ act similarly up to certain extent, as we found in Section IV A. When $N^2 \ll |2 II_\Omega|$, which at high $|\mathcal{D}|$ is analogous to $9/20 \mathcal{D}^2 \ll |2 II_\Omega|$ or $(s_0/d_0)^2 \gg 0.45$, we can approximate (15) as

$$\left(1 + \frac{27}{10} \frac{\mathcal{J}^2}{II_\Omega} \right) N^2 - c'_1 N - \frac{9}{20} \mathcal{D}^2 = 0. \quad (16)$$

Fig. 6 shows that increasing s_0/d_0 we quickly achieve the convergence of quartic (2) and quadratic (16) equations for the case with $\sigma = 0$, $\mathcal{J} = \mathcal{D}/2$. Note that at $s_0/d_0 = 3$ the axisymmetrical value of $a_{11} \approx 0.3$ is already reached (not shown in the figure), while we need higher magnitudes of s_0/d_0 to reach $a_{22} = 0.3$ and $a_{33} = -0.6$.

The last example shows that the quartic equation is effectively reduced to cubic or quadratic equations not only when rotation $\omega = 0$, or dilatation $\mathcal{D} = 0$, or shear $\sigma = 0$ and strain $\mathcal{J} = 0$, but that the simplification is possible at high rotation rate too. Moreover, in RDT regime (the case of asymptotically high strain), Equation (2) is just a biquadratic equation since terms with c'_1 become negligible then. In Sections III A and III B, we employed this biquadratic equation while later we solved full quartic equation due to consideration of a weaker RDT condition (S^* starting at ~ 10). In general, Equation (2) contains information about all the regimes but can be simplified in many cases of interest.

The properties of the old models (both EARSMS and DRSM) Wallin and Johansson,³ EVM, and DRSM in the axisymmetrically compressed and rotating flow are essentially the same as revealed when investigating homogeneously sheared and compressed one-dimensional flow. The minor differences are as follows. Since production now is only due to dilatation, the underprediction of the

FIG. 6. The case with $\sigma = 0$, $\mathcal{J} = \mathcal{D}/2$ characterized by s_0/d_0 — rotation-to-dilatation ratio. Left — N ; right — a_{22} (upper lines) and a_{33} (lower lines). Thin lines — quartic equation (2), thick lines — quadratic (16). Solid, dashed, and dotted-dashed lines correspond to $s_0/d_0 = 3.0, 2.0, 1.0$, respectively. Arrows point in the direction of increasing s_0/d_0 .

turbulence kinetic energy growth rate in the old models becomes more substantial. Eddy-viscosity models as described by (8) do not depend on the rate of rotation, as is the case for axisymmetric rotation and compression, but perform unphysically.¹¹ DRSM converges to EARSF very slowly in our case (extreme compression is required).

To conclude, we stress that the consistent pressure-strain model by Grigoriev *et al.*² correctly captures the physical effects even in idealized cases like that considered in this section. Moreover, the case of axisymmetrically compressed and rotating flow also represents an asymptotic state (depending only on dilatation) for an arbitrary flow case at extreme rotation rates, which can justify a simplified consideration of various setups.

V. INFLUENCE OF VARYING DENSITY

Our ambition is to propose a turbulence model capable of predicting the behaviour of compressible and reacting flows with large density variation. Hence, we need to examine the generation of turbulent fluxes by mean density gradients as well as their influence on the turbulence evolution. In line with the other sections of the paper, we will first slightly generalize a homogeneously compressed and sheared flow assuming cross-stream density stratification and describe the emerging differences. Then we will compare our extended model with a DNS study of combustion in a wall-jet flow by Pouransari *et al.*¹² The consideration of this case lies a bit apart from the rest of the paper and represents non-homogeneously sheared and expanded flow with developed turbulence in an approximately equilibrium state. It enables us to calibrate the model against realistic flow situations.

A. DRSM for homogeneously stratified flow with forcing

If we modify the cases considered above by including spatially varying density of the flow and “local mean acceleration” of the flow ($D_t U_i - g_i$), we need to implement the full model Grigoriev *et al.*⁴ It can be relevant, e.g., when a deflagration or detonation wave has an initial cross-stream inhomogeneity and we want to understand its effect on the turbulence, aside from the effect of streamwise density variation and the modification of purely hydrodynamic instabilities.

To account for the influence of the interaction of $\partial_i \bar{\rho}$ and ($D_t U_i - g_i$), which is represented by term Ψ_{ij} in (5), in addition to Equations (5) and (6), we need to solve the equation for the density flux

$$\begin{aligned} \partial_t \overline{\rho' u'_i} = & -\tau^{-1} \left(\left[(1 + c_S/3 - c_D) \mathcal{D} + (c'_\rho/2 + c''_\rho) \right] \delta_{ik} + c_S S_{ik} + c_\Omega \Omega_{ik} \right) \overline{\rho' u'_k} \\ & - K \left(a_{ik} + \frac{2}{3} \delta_{ik} \right) \partial_k \bar{\rho}. \end{aligned} \quad (17)$$

Obviously, in (17), the driving term is represented by the density gradient. To concentrate on the influence of this term, we have omitted the terms $-\frac{2}{3} c_T \overline{\rho'^2} / \bar{\rho} (D_t U_i - g_i)$ and $-\bar{c}_p \overline{\rho' u'_i \rho' u'_k} / \overline{\rho'^2} \partial_k \bar{\rho}$ on the right-hand side although they can be non-negligible. In addition, their consideration would require a determination of the initial value of $\overline{\rho'^2}$ while we prefer to introduce only one initially arbitrary turbulence quantity, K . Returning to setup (3) of Section III, we note that assuming an initial state with density stratification, i.e., $\rho_0 = \rho(y)$, we will need to include the forcing g_i to counteract the emerging pressure gradient $\partial_y P$. The analysis of the right-hand side of (17) indicates that $\overline{\rho' u'_i}$ does not grow faster than K when approaching the singularity since $\rho = (1 + d_0 t)^{-1} \rho_0(y)$ and $\partial_{x,y} U_x$ have the same dependence on time. Also, if $\rho_0(y)$ is a linear function of y the density flux remains independent of y during the evolution in the case of homogeneous shear and compression, be it planar or cylindrical. Therefore, the flow behaves as homogeneously stratified with $\rho(t, y) = \rho_0(1 + d_0 t)^{-1}(1 + \kappa y)$ if a forcing $g_y \equiv \rho^{-1} \partial_y P$ is applied to sustain the mean dynamics.

To estimate the influence of the density flux on a_{ij} , K , and $\bar{\omega}$, we assess the quantity $\Psi = \overline{\rho' u'_k \partial_k P / \rho}$. Taking $Q = 0$ in (3) we arrive at an adiabatic evolution of P , i.e., $\partial_t P / \rho^\gamma \equiv 0$ or $P = s(y) \rho^\gamma(t, y)$. Hence, $\Psi / \varepsilon = \tau K^{-1} \overline{\rho' u'_y} \partial_y P / \rho^2$ and $\partial_y P / \rho^2 = s(y) \rho^{\gamma-3} (\gamma \partial_y \rho + \rho \partial_y \ln s)$. We assume $s(y) = \text{const}$ in the following. Only in the case with $\gamma = 3$ the influence of Ψ_{ij} / ε in (5) and (6) is independent of y and comparable to the influence of all the other terms at any time and any sign of d_0 . This conclusion is confirmed by evaluating the invariants II_S and $II_{\Gamma\Gamma}$ for the EARSMS version of the model (see the work of Grigoriev *et al.*⁴ for details) since $II_S \sim \tau^2 (1 + d_0 t)^{-2}$ while $II_{\Gamma\Gamma} = -\tau^2 \partial_y \rho \partial_y P / \rho^2 = -\tau^2 s \rho^{\gamma-3} \gamma (\partial_y \rho)^2$ and at $\gamma = 3$ it has the same time dependence as II_S . On the other hand, at $\gamma = 1$, the invariant $II_{\Gamma\Gamma}$ is proportional to $-\tau^2$ and does not have an explicit dependence on time. It means that in the case of significant expansion the turbulence will be mainly governed by the variable density effects and not by shear or strain if γ is close to unity. However, more detailed study is needed to find out if turbulence remains sufficiently strong during such an evolution.

Although the value of $\gamma = 3$ may seem too high, such values of the heat capacity ratio naturally arise in supercritical flows (e.g., Christen *et al.*¹³). In combustion typical values of γ range in the interval from 1 to 3 (the latter quantity is possible for cyclic nitramines RDX). Theoretically, γ is related to the degrees of freedom, n , of the molecule as $\gamma = (n + 2)/n$. $\gamma \rightarrow 1$ implies that many degrees of freedom n are excited, while $\gamma \rightarrow 3$ may correspond to an extremely rapid one-dimensional deformation so that only one of the translational degrees of freedom is effectively active. Anyway, as is typical for the analysis of turbulent flows, the results obtained under a convenient choice of $\gamma = 1.4$ are relevant to get the picture of turbulence behaviour also for other γ 's.

Both the generalized DRSM and EARSMS can be used to investigate the interaction of density and pressure gradients in the RDT regime. As in Secs. I–IV, an EARSMS cannot be expected to capture the rapid transient. Hence, we show the DRSM results for the case of adiabatic compression and expansion with non-zero forcing g_i of a homogeneously stratified flow. Both adiabatically compressed and expanded flows correspond to the stably stratified flow case studied by Lazeroms *et al.*^{14,15} When $\kappa < 0$, the mean density gradient and g_i have negative vertical components and the “baroclinic” production Ψ is negative. Fig. 7 shows the development of the anisotropies and density fluxes normalized by $f_0 = K \rho_0 |\kappa / d_0|$. The influence of the active coupling, when Equation (17) is solved together with (5) and (6), is demonstrated by comparison with the passive case without the coupling terms. The simplest set of parameters $c_S = c_\Omega = 1$, $c_\Psi = 0$, $c_D = 0$ is used and $s_0 = -|d_0|$ for both compression and expansion but $P_0^{(exp)} = 0.1 P_0^{(comp)}$ to reduce the influence of the coupling in the latter case. Indeed, when expanding the turbulent flow, the density flux effects become dominant (if $\gamma < 3.0$) with time and too large negative Ψ can drive K to zero and τ to infinity which can cause problems with positiveness of turbulence kinetic energy and realizability (RDT limit can be violated too). Both for compression and expansion realizability becomes more difficult to satisfy but for expansion the realizability is always violated at some higher expansion rate ρ_0 / ρ . Density fluxes react rather moderately to the coupling and decrease/increase depending on compression/expansion in the case of stable stratification. If we assume unstable stratification (by changing the direction of the pressure gradient assuming $Q(y) \neq 0$) all the trends will be approximately reversed. In this

FIG. 7. Behaviour of a homogeneously stratified fluid under adiabatic rapid compression and expansion predicted by DRSM. Left column — compression with $s_0/d_0 = 1.0$, right column — expansion with $s_0/d_0 = -1$. Upper row — anisotropy tensor: red — a_{11} , blue — a_{22} , green — a_{33} , and magenta — a_{12} . Lower row — density flux: red — $\rho' u'_x$, blue — $\rho' u'_y$, both normalized by $f_0 = K \rho_0 |\kappa/d_0|$. Dashed lines — density flux uncoupled from the other equations (passive scalar), solid lines — density flux coupled to the system (active scalar).

case, the compression will lead to unrealizability at higher γ , unlike for expansion when a_{ij} remains realizable independent of γ .

The trends revealed by Fig. 7 for the case of adiabatic compression and expansion (keeping in mind the unstably stratified configuration too) can be viewed as a reference to assess the possible effects of cross-stream inhomogeneity on turbulence in processes with rapid heating or cooling. An application of the presented DRSM to real detonation and deflagration phenomena is a challenge and requires several additional steps. First of all, non-homogeneity of the streamwise evolution has to be accounted for in a physically consistent way. For example, this can be achieved by assuming slightly modified laws of the evolution of the mean density and mean velocity like $\rho = \rho_0/[1 + d_0 t + \delta(x, t)]$ and $U_x = d_0 x/[1 + d_0 t + \delta(x, t)]$, where $\delta(x, t)$ represents temporal and spatial deviations from homogeneity. However, even this simplified approach requires a separate study since it involves the solution of non-linear equations like $(1 + d_0 t + \delta) \partial_t \delta + 2 d_0 x \partial_x \delta = 0$, staying for the continuity equation, with boundary conditions which have to be additionally determined.

B. Combustion in a wall-jet flow

There is a strong need not only to investigate the model by itself but also to examine its performance against experimental or DNS data. By using the data from an appropriate case we can compare the results with the DRSM presented in Section V A or the explicit algebraic version of the model developed in the work of Grigoriev *et al.*⁴ Moreover, since experiments and DNS mostly concentrate on measurements in the regions where developed steady state is achieved, the EARSIM can be applied in a fixed-point manner, i.e., comparing only a_{ij} and $\zeta_i \equiv \overline{\rho' u'_i} / \sqrt{\rho'^2 K}$ while K , τ , and $\overline{\rho'^2}$, as well as velocity and density gradients, are taken directly from the experiment/DNS.

A DNS study of a turbulent flow with non-premixed combustion by Pouransari *et al.*¹² involves both significant density variation and noticeable acceleration and is suitable to test our model. Pouransari *et al.*¹² concentrate on the analysis of the effects of heat release in a case when fuel is injected as a turbulent wall-jet (the mass ratio of fuel being $\theta_f = 1$) into an oxidizer environment (the mass ratio of oxidizer $\theta_o = 0.5$). The height of the jet at the inlet h and its centerline velocity U_j are taken as characteristic length and velocity scales, respectively. The oxidizer has a co-flow velocity $0.1 U_j$ while the initial temperature of the jet stream, co-flow, and the wall is $T_j = T_w = 750$ K (maximum local temperature is higher than ~ 2100 K). Each of the species behaves as a calorically perfect ideal gas and an Arrhenius type of reaction law is assumed. The simulation parameters are $Re = U_j h / \nu = 2000$ — the Reynolds number, $Sc = \nu / \nu_\theta = 0.7$ — the Schmidt number, $M = U_j / c_s = 0.5$ — the Mach number, $Da = \tau_{conv} / \tau_{react} = 1100$ — the Damköhler number, $Ze = E_a / (R T_j) = 8$ — the Zeldovich number, $Pr = 0.7$ — the Prandtl number, and $Ce = -H / (c_p T_j) = 38$ — the heat release parameter. Note that the heat release of the reaction is similar to the burning of methane but with longer reaction times. The computational domain is of dimensions $(L_x = 35 h) \times (L_y = 17 h) \times (L_z = 7.2 h)$ and the corresponding number of grid points is $(N_x = 320) \times (N_y = 192) \times (L_z = 128)$. The resolution is $\Delta x^+ \approx 10.4$ in streamwise direction and $\Delta z^+ \approx 6$ in spanwise direction, while it varies in wall normal direction from $\Delta y^+ \approx 1$ at the wall through $\Delta y^+ \approx 5$ in the region of maximum flame to $\Delta y^+ \rightarrow 40$ at the edge of the domain. As has been shown in the paper, heat release effects delay the transition to a developed turbulent state, when $Re_\tau = u_\tau h / \nu = 220$ is reached, at the $x/h = 24$ position. For this reason, in accordance with Pouransari *et al.*,¹² we analyze the turbulence variables at the position $x/h = 25$. In a post-processing procedure, the data from 400 boxes at different times have been averaged and a subsequent averaging in spanwise direction has been performed providing us with two-dimensional distributions of mean and turbulence quantities. We employ here Reynolds averaging, not Favre averaging, for the evaluation of the mean velocities and the other quantities. Nevertheless, the EARSMD predictions in both approaches are very close. Although there are minor differences which, in principle, can be of interest, their analysis is outside the purpose of this paper.

The profiles of the mean density $\bar{\rho}$, streamwise, and wall-normal velocities U and V , respectively, as well as density variance $\overline{\rho'^2}$ and turbulence kinetic energy K at $x/h = 25$ in the DNS are shown in Fig. 8. Beyond $y/h = 4.5$ the level of K becomes negligible (the same is true for $\overline{\rho'^2}$ beyond $y/h = 6.0$) which implies that the fully turbulent jet regime is approximately bounded by $y/h = 4.5$ from above. Apparently, the inflection point in the U -profile near $y/h = 2.0$ leads to a maximum in K in agreement with common turbulence results. Moreover, $\bar{\rho}$ drops to its minimum very close to the inflection point in U which confirms that the dominant portion of the combustion process takes place in the vicinity of the turbulent outer shear layer. The peaks in $\overline{\rho'^2}$ at $y/h \approx 0.7$ and $y/h \approx 3.8$ are quite close to the inflection points in the mean density profiles which supports the validity of the density-variance equation

$$\partial_t \overline{\rho'^2} + U_k \partial_k \overline{\rho'^2} = -2 \overline{\rho' u'_k} \partial_k \bar{\rho} - 2 \overline{\rho'^2} \partial_k U_k - 2 (\bar{\rho} \rho' + \rho'^2/2) \partial_k u'_k - \partial_k \overline{\rho'^2 u'_k}, \quad (18)$$

where the first term on the right-hand side represents production due to variable density while the second term is the contribution from the dilatational production. The latter is negative in the case of exothermally reacting wall-jet since dilatation $\partial_k U_k$ is positive due to the heat expansion.¹⁶

Fig. 8 also shows that V does not exceed 5% of U and a closer analysis reveals that S_{11} and S_{22} are not larger than 25% of S_{12} , while S_{33} and $\mathcal{D}$ are approximately 3% and 10% of S_{12} , respectively. We estimate that $\Upsilon_i = \tau \sqrt{\rho'^2/K/\bar{\rho}} (D_t U_i - g_i)$ and $\Gamma_i = \tau \sqrt{K/\rho'^2} \partial_i \bar{\rho}$ do not exceed $|\Upsilon_1| \sim 0.2$ and $|\Gamma_2| \sim 3$. The influence of the mean velocity gradients on the anisotropy tensor can be quantified as proportional to $II_S \sim 10$ while the influence of the density flux (coupled with local mean acceleration) — as proportional to $|\Gamma_2 \Upsilon_1| \sim 0.5$ (note that the invariant $II_{\Gamma \Upsilon} = \Gamma_k \Upsilon_k \sim 0.1$ is even smaller). Thus, the flow is in a moderately strong turbulence regime with $S^* \approx \sqrt{2 II_S} \sim 4$ and although the effects of the mean dilatation and the density fluxes are quite observable (5%–10%), the major influence on the anisotropy tensor is associated with shear component S_{12} aided by the strain components S_{11} and S_{22} . Correspondingly, the normalized density flux $\hat{\zeta}_i$ can be computed as an active scalar (Grigoriev *et al.*⁴) with minor but noticeable influence of the coupling between

FIG. 8. Data from a DNS of a wall-jet of fuel injected into an oxidizer by Pouransari *et al.*¹² Left: density ρ (solid), x-velocity U (dashed), and y-velocity V (multiplied by 10, dotted). Right: density variance ρ'^2 (solid) and turbulence kinetic energy K (dashed). All the values in (a) are normalized by the jet inlet parameters. In (b) the normalization is done with U_j^2 and ρ_j^2 , respectively. $x/h = 25$. $y^+ = y u_\tau / \nu = 102.4 y/h$.

the anisotropy tensor equation and the density flux equation. The detailed analysis of the interplay of the density flux, density variance, mean local acceleration, and velocity-gradient effects is to be given elsewhere.

Fig. 9 demonstrates the behaviour of the components of ζ_i and a_{ij} at $x/h = 25$ with model parameters $c_\rho = -4.0$, $c_S = c_\Omega = 1.0$, $c_\psi = 0.0$, $c_D = 0.0$, and $c_T = 1.5$. At this location $y^+ = 102.4 y/h$ which means that below $y/h < 0.5$ (i.e., $y^+ < 50$) the flow is dominated by near-wall effects, and we cannot hope for a reasonable correspondence between DNS and the model. Indeed, in that region very large magnitudes of $\partial_y U$ are observed (one order larger than in the shear layer) indicating that a

FIG. 9. DNS data of a wall-jet of fuel injected into an oxidizer by Pouransari *et al.*¹² together with model predictions. Upper row: normalized density fluxes ζ_x (left) and ζ_y (right), respectively. Lower row: anisotropies a_{12}, a_{22} (left) and a_{11}, a_{33} (right), respectively. Dashed lines — DNS Pouransari *et al.*,¹² solid lines — the model Grigoriev *et al.*⁴ $x/h = 25$. $y^+ = y u_\tau / \nu = 102.4 y/h$.

near-wall treatment is required, which we here choose not to include in order to keep the comparison as clean as possible for the time being.

In the model, we employ a diffusion correction by modifying c'_1 as $c'_1 = \frac{9}{4} [c_1 - 1 + 2.2 \text{MAX}(1 + \beta_1^{(0)} II_S, 0)]$ where $\beta_1^{(0)} = -\frac{6}{5} \frac{(9/4 c_1)^{(9/4 c_1)}}{(9/4 c_1)^{2-2} II_\Omega}$ according to Wallin and Johansson.³ The main improvement due to its implementation is that a_{12} attains correct slope in regions where $\partial_y U \rightarrow 0$, i.e., near $y/h = 0.5$ and above $y/h = 3.5$. In addition, the correction improves the slope of a_{22} at $y/h > 3.5$. The behaviour of a_{12} and a_{22} is captured significantly better than the behaviour of a_{11} and a_{33} which are under- and overpredicted by the model, respectively (nevertheless summing to give correct a_{22}). This behaviour is to a large extent due to the choice of $q_3 = 1$ ($c_2 = 5/9$) in the model for the rapid pressure-strain correlation (Wallin and Johansson³). Interestingly, at $c_\psi = 0.0$, which implies maximum coupling between a_{ij} and ζ_i , the slopes of a_{11} and a_{33} near $y/h = 1.0$ are predicted correctly despite a shift in the values themselves.

The prediction of both components of ζ_i is quite good even without too careful calibration. Importantly, this calibration can be used in this combustion problem, the nozzle flow (in Grigoriev *et al.*⁴ we used a different one) as well as in the case of supercritical flow corresponding to a DNS study by Peeters *et al.*¹⁷

Thus, we have shown that our explicit algebraic Reynolds stress model has a reasonable performance and really represents physics in a developed turbulence even at a rather low Reynolds numbers. Moreover, since the model remains singularity-free in a wide variety of flow configurations, the effects of three-dimensional swirling can also be investigated, e.g., by extending the model to a subgrid-scale formulation for LES, in line with Marstorp *et al.*¹⁸

Observe that the algebraic method used in Grigoriev *et al.*⁴ can be directly applied for solving a coupled system of the anisotropy tensor and several active scalars. It gives a tool to model complex phenomena governed by two or more competing factors.

VI. CONCLUSION

The first objective of this paper is to justify the DRSM and EARSMS based on the pressure-strain correlation model for compressible flow developed in the work of Grigoriev *et al.*² For this purpose, we applied the models to the case of homogeneously compressed and sheared turbulence which is a natural setup to examine the evolution of turbulence in the limit of asymptotically high strain rate. In this limit, the behaviour of all the parameters depends only on the shear-to-compression ratio and evolves as a function of the compression ratio, not time itself. We have compared the DRSM with the results of Mahesh *et al.*¹ obtained by the rapid distortion theory up to fourfold compression. The DRSM based on the rapid pressure-strain correlation, Grigoriev *et al.*², captures the essentials of the evolution of diagonal anisotropies during the transient process fairly well. This is in sharp contrast to standard compressible extensions of turbulence models including DRSM. Though the prediction of non-diagonal component of the anisotropy tensor is not as convincing, the growth rates of the turbulence kinetic energy given by the present DRSM at different initial conditions almost coincide with the exact RDT calculations.

After a transient period, the DRSM converges to an asymptotic state with constant anisotropies predicted by EARSMS. EARSMS is realizable (in terms of the eigenvalues of the anisotropy tensor) at all shear-to-compression ratios and its behaviour is divided into two distinct regimes. Namely, when shear-to-compression ratio is low or moderate, $s_0/d_0 \lesssim 7$ for the standard set of the model parameters, the normalized strain rate grows continuously and turbulence remains in the RDT regime. During such an evolution all the components of the anisotropy tensor are conserved. In contrast, when shear effects are too strong ($s_0/d_0 \gtrsim 7$), independently of the initial magnitude of strain rate turbulence approaches an asymptotic state with moderate strain ~ 10 and the anisotropies vary significantly during the evolution. In any regime, the turbulence kinetic energy both in DRSM and EARSMS always grows faster than linearly but slower than exponentially versus compression ratio. Specifically, the development of K quickly saturates at a power law growth with an exponent depending exclusively on s_0/d_0 . In general, DRSM and EARSMS turned out to be applicable in a wide range of flow conditions.

The DRSM and EARSIM based on the old pressure-strain correlation model Wallin and Johansson³ substantially underpredict the growth rate of turbulence kinetic energy during the compression. Eddy viscosity models (such as the standard $K - \omega$ model), apart from giving unrealizable anisotropies, also give too rapid exponential growth of turbulence kinetic energy at early stages and, in principle, cannot be calibrated to yield consistent results even at moderate compressions due to unphysical sensitivity to the parameters.

Further, we have shown that the consistency relation for EARSIM needs to be applied in its full form only when shear is low in comparison to dilatation and rotation. Then, using a polynomial equation of a lower order may result in an unrealizable behaviour of the model. To clarify the conditions when the interaction of dilatation and rotation is physical and has to be accounted for in the consistency relation as opposed to conditions when this interaction is just an artifact of the general model, we introduced an astrophysically as well as engineering related case of spinned-up and axisymmetrically compressed turbulent gas cloud (or supercritical fluid). This setup represents an asymptotic state for any case with high, compared to the other characteristics, rotation rate. Moreover, the model representation is independent of the rotation rate unless non-axisymmetric strain is added; hence, a simplified consistency relation can be used in many cases of interest.

Then, we described an extended model which includes the coupling of the density flux with “local mean acceleration” of the flow as proposed in the work of Grigoriev *et al.*⁴ The analysis of these effects is the next step in consideration of variable-density flows, though an additional forcing may be required to sustain the mean dynamics of a corresponding case. In particular, the model can be relevant when considering certain stages of deflagration and detonation processes in modified non-homogeneous flow configurations. We have demonstrated the trends which a DRSM version of the model reveals when applied to an adiabatically compressed or expanded fluid with stable stratification.

The EARSIM formulation of the model with coupled density flux has been applied to perform a fixed-point analysis of a DNS study of combustion in a wall-jet flow. This flow represents a natural non-homogeneous generalization of the other cases considered. Even with a simple calibration our model provided reasonable results for the turbulent density fluxes and turbulence anisotropies in the region where a weak-equilibrium assumption holds. Moreover, the calibration guarantees that our model is free of singularities in a wide variety of three-dimensional flows which implies a possibility to extend the model to be used in a LES context.

We hope that the algebraic models, based on the self-consistent pressure-strain correlation and accounting for the turbulent density flux, can become an important element in RANS and LES tools for the investigation of highly compressible and reacting flows. The possible cases include swirling supercritical flows (supercritical water reactor), gas cycles (Brayton, Ericsson, or Stirling cycles), processes of detonation and deflagration, and astrophysical aggregations undergoing strong rotation and compression. We plan to apply our model to several of these setups to examine its performance when compared to DNS/experimental data.

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- ⁹ The behaviour of ω and a_{ij} is independent of C_μ too but S^* and $\mathcal{P}/\varepsilon$ are inversely proportional to the parameter. Decreasing C_μ will lift up the curve for K but rather soon the behaviour of the turbulence kinetic energy would approach a limiting state. Taking typical value $C_\mu = 0.09$ we see that at $s_0 = 0$ the asymptotic state is unrealizable with $a_{11} = 2$, $a_{22} = a_{33} = -1$, $a_{12} = 0$, and $S^* \approx 23.5$. Increasing $|s_0/d_0|$ up to ~ 3 we finally achieve a realizable asymptotic state with the smallest eigenvalue $\lambda_2 \geq -2/3$ (the realizability of the initial stage depends on the magnitude of d_0) and $S^* \rightarrow 8.8$. Finally, at $s_0/d_0 \rightarrow \infty$, total strain converges to $S^* \rightarrow 4$ and $a_{12} \rightarrow 0.37$, $a_{\alpha\alpha} \rightarrow 0$. Similarly to the transient regime of EARSM and DRSM at high s_0/d_0 , in EVM the asymptotic values of anisotropies and p_K depend on γ (and the other parameters in the ω -equation) but the dependence does not switch off at low s_0/d_0 .
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