On the use of implied yields in real option modelling

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Abstract
In many applications of real options there is an assumption of complete capital markets. For the perpetual optimal timing option this means that if the underlying asset (e.g. a developed project) does not pay out any cash flows, then there is no finite optimal time at which the investment should be undertaken. In contrast, when the markets are incomplete, there could be a possibility of a finite optimal stopping time. We discuss the incomplete case in detail, connect it with yields and “implied yields” and give several examples of incomplete market models where could be a finite optimal time to invest.

Keywords: Real options, incomplete markets, irreversible investments.

JEL Classification: G11, G13, R30.

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1 Background

The focus of this paper is to study and discuss real options valuation and real options analysis models in an incomplete market setting. We show that in an incomplete market model there is a possibility of having a finite optimal time of investment in cases when it is never optimal to invest if the market is complete. We base our study on the waiting to invest option, a fundamental real options problem. Investment timing, or when a firm should undertake an irreversible investment, is a classical problem that managers face. The case when managers have an opportunity to invest in a new real asset today, or at a time in the future, is an example of a simple timing option in the real options literature. This opportunity can be treated as a finite-lived or perpetual American call option. Land development, expansion of firm capacity, and extraction of natural resources are a few popular examples of simple investment timing options that can be analyzed in a real options framework (Guthrie [17]). Indeed, by using real options analysis, it is possible to determine the value of simple timing options, e.g. the option value of waiting to invest, as well as under which circumstances it is optimal to invest and therefore not postpone the investment any further.

It is not uncommon that real option valuation models either directly apply, or only slightly modify, the Black and Scholes [3] and Merton [28] financial option valuation models. However, these models have been derived under several assumptions that in real options applications might be too restrictive and unrealistic. Above all, such models assume that the underlying asset is frequently traded in a perfect capital market, or that other assets perfectly span the risk of the underlying asset (see e.g. Henderson [19]). Under these assumptions, real options are valued in the setting of complete market models. The valuation models of Majd and Pindyck [24], McDonald and Siegel [25], Samuelson and McKean [34], [26] (which is applied to land valuation purposes in Geltner et al [16]) and Titman [37] are also examples of models that are derived from complete market settings. Since real options applications usually concern options where the underlying assets are infrequently traded, or even non-traded, it is important to discuss how to value real options in an incomplete market setting (see Henderson [18]; Borison [4]; Henderson [19]; Floroiu and Pelsser [14]).

A problem from a valuation point of view that can arise in real options analysis is whether the underlying asset that represents the investment project generates positive cash flows or not. Such cash flows, if they exist, are analogous to dividends paid on a stock. Merton [28] showed that American call options will only be exercised before expiration if there are positive dividends. In other words, early exercise of American options is not optimal in the case the underlying asset does not generate any cash flows. For perpetual American options, this result simply implies that it will never be optimal to exercise the option. As a consequence, it will never be optimal to invest in the case a perpetual timing option on an asset that does not generate (positive) cash flows (see e.g. McDonald and Siegel [25]).

In reality, though, there exist real assets that do not generate monetary cash flows analogous to stock dividends. For instance, real estate developers
might instead of building income-producing properties (e.g. multi-family rental properties), rather build small homes that will be sold to households as soon as they have been produced. Clearly such investment projects do not generate any monetary cash flows analogous to stock dividends or net operating incomes from income-generating properties. Geltner et al [15] in fact write that

By the term “built property” we mean the property once it has been developed. We use the word “rent” in its broad meaning. Thus, owner-occupied housing produces rents because the owners living in the housing avoid paying apartment rents and enjoy the utility of home ownership. Under this definition, no building would ever be built if it were not expected to produce rent (even if such rent will only be “monetized” and captured by the developer via the capitalization process of selling the building to a subsequent owner-user). [p. 25, footnote 9]

The above citation clearly shows that under classical completeness assumptions, a monetary or monetized operating current “yield”, of the built property must be assumed to exist in order to have a model that yields a finite exercise date (e.g. housing construction date).

In a critical article about how real options analysis can be applied for real-world corporate decision making, Borison [4] argue that different approaches to ROA have significant problems with inaccurate and inconsistent assumptions that make them effectively unacceptable for practical use in valuation or strategic applications involving corporate investments. A major target for the criticism is that classical approaches to real options analysis assume that the market is complete (all sources of risks can be effectively hedged through a traded tracking portfolio) and that all assets are continuously traded. Titman [37] developed an option valuation model for valuing vacant land spots that is close in its approach to the Cox-Ross-Rubinstein binomial option pricing model (Cox et al [10]), and the two-stage option pricing model of Bartter & Rendleman [33].

In practice, though, the underlying assets in real options analysis are not traded in capital markets, and other assets may only partially span risk, and consequently, valuing claims on non-traded assets introduce new challenges to option valuation modelling (Henderson [18], [19]). Henderson [19] finds that risk aversion and incompleteness in investment timing and option value significantly alter the conclusion of traditional complete real options models. Floroiu and Pelsser [14] also study the valuation of options written on infrequently traded or non-traded assets under incomplete market framework. They show that the introduction of incompleteness in the model makes early exercise of an American perpetuate call options optimal, even though there are no explicit dividends. See also Section 34.2 in Hull [20], where valuation of real options in incomplete models are discussed.

In order to obtain solutions that can result in finite exercise times under completeness in cases when no positive dividend yields exist, it is thus necessary to assume that non-cash flow generating underlying assets actually generate
monetary cash flows. A natural question that arises is if it really is necessary (or needed) to use the concept of monetized, or imputed, cash flows, such as rents, in order to find a solution to the real options valuation problem. In this paper we argue from a theoretical perspective that it is not necessary to use monetized or imputed rents. By assuming that the market is not complete, we derive optimal investment behavior even when the underlying project does not generate any actual (monetary) cash flows. Therefore, we show that when the market is incomplete, an investment timing problem can have a finite exercise date, no matter if a project has a zero or positive dividend yield.

Our main conclusion is that a project that does not generate cash flows can still have a finite investment time in an incomplete model for perpetual American real options (whereas it is never optimal to exercise the option to invest under completeness when the underlying asset does not produce any positive stream of cash flows). In other words, while incompleteness in many option valuation situations is regarded to be an annoying or aggravating circumstance, in this case it provides solution to the observation (questions) that many projects that do not generate any monetary (actual) cash flows still are undertaken. Thus in this case market incompleteness helps rather than hinders.

The rest of this paper is organized as follows. In Section 2 we present the modelling assumptions; in Section 3 we review results regarding the standard approach of valuing American call options applied to real options; Section 4 discusses and calculates the value of an American perpetual call option in incomplete mode. Section 5 concludes this paper and Appendix A contains results regarding perpetual American call options.

2 The basic model and problem

2.1 Modelling assumptions and notation

Let $(\Omega,F,P, (F_t))$ be a filtered probability space. The probability space is assumed to be complete, and the filtration is assumed to satisfy the usual assumptions: $F_0$ contains all $P$-null sets of $F$ and the filtration $(F_t)$ is right-continuous. We recall that a stopping time is a random variable $\tau : \Omega \to [0,\infty]$ such that $\{\tau \leq t\} \in F_t$ for every $t \geq 0$, i.e. given the information in $F_t$ we can decide if the random time $\tau$ has occurred or not.

Regarding the financial part, we assume that there exists a constant risk-free interest rate $r > 0$ in the sense that there exists a traded asset with value $B_t$ at time $t \geq 0$ such that

$$dB_t = rB_t dt \quad \text{with} \quad B_0 = 1.$$

We also assume that there exists a probability measure $Q$ locally absolutely continuous with respect to $P$, and such that the value of a stream of cash flows is the discounted value of the cash flows under $Q$, using $r$ as the discount rate. For the underlying theory, see e.g. Björk [2] or Karatzas & Shreve [22]. Expected values with respect to $P$ are denoted $E$, without any explicit reference to the measure, while $E^Q$ as above denotes the expected value with respect to the
measure \( Q \). We let \( W \) denote a (possibly multivariate) \( P \)-Wiener process and \( Z \) a \( Q \)-Wiener process of the same dimension.

The assumption of the existence of the pricing measure \( Q \) is our no-arbitrage condition. We could alternatively have assumed the existence of a stochastic discount factor process \((\Lambda_t)\), as is done in e.g. Floroiu & Pelsser [14]. In the presence of a bank account the two approaches are essentially equal.

Note that if \( S \) is the price process of a non-dividend paying traded asset then (assuming integrability of the price process) for any \( 0 \leq u \leq t < \infty \)

\[
S_u = E^Q \left[ e^{-r(t-u)} S_t \bigg| \mathcal{F}_u \right] \Leftrightarrow S_u e^{-ru} = E^Q \left[ S_t e^{-rt} \bigg| \mathcal{F}_u \right],
\]

i.e.

\[
S_t e^{-rt} = \frac{S_t}{B_t}
\]
is a \( Q \)-martingale.

### 2.2 The investment problem

Our goal is to analyze the following problem: A firm can invest in a project, where the value of the developed project is given by \( V_t \) at time \( t \geq 0 \) and the investment cost is the constant \( I > 0 \). The firm is allowed to choose the time at which the project is initiated, and wants to do this in an optimal way. Hence, the form wants to solve the problem

\[
sup_{\tau} E^Q \left[ e^{-r\tau}(V_\tau - I) \right]
\]
at time 0, where \( \tau \) is a stopping time. Let us define

\[
F(x) = sup_{\tau} E^Q \left[ e^{-r\tau}(V_\tau - I) \big| V_0 = x \right] = sup_{\tau} E^Q_x \left[ e^{-r\tau}(V_\tau - I) \right].
\]

This is the value of the investment opportunity at time \( t = 0 \) when \( V_0 = x \). Fakeev [13] has shown that the value at time \( t \geq 0 \) when the value of the developed project is equal to \( V_t \) is given by \( F(V_t) \). Hence, it is enough to consider the optimal stopping problem at time \( t = 0 \). For more on optimal stopping see Peskir & Shiryaev [32], and for more on American options see Karatzas [21] and Karatzas & Shreve [22].

If the choice is never to stop, then the optimal value is set to 0. This is consistent with the standard definition

\[
F(x) = \lim_{t \to \infty} sup e^{-rt}(V_t - I) \text{ on } \{ \tau = \infty \}.
\]

In some approaches only finite stopping times are considered, but we want to include the possibility of never stopping (i.e. never investing), and thus allowing for \( \tau = \infty \). Since we will never stop at a \( t \) with \( V_t < I \) we have the equality

\[
\sup_{\tau} E^Q \left[ e^{-r\tau}(V_\tau - I) \right] = \sup_{\tau} E^Q \left[ e^{-r\tau} \max(V_\tau - I, 0) \right].
\]
Hence, our investment problem can be formulated as finding the value of a perpetuate American call option. We let $\tau^\star$ denote, if it exists, an optimal stopping time:

$$E^Q \left[ e^{-r\tau^\star} (V_{\tau^\star} - I) \right] = \sup_{\tau} E^Q \left[ e^{-r\tau} (V_{\tau} - I) \right].$$

A more general class of investment problems allows the cost of investment $I$ to be varying. We do not consider this class of models, but remark that in some cases these models can be rewritten as an optimal stopping problem with constant investment cost.

The solution to the investment problem is known in many cases and here we list some of them.

(a) When $V$ follows a geometric Brownian motion (GBM) under $Q$,

$$dV_t = \mu_Q V_t dt + \sigma V_t dZ_t,$$

where we recall that $Z$ is a $Q$-Wiener process, $\mu_Q \in \mathbb{R}$ and $\sigma > 0$, then there is no finite optimal stopping time if $\mu_Q \geq r$. If $\mu_Q < r$, then the optimal stopping time is given by the hitting time of a level and in this case $\tau^\star$ need not be finite. The GBM model is the standard model, and is used by McDonald & Siegel in their seminal paper [25]. See Section 2.3 below for more on this model.

(b) More generally, we can replace the Brownian motion defining $V$ with a martingale, and specifically by a martingale that is a Lévy process. In Section 3.3 we consider models where the driving noise is given by a Brownian motion and/or a Poisson process. Both these processes are Lévy processes.

(c) Finally, we mention two other processes that can be used as a model for $V$: Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR) processes. We will not consider them, but semi-explicit solutions to the investment problem when $V$ is modelled as an OU process are given in Boyarchenko & Levendorskii [5] and for the CIR case in Ewald & Wang [12] and Leung, Li & Wang [23]. See also Sarkar [35].

Note that in the examples above we consider the dynamics under $Q$. Later, when we discuss modelling in more detail, we start with modelling dynamics under $P$, and then change measure to $Q$.

### 2.3 Geometric Brownian motion

We will now present the full solution to the investment problem when $V$ follows a GBM. Again assume that $V$ has $Q$-dynamics

$$dV_t = \mu_Q V_t dt + \sigma V_t dZ_t,$$

Initially we write $\mu_Q$, but later on we will use more explicit expressions using results from mathematical finance and the Girsanov theorem. The general
solution follows from Theorem A.1 in Appendix A. If \( \mu_Q \geq r \), then there is no optimal stopping time. In this case the optionality is always more valuable 'alive', so it is optimal to never stop. Now consider the case \( \mu_Q < r \). Let

\[
\beta_1 = \frac{1}{2} - \frac{\mu_Q}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu_Q}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}
\]

and

\[
V_c = \frac{\beta_1}{\beta_1 - 1} I.
\]

The optimal value is given by \( V_0 - I \) if \( V_0 \geq V_c \), i.e. it is optimal to immediately invest if the value \( V_0 \) of the developed project today is higher than the critical value \( V_c \). If \( V_0 < V_c \), then we should wait until we invest, and the value in this case is given by

\[
(V_c - I) \left( \frac{V_0}{V_c} \right)^{\beta_1}. \]

The optimal stopping time is equal to

\[
\tau^* = \inf\{t \geq 0 | V_t = V_c\}.
\]

The set on the right-hand side of this expression could be empty, i.e. it can happen that \( \tau^* = \infty \). See below for more on this. One can show that when \( \mu_Q < r \) then \( \beta_1 > 1 \) and \( V_c > I \). Hence, even though the investment is 'in-the-money' at time 0 (i.e. \( V_0 - I > 0 \)) it need not be optimal to invest.

2.3.1 \( V \) is the value of a traded asset

When \( V \) can be considered as the value, or price, of a traded asset, then it follows from the general theory of mathematical finance that we must have

\[
\mu_Q = r - \delta,
\]

where \( \delta \) is the yield (assumed to be constant) the developed project produces. It follows that if \( V \) is the price of a traded asset, then we must have \( \delta > 0 \) in order to have \( \mu_Q < r \) and get a non-trivial, i.e. an optimal stopping time that can be finite, solution to the optimal stopping problem.

2.3.2 \( V \) is the value of a non-traded asset

Now assume that \( V \) is not the price of a traded asset, or cannot be written as a portfolio of traded assets. In this case arbitrage theory is not enough to determine the drift of \( V \) under \( Q \). Let us start with the dynamics of \( V \) under \( P \):

\[
dV_t = \mu V_t dt + \sigma V_t dW_t.
\]

Changing measure from \( P \) to \( Q \) using the Girsanov kernel \(- \lambda \in \mathbb{R}\) we get

\[
dV_t = (\mu - \lambda \sigma) V_t dt + \sigma V_t dZ_t.
\]
Hence, in this case $\mu_Q = \mu - \lambda \sigma$. The reason we have a minus sign before $\lambda$ is that it allows us to interpret $\lambda$ as a market price of risk. In order to get a non-trivial solution to the optimal stopping problem in this case we must have

$$\mu - \lambda \sigma < r \iff \lambda > \frac{\mu - r}{\sigma}.$$ 

Let us now write

$$\mu - \lambda \sigma = r - [r - \mu + \lambda \sigma].$$

Here we can interpret $\delta_{imp} = r - \mu + \lambda \sigma$ as an implied yield. Note that the same condition needs to be satisfied in both the traded and non-traded case, namely that the yield must be strictly positive if a non-trivial solution is to exist. The dependence of $\lambda$ on $\beta_1$ can be studied through the derivative

$$\frac{\partial \beta_1}{\partial \lambda} = \frac{1}{\sigma} \left[ 1 + \frac{\frac{1}{2} - \frac{\mu - \lambda \sigma}{\sigma^2}}{\sqrt{\left(\frac{1}{2} - \frac{\mu - \lambda \sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}} \right] > 0.$$ 

It follows from

$$\frac{\partial V_c}{\partial \beta_1} = -\frac{I}{(\beta_1 - 1)^2} < 0$$

Figure 1: Gain function (solid curve), value of the American call option (dashed curves) and critical values (circled). The parameter values are $\mu = 0.06$, $\sigma = 0.20$, $r = 0.04$, $I = 100$ and the $\lambda$-values are 0.5, 1.0 and 1.5. The lowest value-curve has $\lambda = 1.5$, the middle curve has $\lambda = 1.0$ and the highest curve has $\lambda = 0.5$. To get a non-trivial solution we need to have $\lambda > (\mu - r)/\sigma = 0.10$. 

through the derivative

$$\frac{\partial \beta_1}{\partial \lambda} = \frac{1}{\sigma} \left[ 1 + \frac{\frac{1}{2} - \frac{\mu - \lambda \sigma}{\sigma^2}}{\sqrt{\left(\frac{1}{2} - \frac{\mu - \lambda \sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}} \right] > 0.$$
That
\[ \frac{\partial V_c}{\partial \lambda} = \frac{\partial V_c}{\partial \beta_1} \cdot \frac{\partial \beta_1}{\partial \lambda} < 0. \]
Hence, an increase in \( \lambda \), holding everything else equal, implies a decrease in the critical value. Since we can interpret \( \lambda \) as a market price of risk, an increase in the market price of risk makes the investor more eager to exercise the option; see Figure 1.

2.3.3 The probability of investing

Since the optimal stopping rule when the yield, monetary or implied, is strictly positive, is equal to the first time the value process hits a given level, it is quite easy, using well known facts about Brownian motions, to derive the probability of having a finite investment time when \( V \) is modelled as a GBM. The starting point is

\[ P \left( \sup_{t \geq 0} (\mu t + \sigma W_t) \leq x \right) = 1 - e^{\frac{-\mu x}{\sigma^2}}, \]

which holds when \( \mu < 0 \) and \( x \geq 0 \) (see p. 760 in Shiryaev [36]). This can then be used to prove the following result.

**Proposition 2.1** Let \( \tau^* \) be the optimal stopping time in the GBM model and \( V_c \) the critical level. If \( V_0 < V_c \) and \( \mu - \sigma^2/2 < 0 \) then

\[ P(\tau^* < \infty) = \left( \frac{V_0}{V_c} \right)^{1 - \frac{2\mu}{\sigma^2}}, \]

and else \( P(\tau^* < \infty) = 1. \)

**Proof.** We have

\[ P(\tau^* < \infty) = P \left( \sup_{t \geq 0} V_t \geq V_c \right), \]

where \( V_c \) is the critical level from previously and

\[ V_t = V_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}. \]

It follows that

\[ P \left( \sup_{t \geq 0} V_t \geq V_c \right) = P \left( \sup_{t \geq 0} V_0 e^{(\mu - \sigma^2/2)t + \sigma W_t} \geq V_c \right) \]
\[ = P \left( \sup_{t \geq 0} \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \geq \ln \left( \frac{V_c}{V_0} \right) \right) \]
\[ = 1 - P \left( \sup_{t \geq 0} \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \leq \ln \left( \frac{V_c}{V_0} \right) \right) \]

If \( V_0 \geq V_c \) then we stop immediately, i.e. \( \tau^* = 0 \) in this case. If \( V_0 < V_c \) and \( \mu - \sigma^2/2 \geq 0 \), then we will always hit any level greater than or equal to \( V_0 \)
so again we have $\tau^* < \infty$ with $P$-probability 1. Finally, when $V_0 < V_c$ and $\mu - \sigma^2/2 < 0$ then

$$P(\tau^* < \infty) = e^{2(\mu - \sigma^2/2) \ln(V_c/V_0)} = \left(\frac{V_c}{V_0}\right)^{\frac{2\mu}{\sigma^2}-1} = \left(\frac{V_0}{V_c}\right)^{1 - \frac{2\mu}{\sigma^2}}.$$ 

\[\square\]

Note that we are interested in the probability under $P$ that the critical level is hit. The $Q$-probability for this event has no apparent interpretation.

## 3 Modelling the value process

### 3.1 Complete and incomplete models

A model of a financial market that is free of arbitrage can be either complete or incomplete. It is complete if there exists exactly one pricing measure $Q$, and incomplete if there are strictly more than one. In a complete market any contingent claim that is integrable enough has a unique price. Independent of whether a model is complete or incomplete, a traded asset has the same price under every pricing measure.

**Example 3.1** Let $V$ have dynamics

$$dV_t = \mu V_t dt + \sigma V_t dW_t$$

under $P$, and assume that the developed project does not pay out any cash flows during its lifetime. Consider the following two cases:

- **$V$ is the price of a traded asset, or can be written as a portfolio of traded assets. In this case the drift under $Q$ is $r$, and there is no optimal stopping time.**

- **$V$ is not the price a traded asset. In this case, if we change measure from $P$ to $Q$ by a Girsanov transformation with constant Girsanov kernel $-\lambda$, we get**

$$dV_t = (\mu - \lambda \sigma)V_t dt + \sigma V_t dZ_t.$$  

**Define as above**

$$\delta_{\text{imp}} = r - \mu + \lambda \sigma.$$  

**If**

$$\delta_{\text{imp}} > 0 \iff \mu - \lambda \sigma < r \iff \lambda > \frac{\mu - r}{\sigma}$$

**then there exists a non-trivial optimal stopping time.**

The previous example shows that in an incomplete model it is not enough just to specify the dynamics of $V$, we must also decide if we consider $V$ to be the value of a traded asset or not.
In Dixit & Pindyck [11] it is assumed either that the optimal stopping problem is stated using a subjective discount rate $\rho$ and that the expected value is taken using the probability measure $P$, or that there is a traded asset or a portfolio of traded assets that is perfectly correlated with the value $V$ of the developed project. The last approach is equal to assuming that $V$ is the price of a traded asset.

In order for the investment problem to have a finite optimal time a necessary condition is that the discount rate is higher than the expected growth rate. If the market is complete, then the drift under $Q$ of a traded non-dividend paying asset is equal to the risk-free rate, which is also the discount rate. In order to circumvent this problem, a non-monetary yield is introduced in the literature. In this case, there will be a possibility of a finite optimal stopping time.

The completeness assumption is also present in the concept marketed asset disclaimer (MAD) as presented in Copeland & Antikarov [9]. Using MAD means that we assume that the investment without the optionality is a traded asset, and this assumption is effectively used to complete the market.

However, many times an assumption of incomplete markets is more realistic. There is no canonical way in which incompleteness can occur. We can, however, distinguish between two different cases in an incomplete markets:

(a) The process $V$ is the price of a traded asset.

(b) The process $V$ is not the price of a traded asset.

When (a) holds, we still have the drift $r$ for any non-dividend paying asset, and even though the model is incomplete we will not have a non-trivial solution to our optimal timing problem unless the asset pays a strictly positive dividend yield. Case (b) is the interesting one, where we can get a non-trivial solution whether the project pays out any dividends or not.

An approach that is similar but not equal to ours is to have a model with Knightian uncertainty (or just uncertainty) or ambiguity. Again the crucial part is the fact that there exists a non-single set of stochastic processes such that every one of these can be used as Girsanov kernel to change from the objective measure $P$ to the risk-neutral measure $Q$. One way to handle this type of non-uniqueness of equivalent martingale measures is to look at the worst-case case, i.e. the measure that, if you have bought the asset, assigns the smallest value of the asset. A subclass is the one where the absolute value of the Girsanov kernel is bounded by a non-negative constant $\kappa$; this situation is referred to as $\kappa$-ignorance. Originally $\kappa$-ignorance goes back to Chen & Epstein [7]. Nishimura & Ozaki [30] applies Knightian uncertainty and $\kappa$-ignorance to the optimal timing investment problem. A variation of this is to have a bound on the total volatility of the stochastic discount factor; this is the method suggested by Cochrane & Saa-Requejo [8] and goes by the name of ‘good-deal bounds’. This latter approach is used by Floroiu & Pelsser [14].

Instead of using good-deal bounds, we can try to pin down the exact value of the risk premium. One way of doing this is to use the continuous-time version of CAPM; this is e.g. done in Section 34.3 of Hull [20]. Using CAPM we get the
relation

\[ \lambda = \frac{\rho_{MV}}{\sigma_M} (\mu_M - r) \]

where \( \mu_M \) and \( \sigma_M \) are the drift and volatility of the market portfolio respectively and \( \rho_{MV} \) is the instantaneous correlation between the return on the developed project and the return on the market portfolio.

Our main point in this paper is that from a modelling point of view an incomplete model is much richer and more flexible than a complete one. In a complete model there is essentially only one way in which we can model the value of a project if we want to have a non-trivial solution to our problem: as a traded asset which generates a strictly positive stream of cash flows.

### 3.2 A two-dimensional Brownian motion model

We will now look at an incomplete model driven by a two-dimensional Brownian motion. In this model the value of the developed project is not a traded asset, but that there exists a non-dividend paying traded asset with price process \( S \) that is correlated with the value of the developed project. This is the approach taken in e.g. Floroiu & Pelsser [14] and Henderson [18], [19]. The model is

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_S S_t dt + \sigma_S S_t dW^1_t \\
\frac{dV_t}{V_t} &= \mu_V V_t dt + \sigma_V V_t \left( \rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t \right),
\end{align*}
\]

where \( W = (W^1, W^2)^T \) is a 2-dimensional Brownian motion and \( \rho \in [-1, 1] \) is the constant instantaneous correlation between \( S \) and \( V \) is the sense that (formally)

\[
\text{Corr} \left( \frac{dS_t}{S_t}, \frac{dV_t}{V_t} \right) = \rho dt.
\]

Note that since \( \rho \) is constant both \( P \) and \( V \) are geometric Brownian motions. We assume that the \( Q \)-Wiener process is

\[
\begin{align*}
\frac{dZ^1_t}{Z^1_t} &= \lambda_1 dt + dW^1_t \\
\frac{dZ^2_t}{Z^2_t} &= \lambda_2 dt + dW^2_t.
\end{align*}
\]

Again we have assumed that \( \lambda = (\lambda_1, \lambda_2)^T \) is a vector of constants. Since \( S \) is the price of a traded asset the expected return under any equivalent martingale measure is equal to \( r \). This is equivalent to

\[ \lambda_1 = \frac{\mu_S - r}{\sigma_S}, \]

and we have one degree of freedom in choosing the market price of risk \( \lambda_2 \).

Regarding the correlation, there are some special cases: \( \rho = 0 \) and \( |\rho| = 1 \).

In the first case, the stochastic processes are uncorrelated, and in the other two they are perfectly positively or negatively correlated, respectively. When the processes are uncorrelated we are back to the same situation as when there is no traded asset, and when \( |\rho| = 1 \) we can treat \( V \) as being a traded asset.
When \( \rho \in (-1, 0) \cup (0, 1) \)

we get a new class of models. Whatever the value of \( \rho \), the dynamics of \( V \) under \( Q \) is

\[
dV = \left[ \mu_V - \sigma_V \left( \frac{\mu_S - r}{\sigma_S} \rho + \lambda_2 \sqrt{1 - \rho^2} \right) \right] V \, dt + \sigma_V \, \left( \rho dZ_1^t + \sqrt{1 - \rho^2} dZ_2^t \right).
\]

Note that

\[
\rho dZ_1^t + \sqrt{1 - \rho^2} dZ_2^t
\]

is a standard one-dimensional Brownian motion under \( Q \). The implied yield in this case is

\[
\delta_{\text{imp}} = r - \mu_V + \sigma_V \left( \frac{\mu_S - r}{\sigma_S} \rho + \lambda_2 \sqrt{1 - \rho^2} \right).
\]

Assuming that \( \lambda_2 \) is such that this implied yield is strictly positive, we can proceed as above and use Theorem A.1 again to value the investment opportunity. Assuming that \( |\rho| < 1 \), the condition \( \delta_{\text{imp}} > 0 \) is equivalent to

\[
\lambda_2 > \frac{\mu_V - r}{\sigma_V} - \rho \frac{\mu_S - r}{\sigma_S} \sqrt{1 - \rho^2},
\]

which gives a lower bound on \( \lambda_2 \). One guiding way to find the value of \( \lambda_2 \) is to use good-deal bounds. These were introduced in Cochrane & Saa-Requejo [8], and they also give an example of how to calculate these bounds in a model as the one in this section. Good-deal bounds are also used in Floroiu & Pelsser [14]. With our notation, the bound in Floroiu & Pelsser is given by

\[
\lambda_2 \in \left[ -\sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2}, \sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2} \right],
\]

where \( k > 0 \) is the limit on the total volatility of the stochastic discount factor:

\[
\sqrt{\lambda_1^2 + \lambda_2^2} = \sqrt{\left( \frac{\mu_S - r}{\sigma_S} \right)^2} + \lambda_2^2 \leq k.
\]

Hence, if

\[
\lambda_2 \in \left( \frac{\mu_V - r}{\sigma_V} - \rho \frac{\mu_S - r}{\sigma_S} \sqrt{1 - \rho^2}, \sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2} \right),
\]

which is a non-empty set if \( k \) is large enough, then we have a non-trivial optimal investment time as well as a good-deal bound.
3.3 A Brownian-Poisson model

We can use a Poisson process to represent the unique risk in the value process of developed land. This could e.g. be used to model a situation where there might be a sudden drop, or drops, in the value of the project, and where this drop does not influence the value of the traded asset. Specifically, we assume the following model under the measure \( P \):

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sigma S_t dW_t \\
    dV_t &= \mu V_t dt + \gamma V_t dW_t - \eta V_t dN_t.
\end{align*}
\]

Here \( \eta \in [0,1] \) and \( N \) is a Poisson process with constant \( P \)-intensity \( \nu \) and independent of \( W \). Again we assume that \( S \) is the price process of a non-dividend paying traded asset. We can write

\[
dV_t = (\mu V - \eta \nu) V_t dt + \gamma V_t dW_t - \beta V_t d(N_t - \nu t),
\]

where

\[
N_t - \nu t
\]

is a \( P \)-martingale. Hence, the expected return of \( V \) under \( P \) is

\[
\mu V - \eta \nu
\]

and the correlation between \( dV_t/V_t \) and \( dS_t/S_t \) is

\[
\rho = \frac{\gamma}{\sqrt{\gamma^2 + \beta^2 \lambda}}.
\]

Again, under the measure \( Q \) the price process \( S \) has dynamics

\[
dS_t = r S_t dt + \sigma S_t dZ_t,
\]

where \( Z \) is a \( Q \)-Brownian motion.

When we change measure from \( P \) to \( Q \), the Poisson process changes intensity from \( \nu \) to \( \nu_Q \). We assume that \( \nu_Q > 0 \) is a constant, i.e. that \( N \) is a Poisson process also under \( Q \). One usually writes

\[
\nu_Q = (1 + h) \nu,
\]

where \( h > -1 \) is the, in this case constant, Girsanov kernel. We will however use \( \nu_Q \) instead of \( h \) as our free parameter. The dynamics of \( V \) under \( Q \) is thus given by

\[
dV_t = \left( \mu V - \gamma \frac{\mu S - r}{\sigma S} - \eta \nu_Q \right) V_t dt + \gamma V_t dZ_t - \eta V_t d(N_t - \nu_Q t),
\]

where \( N \) is a Poisson process with \( Q \)-intensity \( \nu_Q \). The intensity \( \nu_Q \) replaces \( \lambda_2 \) in the model in the previous section. Now the implied yield is given by

\[
\delta_{imp} = r - \mu V + \gamma \frac{\mu S - r}{\sigma S} + \eta \nu_Q.
\]
To value the timing option in this case we use Theorem A.2 in Appendix A. Again, it turns out that we need the requirement that $\delta_{\text{imp}} > 0$ in order to get a non-trivial solution to the investment problem. The optimal value when it is not optimal to stop is given by

$$(V_c - I) \left( \frac{V_0}{V_c} \right)^{a_0},$$

which has a close resemblance to the expression in the GBM case (see Appendix A). Here $V_c$ is the critical level at which we should optimally stop, i.e. the optimal stopping time is again a hitting time. In general, there is no explicit expression for either the parameter $a_0$ or the critical level $V_c$ (but see below for the special case $\gamma = 0$ and $\eta = 1$. With $\eta \in (0, 1]$ we have

$$\delta_{\text{imp}} > 0 \quad \Leftrightarrow \quad \nu_Q > \frac{\mu_V - r - \gamma \mu_S - r}{\eta} \sigma_S,$$

which is a lower bound on the free parameter. Note that by setting $\gamma = 0$ we reduce the model to a purely Poisson-driven one in which the dynamics of the traded asset does not play any role.

**Example 3.2** A special case of the Brownian-Poisson model is when $\gamma = 0$ and $\eta = 1$.

In this case, the value $V$ has dynamics

$$dV_t = \mu V_t dt - V_t dN_t$$

under $P$. Hence, the value grows with the constant rate $\mu V$ until the first time the Poisson process jumps, in which case the value goes to zero and stays there forever. We can think of this as a simple model of competition: If we wait too long to invest a competitor can seize the opportunity, and the investment is no longer available for us. The dynamics can be written

$$dV_t = (\mu V - \nu_Q)V_t dt - V_t d(N_t - \nu_Q t),$$

where $N$ is a $Q$-Poisson process with intensity $\nu_Q$. The implied yield in this case is

$$\delta_{\text{imp}} = r - \mu V + \nu_Q.$$

We assume that the parameters in the model satisfy this requirement, and again we use Theorem A.2. In this case there is an explicit solution to the optimal stopping problem. The critical level $V_c$ at which we should stop is given by

$$V_c = \frac{a_0}{a_0 - 1} I,$$

where $a_0$ is given by

$$a_0 = \frac{r + \nu_Q}{\mu V}.$$
Note that 
\[ \delta_{\text{imp}} > 0 \iff a_0 > 1. \]

We now get the critical level
\[ V_c = I \frac{r + \nu Q}{r + \nu Q - \mu V} = I \frac{\delta_{\text{imp}} + \mu V}{\delta_{\text{imp}}} = I \left( 1 + \frac{\mu V}{\delta_{\text{imp}}} \right). \]

The value of the investment if \( V_0 < V_c \) (i.e. it is not optimal to exercise the option) is given by
\[ \text{Value} = (V_c - I) \left( \frac{V_0}{V_c} \right)^{r + \nu Q / \mu V} = (V_c - I) \left( \frac{V_0}{V_c} \right)^{1 + \frac{\delta_{\text{imp}}}{\mu V}}. \]

Note that this formula is also valid if the process has jumped down to 0. \( \square \)

In fact, see Appendix A and Dixit & Pindyck [11] p. 171, the value function in the continuation region when \( \eta = 1 \) solves a similar ODE as in the purely Wiener driven model.

### 3.4 A killed growth model

We can generalize Example 3.2 from the previous section as follows. Now we model \( V \) directly under \( Q \), and let the value have dynamics
\[ dV_t = \mu_Q V_t dt - V_t dN_t, \]
where \( N \) is a counting process with intensity \( \nu_Q(V_t) \) under \( Q \), and \( \mu_Q > 0 \) is a constant. One can show that (see Appendix A) the optimal value is given by
\[ (V_c - I) \left( \frac{V_0}{V_c} \right)^{r / \mu} e^{\frac{1}{2} \nu_0 \int_0^{V_0 - V_c} \frac{\nu_0(u)}{u} du} \]
when \( V_0 < V_c \), and the critical level \( V_c \) is the largest solution strictly greater than \( I \) to the equation
\[ (r + \nu_Q(x)) \cdot (x - I) - \mu x = 0. \]

Again, the value is given by \( V_0 - I \) if \( V_0 \geq V_c \), and the optimal stopping time is given by
\[ \tau^* = \inf \{ t \geq 0 \mid V_t = V_c \}. \]

**Example 3.3** Assume that
\[ \nu_Q(x) = \nu_0 + \nu_1 x, \quad \nu_0, \nu_1 > 0. \]
We can think of this as a model of when the increase in value of the developed project increases the probability of a competitor to enter, and thereby removing the investment opportunity. With this model the optimal value is given by
\[ (V_c - I) \left( \frac{V_0}{V_c} \right)^{r + \nu_0 / \mu} e^{\frac{\nu_1}{2} (V_0 - V_c)}. \]
when $V_0 \in [0, V_c)$, and the critical level is

$$V_c = \frac{\nu_1 I - \nu_0 + \mu Q - r}{2 \nu_1} + \sqrt{\left(\frac{\nu_1 I - \nu_0 + \mu Q - r}{2 \nu_1}\right)^2 + (r + \nu_0)I}.$$  

4 Conclusions

We have shown that by introducing an incomplete model, the optimal investment problem can have a finite optimal investment time even though the value of the developed project doesn’t generate any cash flows. Traditionally, to circumvent the problem of not having a possible finite investment time, a complete model together with an assumption of a strictly positive non-monetary yield has been used. We use the term “implied yield” for the fictitious non-monetary yield that occurs due to the incompleteness of a model. This is similar, but not equal, to the “implicit yield” as described in Dixit & Pindyck [11]. Their yield is a measurement of the utility of having the option instead of the completed project (see Dixit & Pindyck [11] p. 149).

We believe that the use of an incomplete model is better way to understand the fact that many projects that do not produce any cash flows are still undertaken. We have to choose the market price of risk in order to calibrate (or specify) the model. This is not a simple task, and can be done in many ways, but we believe that the advantage of using an incomplete model with an unknown market price of risk is better from a theoretical perspective than assuming a complete model with a strictly positive non-monetary yield.

Further developments would include time-dependent, deterministic or stochastic, parameters, as well as using more complex driving stochastic processes.

A Valuing perpetual American call options

In this section we present the solution to the problem of finding

$$F(x) = \sup_{\tau} E_x [e^{-\rho \tau} \max(X_\tau - K, 0)],$$

where $X$ is a geometric Brownian motion, $\tau$ is a stopping time, $\rho > 0$ is the constant discount rate and $K > 0$ is the strike price. We prefer to formulate the problem in this general setting, and when applied to our model we use the pricing measure $Q$ together with

$$X = V, \; \rho = r \; \text{and} \; K = I,$$

as well as a suitable choice of parameters. We start by considering the case when the underlying process is a geometric Brownian motion. For a proof of the following result, see e.g. Chapter VIII, §2a in Shiryaev [36] or Øksendal [31] pp. 209-211.
Theorem A.1. Let $X$ satisfy
\[ dX_t = \mu X_t dt + \sigma X_t dW_t \]
with $\mu \in \mathbb{R}$ and $\sigma > 0$, and let
\[ F(x) = \sup_\tau E_x [e^{-\rho \tau} \max(X_\tau - I, 0)] , \]
where $\rho > 0$ and $I > 0$.
- When $\mu \geq \rho$, then $F(x) = x$, and there is no optimal stopping time.
- When $\mu < \rho$, then
\[ F(x) = \begin{cases} (X_c - I) \left( \frac{x}{X_c} \right)^{\beta_1} & \text{if } x \in [0, X_c), \\ x - I & \text{if } x \in [X_c, \infty), \end{cases} \]
where
\[ X_c = \frac{\beta_1}{\beta_1 - 1} I \]
and
\[ \beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2\rho}{\sigma^2}} . \]
The optimal stopping time $\tau^*$ is given by
\[ \tau^* = \inf \{ t \geq 0 | X_t = X_c \} . \]

Now let $X$ have dynamics
\[ dX_t = \mu X_t dt + \gamma X_t dW_t - \eta X_t (dN_t - \nu dt) , \]
where $\mu \in \mathbb{R}$, $\gamma > 0$, $\eta \in [0, 1]$ and $N$ a Poisson process with constant intensity $\nu > 0$. Note that the specification of the jump part (only negative jumps) ensures that we will have smooth fit, i.e. the value function $F$ is $C^1$ for every $x \in [0, \infty)$. To find a candidate solution $\hat{F}$ we assume that there exists a level $X_c$ with $X_c \in (K, \infty)$ such that
\[ \frac{1}{2} \gamma^2 x^2 \hat{F}''(x) + (\mu + \eta \nu)x \hat{F}'(x) + \nu[\hat{F}(x(1 - \eta)) - \hat{F}(x)] - \rho \hat{F}(x) = 0 \]
when $x \in [0, X_c)$ and $\hat{F}(x) = x - I$ when $x \geq X_c$. We look for solutions to the ODE on the form
\[ \hat{F}(x) = x^a . \]
Inserting this in the equation yields
\[ \frac{1}{2} \gamma^2 a(a - 1) + (\mu + \eta \nu)a + \nu[(1 - \eta)^a - 1] - \rho = 0. \]

Let
\[ h(a) = \frac{1}{2} \gamma^2 a(a - 1) + (\mu + \eta \nu)a + \nu[(1 - \eta)^a - 1] - \rho. \]

The function \( h \) belongs to \( C^\infty(\mathbb{R}; \mathbb{R}) \), is convex and satisfies
\[ \lim_{a \to \pm \infty} h(a) = \infty. \]

It follows that \( h \) is a smooth U-shaped function. If
\[ h(1) = \mu + \eta \nu + \nu[1 - \eta - 1] - \rho = \mu - \rho < 0 \]
then there exists a unique \( a_0 > 1 \) such that \( h(a_0) = 0 \). The smooth fit condition implies the boundary conditions
\[ \hat{F}(X_c) = X_c - K \]
\[ \hat{F}'(X_c) = 1. \]

Inserting the candidate solution yields
\[ AX_c^{a_0} = X_c - K \]
\[ Aa_0 X_c^{a_0 - 1} = 1, \]

and from this
\[ X_c = \frac{a_0}{a_0 - 1} K \]
\[ A = (X_c - K)^{-a_0}. \]

It is possible to verify that when \( \mu < \rho \), then the optimal solution is indeed given by \( \hat{F} \).

**Theorem A.2** With notation and conditions as above, the optimal value of the Brownian-Poisson model is given by
\[ F(x) = \begin{cases} (X_c - K)\left(\frac{x}{X_c}\right)^{a_0} & \text{if } x \in [0, X_c) \\ x - K & \text{if } x \in [X_c, \infty) \end{cases} \]

with optimal stopping time
\[ \tau^* = \inf\{t \geq 0 \mid X_t = X_c\}. \]

This can be proved as in the proof of Theorem 3.1 in Mordecki [29]. The result can be generalized to the case when the Poisson process is replaced by a Lévy
process with Lévy measure supported on the negative real line. See Mordecki [29] for details.

The equation for $a_0$ above can be explicitly solved when $\eta = 0$, in which case we there is no Poissonian noise, and when $\eta = 1$, in which case the investment opportunity ceases to exist at an exponentially distributed random time. In the latter case we get the equation

$$\frac{1}{2} \gamma^2 a(a - 1) + (\mu + \nu)a - (\rho + \nu) = 0.$$ 

This is in fact the equation we need to solve if we have a purely Wiener-driven model as described in Theorem A.1, and the formulas from that appendix holds if we make the change

$$\sigma \rightarrow \gamma$$

$$\rho \rightarrow \rho + \nu$$

$$\mu \rightarrow \mu + \nu$$

in these formulas.

We will now generalize the Wiener-Poisson model above in the case when $\gamma = 0$ and $\eta = 1$.

**Theorem A.3** The optimal value of the stopping problem

$$F(x) = \sup_{\tau} E_x \left[ e^{-\rho \tau} \max(X_\tau - K, 0) \right]$$

when $(X_t)$ has dynamics

$$dX_t = \mu X_t dt - dX_t - dN_t \text{ with } X_0 \geq 0,$$

where $(N_t)$ is a counting process with intensity $\nu(X_t)$ satisfying $E[\nu(X_t)] < \infty$, and where $0 < \mu < \rho$, is given by

$$F(x) = \begin{cases} (X_c - K) \left( \frac{x}{X_c} \right)^{\rho/\mu} e^{\frac{1}{\mu} \int_{X_c}^{x} \frac{\nu(u)}{u} du} & \text{if } x \in [0, X_c) \\ X_c - K & \text{if } x \in [X_c, \infty). \end{cases}$$

Here $X_c$ is the largest solution strictly greater than $K$ to the equation

$$(\rho + \nu(x)) \cdot (x - K) - \mu x = 0,$$

and the optimal stopping time is

$$\tau^* = \inf \{ t \geq 0 \mid X_t = X_c \}.$$ 

**Proof.** We use the fact that if $(F, \tau^*)$ satisfies

(i) $F(x) = E_x \left[ e^{-\rho \tau^*} \max(X_{\tau^*} - K, 0) \right].$
(ii) $F(x) \geq \max(x - L, 0)$ and

(iii) $e^{-\rho t}F(X_t)$ is a supermartingale,

then $F$ is the optimal value function and $\tau^*$ is an optimal stopping time in the valuation problem (see Mordecki [29] or Armerin [1] for a proof that these conditions are sufficient). First of all

$$E_x \left[ e^{-\rho \tau^*} \max(X_{\tau^*} - K, 0) \right] = E_x \left[ e^{-\rho \tau^*} (X_{\tau^*} - K) 1(\tau^* < \infty) \right]$$

$$= \left\{ X_c = X_{\tau^*} = xe^{\mu \tau^*} \text{ on } \{\tau^* < \infty\} \right\}$$

$$= E_x \left[ \left( \frac{x}{X_c} \right)^{\rho/\mu} (X_c - K) 1(\tau^* < \infty) \right]$$

$$= \left( \frac{x}{X_c} \right)^{\rho/\mu} (X_c - K) P_x (\tau^* < \infty)$$

$$= \left( \frac{x}{X_c} \right)^{\rho/\mu} (X_c - K) e^{-\frac{1}{\mu} \ln \left( \frac{X_c}{x} \right) \nu(xe^{\mu \tau^*}) \int_0^{\tau^*} ds}$$

$$= \left( \frac{x}{X_c} \right)^{\rho/\mu} (X_c - K) e^{\frac{1}{\mu} \int_{x_c}^{x \rho} \frac{\nu(u)}{u} du}$$

$$= F(x),$$

which proves (i). Here we have used that

$$\tau^* < \infty \iff N_{\frac{1}{\mu} \ln \left( \frac{X_c}{x} \right)} = 0$$

when $X_0 = x$. That condition (ii) is satisfied is obvious, and we proceed to prove (iii), i.e. that $e^{-\rho t}F(X_t)$ is a supermartingale. Define

$$f(x) = (\rho + \nu(x)) \cdot (x - K) - \mu x.$$ 

Then $f(K) = -\mu K < 0$ and for every $x \geq K$ we have

$$f(x) \geq \rho(x - K) - \mu x = (\rho - \mu)x - \rho K.$$ 

Since $\rho/(\rho - \mu) > 1$ we have

$$f \left( K \frac{\rho}{\mu - \rho} \right) \geq 0$$

and

$$f(x) > 0 \text{ for every } x > K \frac{\rho}{\rho - \mu}.$$ 

Hence, there exists at least one solution to $f(x) = 0$; we denote by $X_c$ the largest of these. We now recall Ito’s formula for processes of finite variation. If $f \in C^1(\mathbb{R}; \mathbb{R})$ and $(X_t)$ is a real-values stochastic process of finite variation, then

$$df(X_t) = f'(X_t) dX_t^x + f(X_t) - f(X_t).$$
Here $X^c$ is the continuous part of $X$. For a proof see e.g. Section 6.4.1 in Medvegyev [27]. In our case
\[ dX^c_t = \mu X_t dt \]
and
\[ f(X_t) - f(X_{t-}) = f(X_{t-} + \Delta X_t) - f(X_{t-}) = (f(0) - f(X_{t-}))dN_t. \]

We get
\[
d \left( e^{-\rho t} F(X_t) \right) = -\rho e^{-\rho t} F(X_t) dt + e^{-\rho t} dF(X_t)
= -\rho e^{-\rho t} F(X_t) dt + e^{-\rho t} [F'(X_t)\mu X_t dt - F(X_{t-})dN_t]
= e^{-\rho t} F(X_{t-}) \left[ -\rho dt + \mu X_t \frac{F'(X_t)}{F(X_t)} dt - dN_t \right].
\]

Now
\[
\frac{\mu x F'(x)}{F(x)} = (\rho + \nu(x))1(x < X_c) + \frac{\mu x}{x - K}1(x \geq X_c)
= \rho + \nu(x) + \left( \frac{\mu x}{x - K} - \rho - \nu(x) \right)1(x \geq X_c)
\]
It follows that
\[
e^{-\rho t} F(X_t) = F(x) - \int_0^t e^{-\rho u} F(X_u) dM_u +
\int_0^t \frac{\mu X_u - (\rho + \nu(X_u))(X_u - K)}{X_u - K} 1(X_u \geq X_c) d\nu(X_u),
\]
where the process
\[
M_t = N_t - \int_0^t \nu(X_u) du
\]
is a martingale. Since
\[
0 \leq e^{-\rho t} F(X_{t-}) \leq X_t \leq X_0 e^{\mu t},
\]
the process $\int_0^t e^{-\rho u} F(X_u) dM_u$ is also a martingale. Finally, it is easy to check that $E_x \|e^{-\rho t} F(X_t)\| < \infty$ for every $t \geq 0$. This shows that $(e^{-\rho t} F(X_t))$ is a supermartingale, and the proof is complete.

We have not aimed at the most general version of the previous theorem. It is possible to replace $\rho$ and $\mu x$ with functions $\rho(x)$ and $\mu(x)$ respectively, together with suitable conditions on these functions.
References


