Homotopy Theory and TDA
with a View Towards Category Theory

SEBASTIAN ÖBERG

Doctoral Thesis
Stockholm, Sweden 2016

KTH
Institutionen för Matematik
100 44 Stockholm
SWEDEN

TRITA-MAT-A 2016:05
ISRN KTH/MAT/A-16/05-SE
ISBN 978-91-7729-003-2

Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i matematik tisdagen den 7e juni 2016 kl 15.00 i Kollegiesalen, Brinellvägen 8, Kungl. Tekniska högskolan, Stockholm.

© Sebastian Öberg, 2016

Tryck: Universitetsservice US AB

Abstract

This thesis contains three papers. Paper A and Paper B deal with homotopy theory and Paper C deals with Topological Data Analysis. All three papers are written from a categorical point of view.

In Paper A we construct categories of short hammocks and show that their weak homotopy type is that of mapping spaces. While doing this we tackle the problem of applying the nerve to large categories without the use of multiple universes. The main tool in showing the connection between hammocks and mapping spaces is the use of homotopy groupoids, homotopy groupoid actions and the homotopy fiber of their corresponding Borel constructions.

In Paper B we investigate the notion of homotopy commutativity. We show that the fundamental category of a simplicial set is the localization of a subset of the face maps in the corresponding simplex category. This is used to define ∞-homotopy commutative diagrams as functors that send these face maps to weak equivalences. We show that if the simplicial set is the nerve of a small category then such functors are weakly equivalent to functors sending the face maps to isomorphisms. Lastly we show a connection between ∞-homotopy commutative diagrams and mapping spaces of model categories via hammock localization.

In Paper C we study multidimensional persistence modules via tame functors. By defining noise systems in the category of tame functors we get a pseudo-metric topology on these functors. We show how this pseudo-metric can be used to identify persistent features of compact multidimensional persistence modules. To count such features we introduce the feature counting invariant and prove that assigning this invariant to compact tame functors is a 1-Lipschitz operation. For 1-dimensional persistence, we explain how, by choosing an appropriate noise system, the feature counting invariant identifies the same persistent features as the classical barcode construction.

Sammanfattning

Denna avhandling innehåller tre artiklar. Artikel A och Artikel B handlar om homotopiteori och Artikel C handlar om topologisk dataanalys. Alla tre artiklar är skrivna från en kategorisk synvinkel.

I Artikel A konstruerar vi kategorier av korta hängmattor och visar att dess svaga homotopityper är ekvivalenta med avbildningsrum. Samtidigt som vi gör detta så tacklar vi även problemet med att applicera nerv-funktorn till stora kategorier utan att använda sig av multipla universum. Huvudverktyget för att visa kopplingen mellan hängmattor och avbildningsrum är användandet av homotopigruppoider, deras verkan samt den homotopiska fibern av deras respektive Borel-konstruktioner.

I Artikel B undersöker vi konceptet homotopisk kommutativitet. Vi visar att fundamentalkategorin hos en simpliciell mängd är lokaliseringen av en delmängd av sido-avbildningarna i den korresponderande simpliciella kategorin. Detta används för att definiera ∞-homotopiskt kommuterande diagram som funktorer som skickar dessa sido-avbildningar till svaga ekvivalenser. Vi visar att om den simpliciella mängden är nerven av en liten kategori så är sådana funktorer svagt ekivalenta till funktorer som skickar sido-avbildningarna till isomorfier. Slutligen så visar vi på en koppling mellan ∞-homotopiskt kommuterande diagram och avbildningsrum hos modellkategorier via hängmattelokalisering.

I Artikel C studerar multidimensionella persistensmoduler via tama funktorer. Genom att definiera brussystem i kategorin av tama funktorer så får vi en pseudo-metrisk topologi på dessa funktorer. Vi visar hur denna pseduometrik kan användas för att identifiera persistenta egenskaper hos kompakta multidimensionella persistensmoduler. För att räkna antalet sådana persistenta egenskaper så introducerar vi karakteristik-räknings-invarianten och visar att tilldelandet av denna variant till kompakta tama funktorer är en 1-Lipschitz operation. För endimensionell persistens så förklarar vi hur, genom att välja lämpigt brussystem, karakteristik-räknings-invarianten identifierar samma persistenta egenskaper som den klassiska streckkods-konstruktionen.

Contents

Acknowledgements

Firstly I would want to start by thanking my supervisor Wojciech Chachólski for all his help and support with the thesis. This thesis would not have been possible without his indispensable help and astronomical amount of knowledge. Secondly I want to thank Martin Blomgren for his immense insight in the subject of categorical homotopy theory and all his invaluable feedback in writing the licentiate thesis. Thirdly I want to thank my collaborators, Anders Lundman, Martina Scolamiero and Ryan Ramanujam, from the computational topology group at KTH for all the fun and interesting discussions.

Futhermore my thanks also go to Joakim Roos and Anders Lundman for their help in proofreading the licentiate thesis and to Mats Boij and Bauer Tilman for their help in proofreading the doctoral thesis. An extra warm thank you goes to Gustav Sædén Ståhl for his help in proofreading both the licentiate thesis and the doctoral thesis.

In addition I want to give special thanks to Erik Duse for many giving discussions, Erik Aas for learning me about cowsay, Andreas Minne for making me speedrun Mega Man and Samuel Holmin for being a finis. On top of that I want to thank all my coworkers, not mentioned above, who have made my time at the math department enjoyable.

Penultimately I want to thank my family and friends for having the stamina to bear with me through all my mysterious endeavours. In particular I want to thank my mother, Mona, for teaching me that it is never too late to learn something new, my father, Niklas, for teaching me to stay optimistic while striving for perfection, my siblings, Alexander and Olivia, for making me the right combination of competitive and silly and lastly my beloved Jenny for listening to me talking endlessly about math, being supportive and sharing my love for rostad lök.

Finally I want to give my warmest thanks to [*your name here*] for his/her invaluable help with [*your contribution here*] and for always being [*your favourable characteristic here*] when I was working on this thesis.

1 Introduction

Homotopy theory and Topological Data Analysis, or TDA for short, are both subjects that study similar properties of various objects. The objects they are aimed at understanding are however very different. Homotopy theory deals with continuous objects, like simplicial sets or topological spaces, and their algebraic representations. In TDA on the other hand one typically studies finite discrete sets, coming from various measurements, and their algebraic representations. It turns out that the algebraic representations of both the continuous and the discrete objects are very similar. This comes from the fact that homotopy and TDA can both be effectively described using the language of functors, natural transformations and categories: in short, with a view towards category theory.

This thesis consists of this introduction and three papers. The papers have two different themes. The theme of Paper A and Paper B is the intersection of homotopy theory and category theory. The theme of Paper C is Topological Data Analysis, and more specifically multidimensional persistence, written from a categorical point of view.

In Paper A, the licentiate thesis, we give a model for mapping spaces in an arbitrary model category using the short hammocks of Dwyer and Kan from [DK80a] and [DK80b]. Our aim has been to give a comprehensive approach to mapping spaces and develop tools that for example could be used to compare mapping spaces for different structures. For this purpose we study the notion of homotopy groupoid actions. Our key result models the action of the homotopy groupoid of weak equivalences on the mapping spaces. Our methods also explain how to deal with set theoretical difficulties not addressed by Dwyer and Kan in their original papers.

In Paper B we study the concept of homotopy commutative diagrams. Normally this means functors with values in the homotopy category of a model category but in practice the homotopy coherent diagrams of [Vog73] play a much more important role. However, homotopy coherence typically involves specifying a huge amount of data, making the homotopy coherent diagrams difficult to work with. The aim of Paper B is thus to present an alternate, economical way of expressing homotopy coherence. For that we use so called bounded functors indexed by simplex categories. Instead of specifying coherent homotopies in our approach we choose a simplicial set whose fundamental category has the shape of the original indexing category for the homotopy commutative diagram. Such a simplicial set is called a model for the given shape. We show that one such model, given by the nerve of the original indexing category, produces homotopy commutative diagrams that are always weakly equivalent to strictly commutative diagrams of the same shape. We conjecture that this result can be extended to models that are quasi-categories. We also look at how the functor connecting a simplicial set to its fundamental category can be viewed as a pushout square built from considering disjoint unions of degeneracy maps. Lastly we analyse our resulting homotopy commutative diagrams using the hammocks from Paper A, to conclude that they can be interpreted in the context of mapping spaces and homotopies of such.

In Paper C, which is a joint paper together with M. Scolamiero, W. Chacholski, A. Lundman and R. Ramanujam, we move to the world of persistent homology and multidimensional persistence [CZ09]. We study the category of tame functors indexed by r-tuples of rational numbers and with values in the category of vector spaces. These are the functors one gets when using multidimensional persistence in real life applications. Instead of focusing on the classical algebraic connection with graded modules we choose to use topology as our main tool to study these functors. We define a topology on the category of tame functors by using the notion of a noise system, which can be viewed as a generalisation of Serre classes. Our aim with these generalised Serre classes is to remove noise via a process similar to localization, called denoising. In general however we do not know how to do this. Instead we show how this topology leads to a continuous invariant on tame functors. This invariant is then shown to generalise the well-studied barcode from the subject of single persistence.

In the rest of this introduction I will try to give an overview of the contents of the thesis. To do this I start by giving some mathematical background needed to better understand the papers that are in the thesis. After that follows a more thorough overview of each individual paper than the one given above. The reader of the introduction is assumed to know basic category theory and algebraic topology. Knowledge of concepts like model categories and simplicial sets are also strongly advised. Recommended literature for these subjects are [ML98], [Hat02], [DS95] and [Fri12] respectively.

Lastly me, the author, want to add that although I am very proud of this thesis it is my belief that my biggest achievement in mathematics can not be found in any of the papers presented in this thesis. Instead I would insist that my biggest contribution to the world of mathematics is the invention of the following mathematical construction, called *The banana split exact sequence*.

Figure 1: The banana split exact sequence

1.1 Essentially small categories

When working with category theory we often come in contact with statements like "consider all topological spaces" or "consider all groups" and so on. This is from a set-theoretical viewpoint problematic, since these statements are often not realizable in conventional set theory. This is due to a well-known paradox called **Russel's paradox** that was popularized as follows.

> *The barber is the "one who shaves all those, and those only, who do not shave themselves." The question is, does the barber shave himself?* [RS86]

This paradox can be used to prove the fact that a "set of all sets" cannot exist (or similarly that a "set of all groups" cannot exist etc.). To be able then to make statements like "consider all topological spaces" one can use the notion of a **universe**, which is defined as a non-empty set satisfying a number of set-theoretical properties (for a full definition of a universe see Definition 2.0.2 in Paper A). With the choice of a universe one can formalize the foundation of category theory as follows.

Definition 1.1. Fix a universe $\mathcal{U}$. The elements of $\mathcal{U}$ are called **small sets** or simply **sets** and the subsets of $\mathcal{U}$ are called **classes**.

Note that by construction we have that small sets are classes, but classes are in general not small sets. The notion of sets and classes allows us to make precise what assumptions we have on categories in this thesis. For us, a category, $\mathcal{C}$, means a category where the class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a small set for any $X, Y \in \mathrm{ob}\ \mathcal{C}$. Also we make the definition that a **small category** is a category $\mathcal{C}$, such that the objects of $\mathcal{C}$ constitutes a small set. If a category $\mathcal{C}$ is not small we say that $\mathcal{C}$ is a **large category**. The consequence of Russel's paradox on categories is that there can not exist a category of categories. There is however a **category of small categories**, denoted Cat, whose objects are small categories and morphisms are functors between small categories. We can also define the **category of sets**, denoted Sets, whose objects are sets and morphisms are functions between sets.

A large part of the thesis is devoted to applying homotopy theory to categories. One of the more important tools for doing this is the nerve functor. Recall that the **category of simplicial sets**, denoted Spaces, is defined as the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$ where Δ is **the simplex category**, consisting of non-empty finite ordinals $[n]$ for $n \in \mathbf{N}$ and order preserving maps. **The nerve functor** is then defined to be the functor $N\colon \mathrm{Cat} \to \mathrm{Spaces}$ that assigns to any small category I a simplicial set $N(I)$ whose n-dimensional simplices is the set of n-composable morphisms in I for $n > 0$ and where $I_0 := \mathrm{ob}\ I$, the object set of I. However the nerve of a category only makes sense if the category is small. Unfortunately the categories we are interested in are seldom small, which makes using the nerve impossible. Hence we need to introduce another notion of smallness. This notion relies on using weak equivalences between small categories. Let $f\colon J \to I$ be a functor between small categories. Then we say that f is a **weak equivalence** if $N(f)\colon N(J) \to N(I)$ is a weak equivalence of simplicial sets. Here we also see the importance of smallness and the nerve construction. The new notion of smallness can be defined as follows.

Definition 1.2. Let $\mathcal{C}$ be a category and $I \subset \mathcal{C}$ a subcategory. Then we say that I is a **core** of $\mathcal{C}$ if I is small and for any other small subcategory $J \subset \mathcal{C}$ with $I \subset J$ there exists a small subcategory $K \subset \mathcal{C}$ for which $J \subset K$ and the inclusion $I \subset K$ is a weak equivalence. We say that $\mathcal{C}$ is **essentially small** if it has a core.

The definition of essential smallness allows us to define when a functor $f\colon \mathcal{A} \to \mathcal{B}$ between essentially small categories is a weak equivalence and talk about the weak homotopy type of essentially small categories.

Definition 1.3. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a functor between essentially small categories and let $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$ be the respective cores. Then we say that f is a **weak equivalence** if the restriction $N(f|_A)\colon N(A) \to N(B)$ is a weak equivalence of simplicial sets.

Note that we are a bit lax with our definition here. In the thesis we use the concept of a *system of categories* to define a weak equivalence between essentially small categories. However, we want to avoid to go too much into detail to keep the exposition somewhat informal. The curious reader can find the precise definitions in Paper A.

1.2 General homotopy theory

The fundamental concept in homotopy theory for categories is that of a homotopy equivalence. Recall that two functors f and g are **homotopic** if there is a finite sequence of functors $\{h_k\colon \mathcal{B} \to \mathcal{A}\}_{0\leq k\leq n}$ and natural transformations

$$f = h_0 \to h_1 \leftarrow \cdots \to h_{n-1} \leftarrow h_n = g$$

connecting f and g. A functor $f\colon \mathcal{B} \to \mathcal{A}$ is called a **homotopy equivalence** if there is a functor $g\colon \mathcal{A} \to \mathcal{B}$ such that gf is homotopic to $\mathrm{id}_{\mathcal{B}}$ and fg is homotopic to $\mathrm{id}_{\mathcal{A}}$. The concept of essential smallness would not be very helpful if it was not preserved by homotopy equivalence.

Proposition 1.4 ([BC15, Corollary 5.9])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be homotopically equivalent. Then $\mathcal{A}$ is essentially small if and only if $\mathcal{B}$ is.*

Most of the tools in homotopy theory on categories comes from model categories. By a **model category** we mean a category that not only satisfies the standard axioms of model categories **MC1-MC5** (see e.g. [DS95]), but is also closed under arbitrary colimits and limits and has a **functorial fibrant and functorial cofibrant replacement**. The symbol $\mathcal{M}$ is used to denote such a model category. Given a model structure we use the symbols $\xrightarrow{\sim}$, $\twoheadrightarrow$ and $\hookrightarrow$ to denote **weak equivalences**, **fibrations** and **cofibrations** respectively. We denote by $\gamma_{\mathcal{M}} \colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ the **localization** functor of $\mathcal{M}$, where $\mathrm{Ho}(\mathcal{M})$ is the **homotopy category** of $\mathcal{M}$.

The category of simplicial sets, Spaces, can be shown to be a model category (see e.g. [GJ09]). This allows us to define the **weak homotopy type** of an essentially small category $\mathcal{A}$ with core A to be the isomorphism class of $N(A)$ in Ho(Spaces).

Model categories also allow us to expand our definition of homotopic functors. If $f, g \colon \mathcal{C} \to \mathcal{M}$ are functors, we say that a natural transformation $\varphi \colon f \to g$ is a **weak equivalence** (**natural isomorphism**) if for any $c \in \mathcal{C}$ we have that $\varphi_c \colon f(c) \to g(c)$ is a weak equivalence (isomorphism). If f and g are homotopic and all the natural transformations connecting f and g are weak equivalences we say that f and g are **weakly equivalent** (**naturally isomorphic**).

Most of our usage of model categories comes from looking at diagrams that lie in model categories. The homotopy version of a pullback diagram is one of these diagrams of particular interest. A commuting square in $\mathcal{M}$,

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

is called a **homotopy pullback** if the induced map $P \to \mathrm{holim}(C \to B \leftarrow A)$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. Here holim denotes the total right derived functor of the corresponding limit functor (see e.g. [DS95, Definition 9.5]). If further $\mathcal{M}$ is Spaces, C is contractible and c is a vertex in the image of g then we say that P is the **homotopy fiber of** f **over** c and we denote it by $\mathrm{hofib}_c(f)$.

It is also of interest to study more general homotopy diagrams.

Definition 1.5. Let I be a small category. Then a functor $F \colon I \to \mathrm{Ho}(\mathcal{M})$ going from I to the homotopy category of $\mathcal{M}$ is called a **homotopy commutative diagram**. Given a homotopy commutative diagram F we say that a functor $\overline{F} \colon I \to \mathcal{M}$ is a **lift of** F if it is naturally isomorphic to F.

1.3 Homotopy groupoid actions and Borel constructions

In [BC15] the concept of a homotopy groupoid was used to prove one of the more important theorems. This concept, and its continuation, became a very important part of this thesis as well and hence we carefully recall these constructions in this section. The formulations are quite lengthy and might seem technical at a first glance, but hopefully the naturality of the definitions should become apparent to the reader upon further investigation.

The concept of homotopy groupoids is a specialization of the more familiar notion of enrichments.

Definition 1.6. Let $\mathcal{S}$ be a class. An **enrichment indexed by** $\mathcal{S}$, $G_{\mathcal{S}}$, consists of

- simplicial sets $G(r,t)$ for every pair $r,t \in \mathcal{S}$,
- maps $\diamond\colon G(r,s) \times G(s,t) \to G(r,t)$ for every triple $r,s,t \in \mathcal{S}$,
- maps $e_r\colon \Delta[0] \to G(r,r)$ for every $r \in \mathcal{S}$,

such that the following properties are satisfied:

1. For every quadruple $r,s,t,v \in \mathcal{S}$ the following diagram commutes

$$\begin{array}{ccc} G(r,s)\times G(s,t)\times G(t,v) & \xrightarrow{\diamond\times \mathrm{id}} & G(r,t)\times G(t,v) \\ \downarrow{\scriptstyle \mathrm{id}\times\diamond} & & \downarrow{\scriptstyle\diamond} \\ G(r,s)\times G(s,v) & \xrightarrow{\quad\diamond\quad} & G(r,v) \end{array}$$

2. For every pair $r,s \in \mathcal{S}$ the following diagrams commute

$$\begin{array}{ccc} \Delta[0]\times G(r,s) & \xrightarrow{e_r\times\mathrm{id}} & G(r,r)\times G(r,s) \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle\diamond} \\ & & G(r,s) \end{array} \qquad \begin{array}{ccc} G(r,s)\times G(s,s) & \xleftarrow{\mathrm{id}\times e_s} & G(r,s)\times\Delta[0] \\ \downarrow{\scriptstyle\diamond} & \swarrow^{\mathrm{pr}} & \\ G(r,s) & & \end{array}$$

If $\mathcal{S}$ is a set we say that $G_{\mathcal{S}}$ is **small**.

Demanding that the simplicial sets of an enrichment are fibrant and that certain commuting squares are homotopy pullbacks gives us the notion of a homotopy groupoid.

Definition 1.7. Let $G_{\mathcal{S}}$ be an enrichment indexed by a class $\mathcal{S}$. We say that $G_{\mathcal{S}}$ is a **homotopy groupoid indexed by** $\mathcal{S}$ if for every pair $r,t \in \mathcal{S}$ the simplicial set $G(r,t)$ is fibrant and for every triple $r,s,t \in \mathcal{S}$ the following diagrams are homotopy pullbacks

$$\begin{array}{ccc} G(r,s)\times G(s,t) & \xrightarrow{\diamond} & G(r,t) \\ \downarrow{\scriptstyle\mathrm{pr}} & & \downarrow \\ G(r,s) & \longrightarrow & \Delta[0] \end{array} \qquad \begin{array}{ccc} G(r,s)\times G(s,t) & \xrightarrow{\diamond} & G(r,t) \\ \downarrow{\scriptstyle\mathrm{pr}} & & \downarrow \\ G(s,t) & \longrightarrow & \Delta[0] \end{array}$$

Homotopy groupoids are intended to generalize the notion of a group. The conditions of an enrichment can be seen to correspond to the conditions of a monoid and the homotopy pullback conditions of Definition 1.7 can be shown to correspond to the existence of inverses (see Example 2.4.3 of Paper A). One might ask then if there is a corresponding notion of a group action in this context. During the writing of Paper A this was not the case. This prompted the definition of a homotopy groupoid action to be created for Paper A.

Definition 1.8. Let $G_{\mathcal{S}}$ be a homotopy groupoid indexed by a class $\mathcal{S}$. A $G_{\mathcal{S}}$**-space**, $X_{\mathcal{S}}$, consists of a simplical set $X(s)$ for any $s \in \mathcal{S}$ and a map

$$*\colon G(s,t) \times X(t) \to X(s)$$

for every pair $s, t \in \mathcal{S}$ such that the following properties are satisfied:

1. For every triple $r, s, t \in \mathcal{S}$ the following diagram commutes

$$\begin{array}{ccc} G(r,s) \times G(s,t) \times X(t) & \xrightarrow{\diamond \times \mathrm{id}} & G(r,t) \times X(t) \\ \downarrow{\scriptstyle \mathrm{id} \times *} & & \downarrow{\scriptstyle *} \\ G(r,s) \times X(s) & \xrightarrow{\quad * \quad} & X(r) \end{array}$$

2. For every $s \in \mathcal{S}$ the following diagram commutes

$$\begin{array}{ccc} \Delta[0] \times X(s) & \xrightarrow{e_s \times \mathrm{id}} & G(s,s) \times X(s) \\ & {\scriptstyle \mathrm{pr}} \searrow & \downarrow{\scriptstyle *} \\ & & X(s) \end{array}$$

3. For every pair $s, t \in \mathcal{S}$ the following diagram is a homotopy pullback

$$\begin{array}{ccc} G(s,t) \times X(t) & \xrightarrow{\ *\ } & X(s) \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow \\ G(s,t) & \longrightarrow & \Delta[0] \end{array}$$

The map $*\colon G(s,t) \times X(t) \to X(s)$ is called the **homotopy groupoid action of** $G_{\mathcal{S}}$ **on** $X_{\mathcal{S}}$.

Just as in the case of the homotopy groupoid generalizing the notion of a group, the homotopy groupoid action can be seen to generalize the notion of a group action (see Example 2.4.5 of Paper A). The whole purpose of using homotopy groupoids and their actions is the possibility of taking their bar constructions. These are analogues of the Borel constructions. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and let $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. We denote the products for $t_0, \dots, t_n, t \in \mathcal{S}$ and $n \geq 1$ by

$$\mathcal{B}G_{t_0,\dots,t_n} := \prod_{k=0}^{n-1} G(t_k, t_{k+1}) \ , \qquad \mathcal{B}G_t := \Delta[0],$$
$$\mathcal{E}X_{t_0,\dots,t_n} := \mathcal{B}G_{t_0,\dots,t_n} \times X(t_n) \ , \qquad \mathcal{E}X_t := X(t).$$

Further we denote by $\mathrm{pr}_0\colon \mathcal{B}G_{t_0,\dots,t_n} \to \mathcal{B}G_{t_1,\dots,t_n}$ and $\mathrm{pr}_n\colon \mathcal{B}G_{t_0,\dots,t_n} \to \mathcal{B}G_{t_0,\dots,t_{n-1}}$ the projection onto to the last n factors and respectively the first n factors when $n > 1$ or the unique maps when $n = 1$. We can then define the bar construction as follows.

Definition 1.9. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$. Then the **bar construction of** $G_{\mathcal{S}}$, denoted $\mathcal{B}G_{\mathcal{S}}$, is defined degreewise by

$$(\mathcal{B}G_{\mathcal{S}})_n := \coprod_{t_0,\dots,t_n \in \mathcal{S}} \mathcal{B}G_{t_0,\dots,t_n}.$$

The face maps $d_i^{\mathcal{B}} \colon (\mathcal{B}G_{\mathcal{S}})_n \to (\mathcal{B}G_{\mathcal{S}})_{n-1}$ are given componentwise by

$$\begin{aligned} d_i^{\mathcal{B}} &:= \mathrm{id}^{i-1} \times \diamond \times \mathrm{id}^{n-1-i} \;, \quad 0 < i < n \\ d_0^{\mathcal{B}} &:= \mathrm{pr}_0 \;, \quad d_n^{\mathcal{B}} := \mathrm{pr}_n \end{aligned}$$

where id^k is the product of k copies of id and $\diamond$ is the binary map of $G_{\mathcal{S}}$. The degeneracy maps $s_j^{\mathcal{B}} \colon (\mathcal{B}G_{\mathcal{S}})_n \to (\mathcal{B}G_{\mathcal{S}})_{n+1}$ are given componentwise by

$$s_j^{\mathcal{B}} := (\mathrm{id}^j \times e_{t_j} \times \mathrm{id}^{n-j}) \circ \sigma_j \;, \quad 0 \le j \le n$$

where σ_j is the natural isomorphism

$$\mathcal{B}G_{t_0,\dots,t_n} \to \mathcal{B}G_{t_0,\dots,t_j} \times \Delta[0] \times \mathcal{B}G_{t_j,\dots,t_n}$$

and e_{t_j} is the map given from $G_{\mathcal{S}}$.

Given a homotopy groupoid action we also need a version of the Borel construction. We call this $\mathcal{E}X_{\mathcal{S}}$.

Definition 1.10. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. Then the **Borel construction of** $X_{\mathcal{S}}$, denoted $\mathcal{E}X_{\mathcal{S}}$, is defined degreewise by

$$(\mathcal{E}X_{\mathcal{S}})_n := \coprod_{t_0,\dots,t_n \in \mathcal{S}} \mathcal{E}X_{t_0,\dots,t_n}.$$

The face maps $d_i^{\mathcal{E}} \colon (\mathcal{E}X_{\mathcal{S}})_n \to (\mathcal{E}X_{\mathcal{S}})_{n-1}$ are given componentwise by

$$\begin{aligned} d_i^{\mathcal{E}} &:= d_i^{\mathcal{B}} \times \mathrm{id} \;, \quad i < n, \\ d_n^{\mathcal{E}} &:= \mathrm{id}^n \times *, \end{aligned}$$

and the degeneracy maps $s_j^{\mathcal{E}} \colon (\mathcal{E}X_{\mathcal{S}})_n \to (\mathcal{E}X_{\mathcal{S}})_{n+1}$ are given componentwise by

$$s_j^{\mathcal{E}} := s_j^{\mathcal{B}} \times \mathrm{id} \;, \quad 0 \le j \le n.$$

Given a homotopy groupoid $G_{\mathcal{S}}$ and a $G_{\mathcal{S}}$-space $X_{\mathcal{S}}$ we can use projection maps to construct a map between $\mathcal{B}G_{\mathcal{S}}$ and $\mathcal{E}X_{\mathcal{S}}$.

Definition 1.11. Define the map $\pi \colon \mathcal{E}X_{\mathcal{S}} \to \mathcal{B}G_{\mathcal{S}}$ degreewise by

$$\pi_n := \coprod_{t_0,\dots,t_n \in \mathcal{S}} \pi_{t_0,\dots,t_n}$$

where the maps $\pi_{t_0,\dots,t_n} \colon \mathcal{B}G_{t_0,\dots,t_n} \times X(t_n) \to \mathcal{B}G_{t_0,\dots,t_n}$ are the projections onto the first component.

These constructions happen to be bisimplicial sets. Recall that a functor in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spaces})$ is called a **bisimplicial set**. Given a bisimplicial set $F\colon \Delta^{\mathrm{op}} \to \mathrm{Spaces}$ we denote by $\mathrm{diag}(F)$ the simplicial set defined by having n-simplicies

$$\mathrm{diag}(F)_n := (F_n)_n$$

and with face and degeneracy maps being the induced ones. We have the following.

Proposition 1.12. *Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and let $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. Then $\mathcal{B}G_{\mathcal{S}}$ and $\mathcal{E}X_{\mathcal{S}}$ are both bisimplicial sets.*

We will see the importance of these constructions when we summarize the results from Paper A.

1.4 Categories coming from simplicial sets

One of the most important tools in this thesis is the use of functors indexed by simplex categories. The reason for this is that simplex categories are categorical versions of simplicial sets and thus have an intricate underlying geometry. When being used as indexing categories this simplicial geometry can be employed to construct and analyse functors in more ways than before. Thus we take some time to remind ourselves about simplicial sets and simplex categories.

Certain simplicial sets are more interesting than others. Recall that the **standard n-simplex** is the simplicial set given by the functor $\mathrm{mor}_{\Delta}(-, [n])$ and is denoted by $\Delta[n]$. The **k-th horn of** $\Delta[n]$ is the simplicial set $\Lambda_k[n]$ obtained from the boundary of $\Delta[n]$ by removing the k-th face. We say that a simplicial set K is a **quasi-category** if for any diagram of the form

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

we can always find a diagonal morphism that makes the diagram commute for $0 < k < n$. The full subcategory of Δ with objects $[0], [1], \dots, [n]$ is denoted $\Delta_{\leq n}$. The inclusion $\Delta_{\leq n} \hookrightarrow \Delta$ induces a functor $i_*\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}_{\leq n}, \mathrm{Sets})$ that has a left adjoint $i^*\colon \mathrm{Fun}(\Delta^{\mathrm{op}}_{\leq n}, \mathrm{Sets}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$. The **$n$-skeleton of a simplicial set** K is defined as $\mathrm{sk}_n(K) := i^* i_* K$.

Given a simplicial set $K \in \mathrm{Spaces}$, the **simplex category of** K is the category whose objects are simplicies of K, i.e. maps $\sigma\colon \Delta[n] \to K$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} \Delta[m] & \xrightarrow{\alpha} & \Delta[n] \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & K & \end{array} .$$

The simplex category of K is also denoted by K. It should be clear from the context if we are referring to the simplicial set or its corresponding simplex category. Morphisms in K are generated by **face maps**

$$\begin{array}{ccc} \Delta[n-1] & \xrightarrow{d_i} & \Delta[n] \\ & {\scriptstyle d_i\sigma}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & K & \end{array} \quad , \quad 0 \le i \le n,\ \forall\ \sigma\colon \Delta[n] \to K$$

and **degeneracy maps**

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{s_i} & \Delta[n-1] \\ & {\scriptstyle s_i\sigma}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & K & \end{array} \quad , \quad 0 \le i \le n,\ \forall\ \sigma\colon \Delta[n] \to K.$$

For any $n \in \mathbf{N}$ the subsets

$$\Upsilon_K^n := \left\{ \begin{array}{ccc} \Delta[m-1] & \xrightarrow{d_i} & \Delta[m] \\ & {\scriptstyle d_i\sigma}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & K & \end{array} \;\middle|\; 0 \le i < m \le n,\ \forall\ \sigma\colon \Delta[m] \to K \right\}$$

will be of special interest for Paper B. Note that the case $i = m$ is carefully excluded for all $m \le n$. The simplex category of a fixed simplicial set K is not the only useful category one can get from K. Another one can be acquired by noting that the nerve functor has a left adjoint (see e.g. [Lur09, Proposition 1.2.3.1.]).

Definition 1.13. The left adjoint of the nerve functor $N\colon \mathrm{Cat} \to \mathrm{Spaces}$ is denoted $\pi\colon \mathrm{Spaces} \to \mathrm{Cat}$. Given a simplicial set K, the category $\pi(K)$ is called the **fundamental category of** K.

The adjunction $\pi\colon \mathrm{Spaces} \rightleftarrows \mathrm{Cat} : N$ can be used to construct family of interesting maps. For any simplicial set K we can take the adjunct of the functor $\mathrm{id}\colon \pi(K) \to \pi(K)$ to aquire a simplicial map $\varepsilon_0\colon K \to N\pi(K)$. We can use the following forgetful functor together with ε_0 to directly connect a simplicial set to its fundamental category.

Definition 1.14. Let I be a small category. Then we define the forgetful functor $\varepsilon_N\colon N(I) \to I$ as

$$\sigma = \ (i_0 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{n-1}} i_n) \ \mapsto i_n\ , \ \ (s_j\sigma \xrightarrow{s_j} \sigma) \ \mapsto \mathrm{id}_{i_n}\ , \ \ (d_j\sigma \xrightarrow{d_j} \sigma) \ \mapsto \begin{cases} id_{i_n} & \text{if } j > 0 \\ \alpha_{n-1} & \text{if } j = 0 \end{cases}.$$

We define the functor $\varepsilon\colon K \to \pi(K)$ to be given by the composition

$$\varepsilon\colon \qquad K \xrightarrow{\varepsilon_0} N(\pi(K)) \xrightarrow{\varepsilon_N} \pi(K)\ .$$

1.5 Bounded functors and mapping spaces

We mentioned in the previous section that we want to use simplex categories as indexing categories for our functors in this thesis. However not all functors are of interest. The functors we are interested in have to invert the right set of morphisms.

Definition 1.15. Let I be a small category, $W \subset \mathrm{mor}(I)$ a set of morphisms in I and $F\colon I \to \mathcal{M}$ a functor. We say that F is **weakly inverting** W if $F(\alpha)$ is a weak equivalence for all $\alpha \in W$ and **strictly inverting** W if $F(\alpha)$ is a isomorphism for all $\alpha \in W$.

Given a functor that weakly inverts a set of morphism it is often more desirable to have a functor that strictly inverts these morphism but still is weakly equivalent to our original functor. Such a functor is called a rigidification.

Definition 1.16. We say that a functor $F\colon I \to \mathcal{M}$ weakly inverting W can be **rigidified with respect to** W if there exists a functor weakly equivalent to F that is strictly inverting W. Such a functor is called a **rigidification of** F **with respect to** W.

The reason we want to use the simplex categories as our indexing categories is because of their inherent geometry. Not all functors can make use of this geometry. Bounded functors were introduced in [CS02] to effectively work together with the underlying geometry of simplex categories.

Definition 1.17. Let K be a simplicial set and let $\mathcal{C}$ be any category. We say that a functor $F\colon K \to \mathcal{C}$ is **bounded** if F strictly inverts the set of all degeneracy morphisms $s_i : \Delta[n] \to \Delta[n{+}1]$ in K. The full subcategory of $\mathrm{Fun}(K, \mathcal{C})$ of bounded functors is denoted $\mathrm{Fun}^b(K, \mathcal{C})$.

A useful observation is that the degeneracy morphisms do not form an essential loop inside a simplicial set K. This means that any bounded functor is naturally isomorphic to a functor which sends degeneracy morphisms to identities [CS02, Proposition 10.3]. We can therefore assume this about all considered bounded functors.

Example 1.18. Consider the simplicial set $\Delta[0]$. One can show that the simplex category of $\Delta[0]$ is the entire Δ. This means that every functor indexed by $\Delta[0]$ is a cosimplicial object. Geometric intuition however would indicate that we should only consider constant functors of $\Delta[0]$ since we think of $\Delta[0]$ as a single point. This is what one gets if one instead considers bounded functors.

Just as with categories we would like to have a notion of homotopy on functors. This means that we will need to find a suitable candidate for a model structure for the functors we consider. One of the main advantages of looking at bounded functors is that they have their own model structure.

Theorem 1.19 ([CS02, Theorem 21.1])**.** *Let $\mathcal{M}$ be a model category. Then* $\mathrm{Fun}^b(K, \mathcal{M})$ *is a model category.*

Another advantage of working with bounded functors is that they give us an easy way of defining mapping spaces for any model category. This is done by first defining a mapping space on the functor category $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. For any $n \in \mathbf{N}$ let $p : \Delta[n] \to \Delta[0]$ denote the unique map. Then we denote the effect of the functor $N(p)^* : \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \to \mathrm{Fun}^b(N(\Delta[n]), \mathcal{M})$ by $F[n] := N(p)^*(F)$. **The mapping space for** $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ can be defined as follows.

Definition 1.20. Let F and G be functors in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then the simplicial set $\mathrm{map}_{\mathrm{Fun}}(F, G)$ is defined degree-wise by

$$\mathrm{map}_{\mathrm{Fun}}(F, G)_n := \mathrm{Nat}(F[n], G[n])$$

for $n \in \mathbf{N}$ and by sending any morphism $\alpha\colon \Delta[n] \to \Delta[m]$ to the morphism $\mathrm{map}_{\mathrm{Fun}}(F, G)_\alpha\colon \mathrm{map}_{\mathrm{Fun}}(F, G)_m \to \mathrm{map}_{\mathrm{Fun}}(F, G)_n$ defined by

$$\mathrm{map}_{\mathrm{Fun}}(F, G)_\alpha := N(\alpha)^*.$$

Let $F, G, H \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. The **composition map**

$$\circ\colon \mathrm{map}_{\mathrm{Fun}}(F, G) \times \mathrm{map}_{\mathrm{Fun}}(G, H) \to \mathrm{map}_{\mathrm{Fun}}(F, H)$$

is defined degree-wise by the composition of natural transformations. Further for every $F \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ the **unit map** is defined to be the simplicial map $e_F\colon \Delta[0] \to \mathrm{map}_{\mathrm{Fun}}(F, F)$ given by

$$e_F(\Delta[0]) := \mathrm{id}_{F[0]} \in \mathrm{map}_{\mathrm{Fun}}(F, F)_0.$$

This mapping space construction is the perfect example of an enrichment.

Proposition 1.21 ([BC15, Proposition 11.2 (1)])**.** $\mathrm{map}_{\mathrm{Fun}}$ *is an enrichment indexed by the objects of* $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$*.*

To be able to use this construction for the mapping space in $\mathcal{M}$ we need to be able to represent objects of $\mathcal{M}$ by objects in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. The natural choice is to consider homotopically constant functors. We say that a functor $F \in \mathrm{Fun}^b(K, \mathcal{M})$ is **homotopically constant** if it is weakly equivalent to some functor $cX\colon K \to \mathcal{M}$ with constant value $X \in \mathcal{M}$. We denote by $\mathrm{Cons}(N(K), \mathcal{M})$ the full subcategory in $\mathrm{Fun}^b(N(K), \mathcal{M})$ with objects being cofibrant, fibrant and homotopically constant. To find a representation of objects of $\mathcal{M}$ in $\mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ we start by choosing a functorial fibrant replacement, $R_{\mathcal{M}}$, in $\mathcal{M}$. Then let P_{Fun} denote the functorial cofibrant replacement in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. We define the functor $Q\colon \mathcal{M} \to \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ by the composition

$$\mathcal{M} \xrightarrow{R_{\mathcal{M}}} \mathcal{M} \xrightarrow{c} \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \xrightarrow{P_{\mathrm{Fun}}} \mathrm{Cons}(N(\Delta[0]), \mathcal{M}), \qquad Q = P_{\mathrm{Fun}} \circ c \circ R_{\mathcal{M}}$$

where $c\colon \mathcal{M} \to \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ is the **constant functor**. **The mapping space** for any given model category can now be defined as follows.

Definition 1.22. Let $\mathcal{M}$ be a model category and let $X, Y \in \mathcal{M}$. Define

$$\mathrm{map}_{\mathcal{M}}(X, Y) := \mathrm{map}_{\mathrm{Fun}}(QX, QY).$$

The following result tells us that this definition gives us the properties one would expect from mapping spaces.

Proposition 1.23 ([CS08, Proposition 9.2])**.** *Let X, Y be any objects in $\mathcal{M}$. Then $\mathrm{map}_{\mathcal{M}}(X, Y)$ is fibrant. Moreover there is a bijection, natural in X and Y, between the set of connected components $\pi_0\mathrm{map}_{\mathcal{M}}(X, Y)$ and the set of morphisms $\mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$ in $\mathrm{Ho}(\mathcal{M})$.*

The last construction using bounded functors we want to consider is the space of weak equivalences. For F and G in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ we define $\mathrm{Natwe}(F, G)$ to be the set of all natural transformations from F to G that are weak equivalences.

Definition 1.24. Let $F, G \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$. Then the simplicial set $\mathrm{we}(F, G)$ is defined as the subspace of $\mathrm{map}_{\mathrm{Fun}}(F, G)$ given degree-wise by

$$\mathrm{we}(F, G)_n := \mathrm{Natwe}(F[n], G[n]) \quad \subseteq \; \mathrm{map}_{\mathrm{Fun}}(F, G)_n$$

for $n \in \mathbf{N}$. The composition map $\circ\colon \mathrm{we}(F, G) \times \mathrm{we}(G, H) \to \mathrm{we}(F, H)$ and the map $e_F : \Delta[0] \to \mathrm{we}(F, F)$ for $F, G, H \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ are defined as in $\mathrm{map}_{\mathrm{Fun}}$.

1.6 A categorical description of TDA

So far we have looked at how homotopy theory can be used to describe categorical objects. In this section we are instead going to look at how category theory can be used to describe real life objects. In Topological Data Analysis, or TDA, the goal is to use topological tools to study datasets from real life data, usually in the form of a point cloud. The main tool one uses is called **persistent homology** and it is the focus of the last part of the thesis. We will especially focus on the multidimensional case of persistent homology, called **multidimensional persistence**.

The pipeline for using multidimensional persistence for data analysis starts with a data set with multiple measurements. The idea is that the data somehow hides topological information which one wants to extract. Often one assumes that the data is sampled from some fixed topological object or a sequence of such. In order to model these topological objects one uses for example the Čech or Vietoris-Rips constructions (see [Car09]) which result in a functor $X : \mathbf{Q}_+^r \to \mathrm{Spaces}$, where each choice of an r-tuple of non-negative rational numbers gives one of these topological objects. Applying the i-th homology with respect to a field K to every such choice gives us further a functor $H_i(X, K)\colon \mathbf{Q}_+^r \to \mathrm{Vect}_K$, which is a so called ***r*-dimensional persistence module**. All of the persistence modules that arise in this way share a number of properties which we will try to describe. It is important to note that Vect_K is **bicomplete**, meaning that it contains all small limits and colimits (see e.g. [ML98, V.1 Theorem 3]). Also recall that the morphisms of the category $\mathbf{Q}^r_+$ are given by the partial order $<$. That is, given objects $v, w \in \mathrm{ob}\, \mathbf{Q}^r_+$ we write that $v < w$ if $v_i \leq w_i$ for all $i = 0, \dots, r$ and $v \neq w$. We have the same partial order on $\mathbf{N}^r$. Since we will only consider non-negative rational numbers in this thesis we will simply write $\mathbf{Q}^r$ for the category $\mathbf{Q}^r_+$.

Let $\mathcal{C}$ be any bicomplete category. Further let α be a positive rational number and let the same symbol $\alpha\colon \mathbf{N}^r \to \mathbf{Q}^r$ denote the unique functor that maps an object v in $\mathbf{N}^r$ to αv (the multiplication of all the coordinates of v by α) in $\mathbf{Q}^r$. This gives us an induced functor $\alpha^*\colon \mathrm{Fun}(\mathbf{Q}^r, \mathcal{C}) \to \mathrm{Fun}(\mathbf{N}^r, \mathcal{C})$. If F is any functor in $\mathrm{Fun}(\mathbf{N}^r, \mathcal{C})$ we have the following diagram

$$\begin{array}{ccc} \mathbf{N}^r & \xrightarrow{F} & \mathcal{C} \\ \downarrow{\scriptstyle \alpha} & \nearrow_{\alpha^! F} & \\ \mathbf{Q}^r & & \end{array}$$

where we recognise $\alpha^! F\colon \mathbf{Q}^r \to \mathcal{C}$ as the left Kan extension of F along α. This left Kan extension always exists since $\mathcal{C}$ is cocomplete and $\mathbf{N}^r$ is small (see e.g. [ML98, X.3 Corollary 2]). Thus we get a functor $\alpha^!\colon \mathrm{Fun}(\mathbf{N}^r, \mathcal{C}) \to \mathrm{Fun}(\mathbf{Q}^r, \mathcal{C})$ which is left adjoint to α^*. We can now define a very important class of functors.

Definition 1.25. Let α be in $\mathbf{Q}$. A functor $G\colon \mathbf{Q}^r \to \mathcal{C}$ is called α-**tame** if it is naturally isomorphic to a functor of the form $\alpha^! F$ for some $F\colon \mathbf{N}^r \to \mathcal{C}$. A functor is **tame** if it is α-tame for some $\alpha \in \mathbf{Q}$. We will use the symbol $\mathrm{Tame}(\mathbf{Q}^r, \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}(\mathbf{Q}^r, \mathcal{C})$ whose objects are tame functors.

The r-dimensional persistence modules that we get from the multidimensional persistence pipeline are always tame and thus belong to the category $\mathrm{Tame}(\mathbf{Q}^r, \mathrm{Vect}_K)$. Studying this category is hence the goal of multidimensional persistence.

One advantage of studying tame functors instead of general functors is that we can borrow concepts from the category of $\mathrm{Fun}(\mathbf{N}^r, \mathrm{Vect}_K)$ that would otherwise be hard to define in $\mathrm{Fun}(\mathbf{Q}^r, \mathrm{Vect}_K)$. One such concept is rank. Let F be a functor in $\mathrm{Fun}(\mathbf{N}^r, \mathrm{Vect}_K)$. The **radical of** F is a subfunctor $\mathrm{rad}(F) \subset F$ whose value $\mathrm{rad}(F)(v)$ is the subspace of $F(v)$ given by the sum of all the images of $F(u < v)\colon F(u) \to F(v)$ for all $u < v$. One can show that the quotient $F/\mathrm{rad}(F)$ induces an isomorphism

$$F/\mathrm{rad}(F) \simeq \oplus_{v \in \mathbf{N}^r} (U_v \otimes V_v)$$

where $\{V_v\}_{v \in \mathbf{N}^r}$ is a sequence of vector spaces and $U_v\colon \mathbf{N}^r \to \mathrm{Vect}_K$ is the unique functor such that $U_v(v) = K$ and $U_v(w) = 0$ if $w \neq v$. Fixing such an isomorphism we say that F is of **finite rank** if $\{v \in \mathbf{N}^r \mid V_v \neq 0\}$ is a finite set and V_v is finite dimensional for any v. If F is of finite rank we define its **rank** to be given by $\text{rank}(F) := \sum_{v\in\mathbf{N}^r} \dim_K V_v$. It is now straightforward to define the rank for a tame functor.

Definition 1.26. Let G be a functor in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ and let $\alpha \in \mathbf{Q}$ be such that G is α-tame. Then the **rank of** G is given by

$$\text{rank}(G) := \text{rank}(\alpha^* G).$$

One can show that this definition is not dependent on the choice of α.

In most applications it is natural to assume that our functors are compact. Recall that an object A in an abelian category is **compact** if, for any sequence of monomorphisms $A_1 \subset A_2 \subset \cdots \subset A$ such that $A = \text{colim} A_i$, there is k for which $A_k = A$. Since Vect_K is a abelian category so is $\text{Fun}(\mathbf{Q}^r, \text{Vect}_K)$. To see why the assumption of compactness is natural we have the following proposition.

Proposition 1.27 (Proposition 14.12, Paper C)**.** *Let* $F\colon \mathbf{N}^r \to \text{Vect}_K$ *be a functor. The following are equivalent:*

(a) F is compact in $\text{Fun}(\mathbf{N}^r, \text{Vect}_K)$*;*

(b) F is of finite rank;

Thus saying that a functor is compact is tantamount to saying that it is of finite rank. In real life applications all the functors one encounters are of finite rank.

The category $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ has very similar properties to the category of graded modules over the r-graded polynomial ring $K[x_1, \ldots, x_r]$. In the case $r = 1$ this translates into the existence of a complete discrete invariant for one-dimensional compact and tame persistence modules, called the barcode, which we now recall how to construct.

Let $w \leq u$ be two elements in $\mathbf{Q}^r$. The symbol $K(v, -)\colon \mathbf{Q}^r \to \text{Vect}_K$ denotes the composition of the representable functor $\text{mor}_{\mathbf{Q}^r}(v, -)\colon \mathbf{Q}^r \to \text{Sets}$ with the linear span functor $K\colon \text{Sets} \to \text{Vect}_K$. There is a unique inclusion $K(u, -) \subset K(w, -)$. The cokernel of this inclusion is denoted by $[w, u)$. Functors on this form can be shown to be both tame and compact. The barcode invariant can be described as follows.

Proposition 1.28 (Proposition 5.6, Paper C)**.** *Any compact object G in* $\text{Tame}(\mathbf{Q}, \text{Vect}_K)$ *is isomorphic to a finite direct sum of functors of the form* $[w, u]$ *and* $K(v, -)$*. Moreover the isomorphism types of these summands are uniquely determined by the isomorphism type of G. We call this direct sum the* **barcode of** *G.*

The barcode was a very important invariant in the development of the case $r = 1$, also know as **single persistence** and has since proven itself to be a valuable tool for analysing data from a variety of different research areas (see e.g. [CCR13],

[PH14], [dSGM05] or [CISZ07]). For $r > 1$ however it is known that no such complete discrete invariant can exist, as in this case the moduli space of r-dimensional compact and tame persistence modules is a positive dimensional algebraic variety, (see [CZ09]). Furthermore, this variety is complicated enough that there is simply no realistic hope to find easily visualizable and continuous invariants completely describing their compact objects. This is where Paper C of this thesis comes in.

2 Overview of Papers

2.1 Paper A
Modeling mapping spaces with short hammocks

In Paper A we introduce the hammock categories, which are highly inspired by the notion of hammock localizations in [DK80a] and [DK80b]. If $\mathcal{M}$ is a model category and X, Y are objects of $\mathcal{M}$ such that Y is fibrant and cofibrant the **hammock category** $\mathrm{Ham}^R_{\mathcal{M}}(X, Y)$ is defined to be the category whose objects, called **short hammocks**, are zig-zags of the form

$$X \xleftarrow{\sim} Z \longrightarrow Y$$

where the indicated morphism is a weak equivalence. A morphism from $X \xleftarrow{\sim} Z \longrightarrow Y$ to $X \xleftarrow{\sim} Z' \longrightarrow Y$ is given by $h\colon Z \to Z'$ such that the following diagram commutes

$$\begin{array}{ccccc} & & Z & & \\ & \overset{\sim}{\swarrow} & \downarrow{\scriptstyle h} & \searrow & \\ X & & & & Y \\ & \underset{\sim}{\nwarrow} & Z' & \nearrow & \end{array} .$$

The hammocks of [DK80a] and [DK80b] have been widely studied and in [DK80a] they showed that taking the nerve of a category of hammocks would yield the correct weak homotopy type for a mapping space from X to Y, or $\mathrm{map}_{\mathcal{M}}(X, Y)$. This result has been made more precise in [DK80b], [Dug06] and [Man99].

However in neither of these papers the problematic treatment of the nerve of large categories was properly dealt with. In particular the category $\mathrm{Ham}^R_{\mathcal{M}}(X, Y)$ is a large category and we have seen that taking its nerve is therefore set theoretically impossible. In Paper A we remedy this by proving the following theorem.

Theorem 2.1. *The category* $\mathrm{Ham}^R_{\mathcal{M}}(X, Y)$ *is essentially small and has weak homotopy type of* $\mathrm{map}(X, Y)$.

The proof of this theorem relies on the construction of homotopy groupoids, homotopy groupoid actions and their corresponding Borel constructions. The definitions in Paper A of a homotopy groupoid action and its Borel construction, Definition 1.8 and 1.10, are, to the author's knowledge, entirely novel. With these concepts we prove that the space of weak equivalences defines a homotopy groupoid action on $\mathrm{map}_{\mathrm{Fun}}$.

Proposition 2.2. $\mathrm{we}(-,-)$ *is a small homotopy groupoid indexed by* $\mathrm{Cons}(N(\Delta[0]),\mathcal{M})$*. Furthermore if* $H \in \mathrm{Fun}^b(N(\Delta[0]),\mathcal{M})$ *is a fibrant functor then* $\mathrm{map}_{\mathrm{Fun}}(-,H)$ *is a* we*-space indexed by* $\mathrm{Cons}(N(\Delta[0]),\mathcal{M})$ *with the homotopy groupoid action being the composition.*

Since the Borel constructions are bisimplicial sets this allows one to study the map

$$\mathrm{diag}(\pi)\colon \mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-,QY)_{\mathcal{S}}) \to \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}}).$$

It is then shown that the homotopy fiber of this map over QX is both weakly equivalent to $\mathrm{map}_{\mathcal{M}}(X,Y)$ and $\mathrm{Ham}^R_{\mathcal{M}}(X,Y)$, proving the theorem. Lastly, as a bonus, Theorem 2.1 is applied to make a set-theoretically correct re-proving of Retakh's theorem from [Ret86].

2.2 Paper B
Rigidifying homotopy commutative diagrams

In Paper B we turn our attention to homotopy commutative diagrams. In [DKS89] Dwyer, Kan and Smith formulated the notion of a homotopy commutative diagram as we saw in Definition 1.5. The question of the existence of lifts to such diagrams proved to be very involved and prompted the definition of **homotopy coherent diagrams** in [Vog73]. However, as was pointed out by Jacob Lurie in [Lur09, Section 1.2.6], "*the amount of data required to specify a homotopy coherent diagram is considerable, so the concept is quite difficult to employ in practical situations*". The aim of Paper B was thus to propose a more concise way of saying that a diagram is homotopy commutative, that would not need the language of homotopy coherent diagrams. To do this we look at the related question of localization for simplex categories. More precisely we show that the the localization of a simplicial set K with respect to Υ_K is given by the fundamental category of K.

Proposition 2.3. *Given a simplicial set* K*, the functor* $\varepsilon\colon K \to \pi(K)$ *is the localization of* K *with respect to* Υ_K*.*

From this proposition one gets the important corollary that $\varepsilon_N\colon N(I) \to I$ **is the localization of** $N(I)$ **with respect to** $\Upsilon_{N(I)}$. This result tells us that looking at functors indexed by a small category I is the same as looking at functors indexed by the simplex category of $N(I)$ that assign any morphism in $\Upsilon_{N(I)}$ to isomorphisms. The same is also true if we replace $N(I)$ with any K such that $\pi(K) = I$. We call such a K a **model for** I. This result motivated us to make the following definition of homotopy commutativity.

Definition 2.4. Let $\mathcal{M}$ be a model category, I a small category, K a model for I (i.e. $\pi(K)$ is equivalent to I), and $n \in \mathbf{N}$.

- An **n-homotopy commutative diagram of shape** I is a bounded functor $F\colon K \to \mathcal{M}$ weakly inverting Υ_K^n.
- An **∞-homotopy commutative diagram of shape** I is a bounded functor $F\colon K \to \mathcal{M}$ weakly inverting Υ_K.

The analogue of finding a lift of a homotopy commutative diagram, as in Definition 1.5, would in our context then translate into finding a rigidification of a ∞-homotopy commutative diagram. In Paper B we prove the following rigidification result.

Proposition 2.5. *Let I be a small category and let $F\colon N(I) \to \mathcal{M}$ be a ∞-homotopy commutative diagram. Then F can be rigidified.*

We also conjecture that a ∞-homotopy commutative diagram $F\colon K \to \mathcal{M}$ can always be rigidified whenever K is a quasi-category. We motivate this conjecture by showing that it is true in the case when the fundamental category is a point.

Proposition 2.6. *Let K be a quasi-category such that $\pi(K) = [0]$ and let $F\colon K \to \mathcal{M}$ be a ∞-homotopy commutative diagram. Then F can be rigidified.*

The functor ε was defined by composing the functor $\varepsilon_0\colon K \to N(\pi(K))$ with a forgetful functor. Thus in the paper we also take some time to study ε_0 by showing that we can find a set J such that the functor ε_0 can be identified as being the pushout of a square

$$\begin{array}{ccc} \coprod_{j\in J} \Delta[n_j+1] & \longrightarrow & K \\ {\scriptstyle \coprod s_{i_j}}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ \coprod_{j\in J} \Delta[n_j] & \longrightarrow & N(\pi(K)) \end{array}$$

where $s_{i_j}\colon \Delta[n_j+1] \to \Delta[n_j]$ are all degeneracy maps. Lastly we connect our result with Paper A by considering the **hammock category** $\mathrm{Ham}^L_{\mathcal{M}}(X,Y)$, that is defined to be the category whose objects are zig-zags of the form

$$X \longrightarrow Z \xleftarrow{\sim} Y$$

where the indicated morphism is a weak equivalence. The morphisms of $\mathrm{Ham}^L_{\mathcal{M}}(X,Y)$ are defined in a analogous way as for $\mathrm{Ham}^R_{\mathcal{M}}(X,Y)$. We show that $\mathrm{Ham}^L_{\mathcal{M}}(X,Y)$ is homotopy equivalent to $\mathrm{Ham}^R_{\mathcal{M}}(X,Y)$ and thus also model the mapping space $\mathrm{map}_{\mathcal{M}}\big(X,Y\big)$. We also show that an n-homotopy commutative diagram, $F\colon K \to \mathcal{M}$, for $n > 1$ can be interpreted as a functor that sends edges $e\colon v_0 \to v_1$ in K_1 to objects of $\mathrm{Ham}^L_{\mathcal{M}}\big(F(v_0), F(v_1)\big)$ and similarly that sends 2-simplices to **hammock homotopies** linking formal composition with actual maps.

2.3 Paper C
Multidimensional Persistence and Noise

In Paper C we present a new perspective on multidimensional persistence and introduce a tool for creating numerous new invariants for multidimensional persistence modules. Instead of focusing on the algebraic connection with graded modules we use topology as our main tool to analyse the category $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$. The definition of a topology relies on the definition of a noise system.

Definition 2.7. A **noise system** in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is a collection $\{\mathcal{S}_\varepsilon\}_{\varepsilon\in\mathbf{Q}}$ of sets of tame functors, indexed by rational non-negative numbers ε, such that:

- the zero functor belongs to $\mathcal{S}_\varepsilon$ for any ε;
- if $0 \le \tau < \varepsilon$, then $\mathcal{S}_\tau \subseteq \mathcal{S}_\varepsilon$;
- if $0 \to F \to G \to H \to 0$ is an exact sequence in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$, then
  - if G is in $\mathcal{S}_\varepsilon$, then so are F and H;
  - if F is in $\mathcal{S}_\varepsilon$ and H is in $\mathcal{S}_\tau$, then G is in $\mathcal{S}_{\varepsilon+\tau}$.

We think of a noise system as a definition of what things we want to discard in the category of tame functors. In Section 6 of Paper C we showcase the versatility of this definition by providing a lot of different examples of noise systems.

Assume for the rest of this introduction that we have a fixed noise system. With the noise system in place we say that, for any ε in $\mathbf{Q}$, a natural transformation $\varphi\colon F \to G$ in $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$ is an **ε-equivalence** if there are τ and μ in $\mathbf{Q}$ such that $\tau + \mu \le \varepsilon$, $\ker(\varphi)$ belongs to $\mathcal{S}_\tau$ and $\text{coker}(\varphi)$ belongs to $\mathcal{S}_\mu$. This allows us to define distances within $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$.

Definition 2.8. Let ε be in $\mathbf{Q}$. Two tame functors F and G are **ε-close** if there are natural transformations $F \leftarrow H : \varphi$ and $\psi\colon H \to G$ such that φ is a τ-equivalence, ψ is a μ-equivalence, and $\tau + \mu \le \varepsilon$. Further, if F and G are ε-close for some $\varepsilon \in \mathbf{Q}$ we define $d(F, G) := \inf\{\delta \in \mathbf{Q} \mid F \text{ and } G \text{ are } \delta\text{-close}\}$ and otherwise $d(F, G) := \infty$.

We prove that this does in fact define an extended pseudo-metric on the category of tame functors. Recall that an **extended pseudo-metric** is a metric d, with possibly infinite values, where we can have $d(x, y) = 0$ even when $x \neq y$.

Proposition 2.9. *The function d, defined in 2.8, is an extended pseudometric on the set of tame functors with values in* Vect_K.

Hence by defining the **open ball** $B(F, \tau)$, around a tame functor F with radius $\tau \in \mathbf{Q}$, to consist of all tame functors which are ε-close to F for some $\varepsilon < \tau$ we get a base for a **topology on** $\text{Tame}(\mathbf{Q}^r, \text{Vect}_K)$.

This topology is then used to produce new invariants for multidimensional persistence modules.

Definition 2.10. Let $F\colon \mathbf{Q}^r \to \mathrm{Vect}_K$ be a tame and compact functor. For t in $\mathbf{Q}$, define

$$\mathrm{bar}(F)_t := \begin{cases} \mathrm{rank}(F) & \text{if } t = 0 \\ \min\{\mathrm{rank}(G) \mid G \in B(F,t)\} & \text{if } t > 0 \end{cases}.$$

Since F is tame and compact, $\mathrm{bar}(F)_t$ is a natural number. Note that $\mathrm{rank}(F) \geq \mathrm{bar}(F)_t$ for any t. Furthermore if $0 < t \leq s$, then $B(F,t) \subset B(F,s)$ and hence $\mathrm{bar}(F)_t \geq \mathrm{bar}(F)_s$. Thus the association $t \mapsto \mathrm{bar}(F)_t$ defines a functor $\mathrm{bar}(F)\colon \mathbf{Q} \to \mathbf{N}^{op}$ which we call the **feature counting invariant** of F. By defining a pseudo-metric, also denoted d, on the functors of the form $f\colon \mathbf{Q} \to \mathbf{N}^{op}$ we lastly prove our main result about the feature counting invariants. Namely that they are 1–Lipschitz with respect to the topology on $\mathrm{Tame}(\mathbf{Q}^r, \mathrm{Vect}_K)$.

Proposition 2.11. *Let $F, G : \mathbf{Q}^r \to \mathrm{Vect}_K$ be tame and compact. Then*

$$d(\mathrm{bar}(F), \mathrm{bar}(G)) \leq d(F,G).$$

This result is of particular interest since it is the first continuous invariant, to the authors knowledge, constructed for multidimensional persistence that does not rely on using a reduction down to the one dimensional case. The feature counting invariants are then also shown to generalize the barcode from the one dimensional case.

References

[BC15] M. Blomgren and W. Chachólski, *On the classification of fibrations*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 519–557. MR 3271269

[Car09] Gunnar Carlsson, *Topology and data*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 2, 255–308. MR 2476414

[CCR13] Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan, *Topology of viral evolution*, Proceedings of the National Academy of Sciences **110** (2013), no. 46, 18566–18571.

[CISZ07] Gunnar Carlsson, Tigran Ishkhanov, Vin Silva, and Afra Zomorodian, *On the local behavior of spaces of natural images*, International Journal of Computer Vision **76** (2007), no. 1, 1–12.

[CS02] Wojciech Chachólski and Jérôme Scherer, *Homotopy theory of diagrams*, Mem. Amer. Math. Soc. **155** (2002), no. 736, x+90. MR 1879153 (2002k:55026)

[CS08] Wojciech Chachólski and Jerome Scherer, *Representations of spaces*, Algebr. Geom. Topol. **8** (2008), no. 1, 245–278. MR 2443229 (2010f:55022)

[CZ09] Gunnar Carlsson and Afra Zomorodian, *The theory of multidimensional persistence*, Discrete Comput. Geom. **42** (2009), no. 1, 71–93. MR 2506738

[DK80a] W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35. MR 578563 (81h:55019)

[DK80b] ———, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440. MR 584566 (81m:55018)

[DKS89] W. G. Dwyer, D. M. Kan, and J. H. Smith, *Homotopy commutative diagrams and their realizations*, J. Pure Appl. Algebra **57** (1989), no. 1, 5–24. MR 984042 (90d:18007)

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR 1361887

[dSGM05] Vin de Silva, Robert Ghrist, and Abubakr Muhammad, *Blind swarms for coverage in 2-D*, Proceedings of Robotics: Science and Systems (Cambridge, USA), June 2005.

[Dug06] Daniel Dugger, *Classification spaces of maps in model categories*, arXiv preprint math/0604537 (2006).

[Fri12] Greg Friedman, *Survey article: an elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. **42** (2012), no. 2, 353–423. MR 2915498

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition [MR1711612]. MR 2840650

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Man99] Michael A. Mandell, *Equivalence of simplicial localizations of closed model categories*, J. Pure Appl. Algebra **142** (1999), no. 2, 131–152. MR 1715404 (2000i:55050)

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872 (2001j:18001)

[PH14] Jose A. Perea and John Harer, *Sliding windows and persistence: An application of topological methods to signal analysis*, Foundations of Computational Mathematics **15** (2014), no. 3, 799–838.

[Ret86] V. S. Retakh, *Homotopy properties of categories of extensions*, Uspekhi Mat. Nauk **41** (1986), no. 6(252), 179–180. MR 890505 (88f:18020)

[RS86] Bertrand Russell and John G. Slater, *The philosophy of logical atomism and other essays, 1914-19 / bertrand russell; edited by john g. slater*, mcmaster university ed. ed., Allen & Unwin London ; Boston, 1986 (English).

[Vog73] R. M. Vogt, *Homotopy limits and colimits*, Proceedings of the International Symposium on Topology and its Applications (Budva, 1972), Savez Društava Mat. Fiz. i Astronom., Belgrade, 1973, pp. 235–241. MR 0334197