Random start American perpetual options

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Abstract

The valuation of American perpetual options with the property that they are only possible to exercise after a random time has occurred are considered. One situation where this feature is present is when we want to value the real option of when to build on vacant land and we are waiting for a permit. This and a version of an abandonment option are our two applications of this model.

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1 Introduction

During the lifetime of many investments there are events which the investor cannot control, but which are crucial in the development of the project. It could be waiting for a permit or unexpectedly be exposed to some form of news. In this paper we consider a situation where there is an exogenously given random time marking the first time at which an action can be taken. This means that even though the investor wants to take the action, e.g. initiate a project, he is not necessarily allowed to do so.

There is a resemblance between this type of random time and the default time of a bond, and we will partly use similar models as the ones used in credit risk theory. A simple way of modelling the random time is to assume that it is constant. Although this in many cases is not a very realistic model, it is a simple one, and we include it as one of our examples. A more realistic model is to assume that the random time is non-deterministic. Since we use a risk-neutral probability measure to calculate the value of options, we need to specify the properties of the random time under both the objective as well as under the risk-neutral measure. We must also determine how the random time is related to any other stochastic elements, typically some stochastic process(es) that is the underlying of the option, both under the objective as well as under the risk-neutral probability measure.

As applications we consider the optimal time to initiate a project (e.g. to start building on vacant land) given the constraint of a pending application to start, and a version of an abandonment option. These are two examples of real options, i.e. investment opportunities where there is an element of optionality. For a broad introduction to real options, see Dixit & Pindyck [4].

The rest of the paper is organised as follows. In Section 2 we discuss the general modelling assumptions. Section 3 contains the model applied to two random start investment problems, Section 4 outline some generalisations of the models described in Section 2, and Section 5 concludes and summarises.

2 The model

2.1 Generalities

Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t))\) be a complete filtered probability space where the filtration satisfies the usual assumptions of being right-continuous and \(\mathcal{F}_0\) containing all \(P\)-null sets of \(\mathcal{F}\). A random time \(\tau\) is a non-negative random variable:

\[\tau : \Omega \rightarrow [0, \infty].\]

A random time \(\tau\) is a stopping time with respect to the filtration \((\mathcal{F}_t)\) if it fulfills

\[\{\tau \leq t\} \in \mathcal{F}_t \text{ for every } t \geq 0.\]
We assume that there exists a bank account with constant rate \( r \geq 0 \) whose value evolves according to
\[
\frac{dB(t)}{dt} = rB(t)dt \quad \text{with} \quad B(0) = 1.
\]

We further assume the existence of a pricing measure \( Q \) locally equivalent to \( P \) such that the value of a stream of cash flows is the discounted expected value under \( Q \), where the risk-free rate is used as discount rate.

Given is also a real-valued and continuous time-homogeneous Markov process \((X_t)\) which is adapted to \((\mathcal{F}_t)\). We will restrict our models to the geometric Brownian motion case, but see Section 4.1 below, and to non-negative gain functions \( G : \mathbb{R} \to \mathbb{R}_+ \) independent of time. The geometric Brownian motion is in our applications the value of a developed project. Given these parts, our first task will be to calculate the value of the standard American perpetual option.

The value at time \( t \geq 0 \) of this American contract is
\[
U_t = \text{ess sup}_{\tau \in S_t} E_Q^x \left[ e^{-r(\tau-t)} G(X_{\tau}) \bigg| \mathcal{F}_t \right],
\]
where \( S_t \) is the set of stopping times greater than or equal to \( t \). Fakeev [6] has shown that when \((X_t)\) is a time-homogenous Markov process, then \( U_t = V(X_t) \) where \( V \) is the function
\[
V(x) = \text{sup}_{\tau} E_Q^x \left[ e^{-r\tau} G(X_{\tau}) \right],
\]
the supremum is taken over all stopping times and where
\[
E_Q^x [\cdot] = E_Q [\cdot | X_0 = x].
\]

Hence, it is enough to calculate the function \( V \). We allow for \( \tau = \infty \), and define
\[
e^{-r\tau} G(X_{\tau}) = \lim_{t \to \infty} e^{-rt} G(X_t) \quad \text{on} \quad \{ \tau = \infty \}.
\]

A stopping time \( \tau^* \) such that
\[
V(x) = E_Q^x \left[ e^{-r\tau^*} G(X_{\tau^*}) \right]
\]
is called an optimal stopping time. For more on optimal stopping see e.g. Peskir & Shiryaev [15], and for more on optimal stopping and American options in models driven by a Brownian motion see Karatzas [9] and Karatzas & Shreve [10]. We finally let \( \tau_S \) denote the random time at which we at the earliest can exercise an American perpetual option.

To solve our type of problems we proceed according to the following program:

1. Calculate the value function \( V \) for the standard perpetual American option with gain function \( G \):
\[
V(x) = \text{sup}_{\tau} E_Q^x \left[ e^{-r\tau} G(X_{\tau}) \right].
\]
2. If \( t \geq \tau_S \), then the value at \( t \) of the random start option with gain function \( G \) is given by \( V(X_t) \).

3. If \( t < \tau_S \), the value is given by
   \[
   E^{Q} \left[ e^{-r(\tau_S - t)} V(X_{\tau_S}) \mid \mathcal{F}_t \right] 1(t < \tau_S).
   \] (1)

   Note that we in general need to keep track of the time \( t \) here, since it might influence when the time \( \tau_S \) occurs.

More compactly, we can write the value of the random start American perpetual option at time \( t \) as
   \[
   V(X_t) 1(t \geq \tau_S) + E^{Q} \left[ e^{-r(\tau_S - t)} V(X_{\tau_S}) \mid \mathcal{F}_t \right] 1(t < \tau_S).
   \]

Since the stochastic process \( V(X_t) \) is a supermartingale, see e.g. Section I.2.2 in Peskir & Shiryaev [15], we have
   \[
   V(X_t) - E^{Q} \left[ e^{-r(\tau_S - t)} V(X_{\tau_S}) \mid \mathcal{F}_t \right] \geq 0.
   \]

Hence, we can interpret \( V(X_t) - E^{Q} \left[ e^{-r(\tau_S - t)} V(X_{\tau_S}) \mid \mathcal{F}_t \right] \) as the non-negative cost of not being able to exercise the option when \( t < \tau_S \). In our applications the function \( V \) is known and the goal of the major part of this paper is to show how we can compute the value of the option at times \( t \) when \( t < \tau_S \); i.e. evaluate expressions of the type given in Equation (1) above.

We have to specify the properties of \( \tau_S \) under the objective measure \( P \), and then determine what happens to these properties when we change measure from \( P \) to \( Q \). The investor in our model has no possibility of influencing the time \( \tau_S \), so we assume that \( \tau_S \) and the underlying process \( (X_t) \) are independent under \( P \). We assume that the randomness generated by \( \tau_S \) can not be traded, so we have an incomplete model. This means that there is not one unique, but infinitely many, potential pricing measures \( Q \), and we need to choose one of these. One way of doing this is to assume that the distribution of \( \tau_S \) under \( Q \) is the same as under \( P \) and that \( \tau_S \) is independent of \( (X_t) \) under \( Q \) as well. Choosing \( Q \) to have these properties means that we use what is called the minimal martingale measure, and this is the approach we will use. It has previously been used by e.g. Møller [12] in applications to insurance and by Armerin & Song [1] in a real options model. See Föllmer & Schweizer [7] and references therein for more on the minimal martingale measure. Explicitly, we make the following assumptions on the random time \( \tau_S \):

- \( P(\tau_S > t) = Q(\tau_S > t) > 0 \) for every \( t \geq 0 \),
- \( P(\tau_S < \infty) = Q(\tau_S < \infty) = 1 \), and
- \( \tau_S \) is independent of \( X \) under both \( P \) and \( Q \).

Note that a constant starting \( \tau_S \) time does not fulfill the first of these requirements, and must thus be considered seperately.
2.2 The hazard function

To be able to calculate the value of the American random start option we need the result in Lemma 2.1 below. We let

\[ F_t = Q(\tau \leq t), \]

and introduce

\[ \Gamma_t = -\ln(1 - F_t) \iff F_t = 1 - e^{-\Gamma_t}, \]

where the assumption \( Q(\tau > t) > 0 \) for every \( t \geq 0 \) from above guarantees that \( \Gamma \) is well defined for every \( t \geq 0 \). Now fix \( T > 0 \). Using the previous notation, we have the following result:

**Lemma 2.1** Assume that \( Z \) is an \((\mathcal{F}_t)\)-predictable process such that the random variable \( Z_{\tau\leq T} 1(\tau \leq T) \) is integrable. Then we have, for every \( t \leq T \),

\[
E^{Q}[Z_{\tau \leq T} 1(\tau \leq T) | \mathcal{F}_t] = e^{\Gamma_t} E^{Q}\left[ \int_{(t,T]} Z_u dF_u \bigg| \mathcal{F}_t \right] 1(t < \tau)
\]

\[ = e^{\Gamma_t} \int_{(t,T]} E^{Q}[Z_u | \mathcal{F}_t] dF_u 1(t < \tau). \]

For a proof see Bielecki & Rutkowski [2] or Jeanblanc et al [8]. If \( Z \) is non-negative, then we can let \( T \to \infty \) and get

\[
E^{Q}[Z_{\tau < \infty} 1(\tau < \infty) | \mathcal{F}_t] = e^{\Gamma_t} E^{Q}\left[ \int_{(t,\infty)} Z_u dF_u \bigg| \mathcal{F}_t \right] 1(t < \tau)
\]

\[ = e^{\Gamma_t} \int_{(t,\infty)} E^{Q}[Z_u | \mathcal{F}_t] dF_u 1(t < \tau). \]

Here we have used the fact that by assumption \( Q(\tau < \infty) \). When \( \Gamma \) can be written

\[ \Gamma_t = \int_0^t \gamma_s ds \]

for a function \( \gamma \), then we say that \( \tau \) has intensity \( \gamma \) and we have

\[ dF_t = \gamma_t e^{-\int_0^t \gamma_s ds} dt \]

in this case. We can then write

\[
E^{Q}[Z_{\tau < \infty} | \mathcal{F}_t] 1(t < \tau) = \int_t^\infty E^{Q}[Z_u | \mathcal{F}_t] \gamma_u e^{-\int_t^u \gamma_s ds} du 1(t < \tau)
\]

Using this result when

\[ Z_t = e^{-rt} f(X_t) \]

for a function \( f : \mathbb{R} \to \mathbb{R}_+ \) yields the following result.
Proposition 2.2 With notation and assumptions above, we have
\[ E^Q \left[ e^{-r(\tau_S-t)} f(X_{\tau_S}) \big| \mathcal{F}_t \right] = \int_t^\infty E^Q \left[ f(X_u) \big| \mathcal{F}_t \right] \gamma_u e^{-\int_u^t (r+\gamma_s)ds} du \]
when \( t < \tau_S \).

Note that the left-hand side with \( f = V \) in the expression above is Equation (1). The case when \( \tau_S = T > 0 \) is deterministic is not covered by the previous Proposition. In this case we use that
\[ E^Q \left[ e^{-r(\tau_S-t)} f(X_{\tau_S}) \big| \mathcal{F}_t \right] = e^{-r(T-t)} E^Q \left[ f(X_T) \big| \mathcal{F}_t \right] \]
for \( t < T \).

3 Applications

3.1 Introduction

We will now give examples of the technique described so far. In all examples below we use the following model. Here \( X_t \) denotes the present value at time \( t \geq 0 \) of a developed project or investment.

- Under \( P \) the value follows the geometric Brownian motion
  \[ dX_t = \mu X_t dt + \sigma X_t dW_t \]
  with \( X_0 > 0, \mu \in \mathbb{R} \) and \( \sigma > 0 \). The process \( W \) is a standard Brownian motion under \( P \).

- Under \( Q \) the value follows the geometric Brownian motion
  \[ dX_t = (r-\delta)X_t dt + \sigma X_t dW^Q_t, \]
  where \( W^Q \) is a \( Q \)-Brownian motion. Here \( \delta > 0 \) is the cash flow yield generated by the investment.

- The intensity function is a constant \( \gamma > 0 \); i.e. \( \tau_S \) is exponentially distributed with mean \( 1/\gamma \) under both \( P \) and \( Q \).

In the examples below the two constants
\[ \beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left[ \frac{1}{2} - \frac{r-\delta}{\sigma^2} \right]^2 + \frac{2r}{\sigma^2}} > 1 \]
and
\[ \beta_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[ \frac{1}{2} - \frac{r-\delta}{\sigma^2} \right]^2 + \frac{2r}{\sigma^2}} < 0 \]
will be used. They are the solutions to the quadratic equation
\[ \frac{1}{2} \sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r = 0, \] (2)
which in turn comes from the fact that we use the geometric Brownian motion above when modelling the underlying value. In the continuation region (i.e. the values of \( X_t \) for which we choose not to stop) the value function \( V \) satisfies a second order ODE, and in general the solutions to this ODE are the linear combinations of the two functions \( x^{\beta_1} \) and \( x^{\beta_2} \). The following Proposition will be used to calculate the value of random start options.

**Proposition 3.1** Let \( X \) be the geometric Brownian motion
\[ dX_t = (r - \delta)X_t dt + \sigma X_t dW_t^Q, \]
where \( W^Q \) is a standard Brownian motion under \( Q \), and let \( \tau_S \) be exponentially distributed with mean \( 1/\gamma > 0 \) and independent of \( X \) under \( Q \). For any \( a, b \in \mathbb{R} \) and \( t < \tau_S \) we have
\[
E^Q \left[ e^{-r(\tau_S-t)} X_{\tau_S}^a 1(X_{\tau_S} \leq b) \mid \mathcal{F}_t \right] = \gamma X_t^a J \left( r + \gamma - a(r - \delta + (a - 1)\sigma^2/2), \frac{1}{\sigma} \ln \left( \frac{b}{X_t} \right), \frac{\sigma - r - \delta}{\sigma} - a\sigma \right),
\]
where
\[
J(k, L, M) = \int_0^\infty \Phi \left( M \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} dx
\]
\[
= \begin{cases} 
\frac{1}{2\pi} e^{-L(M-\sqrt{M^2+2k})} \left( \frac{M}{\sqrt{M^2+2k}} + 1 \right) & \text{if } L < 0 \\
\frac{1}{2} + \frac{1}{2\pi} e^{-L(M+\sqrt{M^2+2k})} \left( \frac{M}{\sqrt{M^2+2k}} - 1 \right) & \text{if } L \geq 0,
\end{cases}
\]
and where \( \Phi \) is the distribution function of a standard normally distributed random variable. When \( a = \beta_i \) for \( i = 1, 2 \) (with \( \beta_1 \) and \( \beta_2 \) as above), then
\[
E^Q \left[ e^{-r(\tau_S-t)} X_{\tau_S}^{\beta_i} 1(X_{\tau_S} \leq b) \mid \mathcal{F}_t \right] = \gamma X_t^{\beta_i} J \left( r, \frac{1}{\sigma} \ln \left( \frac{b}{X_t} \right), -\text{sgn}(\beta_i) \sqrt{ \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2} } \right).
\]
For a proof of the Proposition, see Appendix A.

### 3.2 The objective measure \( P \)

We are mainly interested in the pricing measure \( Q \), but when we calculate the expected time until the investment is made we need to use the objective measure \( P \). In order to compare the time until we exercise the investment option in the
non-random case (i.e. when $\tau_S = 0$) with the random start case we need the distribution of $X_{\tau_S}$ under $P$. The solution to the GBM
\[ dX_t = \mu X_t dt + \sigma X_t dW_t \]
is
\[ X_t = X_0 e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t}, \quad t \geq 0. \]

It follows that
\[ \ln X_{\tau_S} = \ln X_0 + \left( \mu - \frac{\sigma^2}{2} \right) \tau_S + \sigma W_{\tau_S} \]
where we set $W_{\tau_S}/\sqrt{\tau_S} = 0$ when $\tau_S = 0$. Since $\tau_S$ is independent of $W$ under $P$ we have $W_{\tau_S}/\sqrt{\tau_S} \sim N(0,1)$ and we can write
\[ \ln X_{\tau_S} \overset{D}{=} \ln X_0 + \left( \mu - \frac{\sigma^2}{2} \right) \tau_S + \sigma \sqrt{\tau_S} \cdot Z, \]
where $Z \sim N(0,1)$ is independent of $\tau_S$. Hence, we recover the well known fact that under $P$ the random variable $\ln X_{\tau_S}$ has a normal mean-variance mixture distribution. When $\tau_S = T$ is a deterministic time, then
\[ \ln X_T \sim N \left( \ln X_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right), \]
and when $\tau_S$ is exponentially distributed, then $\ln X_{\tau_S}$ is skew-Laplace distributed. We recall that a random variable is skew-Laplace distributed if its density function is given by
\[ f(x) = \frac{\sqrt{2}}{\Sigma} \cdot \frac{\kappa}{1 + \kappa^2} \left\{ \begin{array}{ll}
 e^{-\frac{\Sigma}{2}\kappa |x-\theta|} & \text{if } x \geq \theta \\
 e^{-\frac{\Sigma}{2}\kappa |x-\theta|} & \text{if } x < \theta
\end{array} \right. \]
for some $\theta, \kappa \in \mathbb{R}$ and $\Sigma \geq 0$ (when $\Sigma = 0$ we define the function as the limit when $\Sigma \downarrow 0$). We write $AL(\theta, \Sigma, \kappa)$ to denote this distribution. If $\tau_S$ is exponentially distributed with mean $1/\gamma$, then $\ln X_{\tau_S}$ is skew-Laplace distributed with parameters
\[ \begin{align*}
\theta &= \ln X_0 \\
\Sigma &= \frac{\sigma}{\sqrt{\gamma}} \\
\kappa &= \frac{\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu}{\sqrt{2\gamma\sigma}},
\end{align*} \]
where
\[ \nu = \mu - \frac{\sigma^2}{2} \]
is the expected growth rate of the GBM. Expressing the parameters in the skew-Laplace’s density function using $\sigma$, $\gamma$, $\nu$ and $\ln X_0$ we get

$$f_{\tau_S}(x) = \frac{\sigma^2 \gamma}{\sqrt{\nu^2 + 2\gamma \sigma^2}} \begin{cases} e^{-\frac{\sqrt{\nu^2 + 2\gamma \sigma^2} - \nu|\ln X_0|}{\nu^2 + 2\gamma \sigma^2}} & \text{if } x \geq \ln X_0 \\ e^{-\frac{\sqrt{\nu^2 + 2\gamma \sigma^2} - \nu|\ln X_0|}{\nu^2 + 2\gamma \sigma^2}} & \text{if } x < \ln X_0 \end{cases}$$

For more on normal mean-variance mixture models and the skew-Laplace distribution see Kotz et al [11].

### 3.3 Valuation of a project – an optimal timing option

This is the main example we have in mind when studying random start American options. When buying land in order to build on it, usually a building permit is needed. Hence, even though the investor wants to build on the land he is not allowed to do so until he has received the permit. In this application $\tau_S$ is the time at which the building permit is given. The gains function in this case is given by

$$G(x) = x - I,$$

where $I$ is the investment cost of the project. Since it is never optimal to exercise the option when the value of the project is smaller than 0, this problem is equivalent to the one where

$$G(x) = \max(x - I, 0),$$

i.e. when we have a perpetual American call option. Hence, the problem we initially want to solve is

$$V(x) = \sup_\tau E^Q_\tau \left[ \max(X_\tau - I, 0) \right].$$

With dynamics of $X$ as above we have

$$V(x) = \begin{cases} (L_c - I) \left(\frac{x}{L_c}\right)^{\beta_1} & \text{when } x \in [0, L_c) \\ x - I & \text{when } x \in [L_c, \infty) \end{cases},$$

where the critical level $L_c$ is given by

$$L_c = \frac{\beta_1}{\beta_1 - 1} I.$$

For a proof of this see e.g. Chapter VIII, §2a in Shiryaev [16] or pp. 209-211 in Øksendal [14]. To calculate the value at a time $t < \tau_S$ of the random start option, we write the value $V$ as

$$V(x) = (L_c - I) \left(\frac{x}{L_c}\right)^{\beta_1} 1(x < L_c) + (x - I) 1(x \geq L_c)$$

$$= (L_c - I) \left(\frac{x}{L_c}\right)^{\beta_1} 1(x < L_c) + x - I + x 1(x < L_c) - I 1(x < L_c).$$
We now use Proposition 3.1 to find the value of this random start option at a time \( t < \tau_S \) with \( X_t = x \). By using Proposition 3.1 on the first, fourth and fifth of the five terms in the expression for \( V(x) \) we get

\[
\text{Value} = E^Q_{t,x}\left[e^{-r(\tau_S - t)} \max(X_{\tau_S} - I, 0)\right]
\]

\[
= (L_c - I) \left( \frac{x}{L_c} \right)^{\beta_1} \gamma J \left( \gamma, \frac{1}{\sigma} \ln \left( \frac{L_c}{x} \right), -\sqrt{\left[ \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right]^2 + 2\frac{r}{\sigma^2}} \right) + x \frac{\gamma}{\gamma + \delta} - I \frac{\gamma}{\gamma + r}
\]

\[
+ \frac{\gamma x J}{\gamma + \delta} \left( \frac{1}{\sigma} \ln \left( \frac{L_c}{x} \right), -\frac{r - \delta + \sigma^2/2}{\sigma} \right)
\]

\[
- \gamma I J \left( \gamma + r, \frac{1}{\sigma} \ln \left( \frac{L_c}{x} \right), -\frac{r - \delta - \sigma^2/2}{\sigma} \right).
\]

This formula is not so intuitive, but looking at a concrete example as in Figure 1 we get a better picture. For small values of the geometric Brownian motion the value of the random start option is not much smaller than the value of the standard American perpetuate call option. As the value of the underlying increases towards the critical value, the two option values starts to diverge, and at one point the value of random start option crosses the gain function; this of course will never happen to the standard American option. The value at time \( t \)

Figure 1: Gain function (solid curve), value of the standard American call option (dotted curve) and value of the random start American call option (dashed curve) all with \( I = 100 \). The parameter values are \( r = 0.01, \delta = 0.02, \sigma = 0.15 \) and \( \gamma = 0.10 \).
of the random start option is equal to value on the dashed curve if \( t < \tau_S \), and is equal to the value of dotted curve if \( t \geq \tau_S \).

We now proceed to calculate, at time 0, the mean time until the project is initiated. We start with some notation. Let \( \tau^*_c \) denote the optimal stopping time in the standard perpetuate American call option case, i.e.

\[
\tau^*_c = \inf \{ t \geq 0 \mid X_t \geq L_c \}.
\]

We let

\[
\tau^* = \inf \{ t \geq \tau_S \mid X_t \geq L_c \}
\]

be the optimal stopping time for the random start option, and we finally let

\[
\tau^*_S = \tau^* - \tau_S
\]

denote the time we wait until we optimally start the project after the random time \( \tau_S \) has occurred. With this notation we have

\[
\tau^*_S | X_{\tau_S} = x \overset{d}{=} \tau^*_c | X_0 = x.
\]

We are interested in the actual mean time until the option is exercised, so we use the objective measure \( P \) here, and we want to calculate

\[
E_x [\tau^*] = E_x [\tau_S] + E_x [\tau^*_S].
\]

We have \( E_x [\tau_S] = 1/\gamma \) and use relation (3) to calculate \( E_x [\tau^*_S] \). We recall that under \( P \) the value process \( X \) has dynamics

\[
dX_t = \mu X_t dt + \sigma X_t dW_t.
\]

We now assume that

\[
\nu = \mu - \frac{\sigma^2}{2} > 0.
\]

The reason for doing this is that if this inequality does not hold, then if \( x < L_c \) the expected time until we hit the critical level is infinite. One can show that if \( \nu > 0 \), then

\[
E_x \left[ e^{-\alpha \tau^*_c} \right] = \left\{ \begin{array}{ll} 
\frac{1}{(x/L_c)^{2\alpha/\nu} 2\alpha/\nu^{2\alpha/\nu}} & \text{when } x \geq L_c \\
\frac{1}{\ln(L_c/x)} & \text{when } x < L_c 
\end{array} \right.
\]

(see e.g. Borodin & Salminen [3] p. 622). From this it follows that

\[
E_x [\tau^*_c] = \left\{ \begin{array}{ll} 
\frac{1}{\nu} \ln (L_c/x) & \text{when } x \leq L_c \\
0 & \text{when } x > L_c,
\end{array} \right.
\]

and using relation (3) we get

\[
E_x [\tau^*_S] = E_x [E_x [\tau^*_S | X_{\tau_S}]] = E_x \left[ \frac{1}{\nu} \ln(L_c/X_{\tau_S}) 1(X_{\tau_S} \leq L_c) \right] = -\frac{1}{\nu} E_x \left[ \ln(X_{\tau_S}/L_c) 1(\ln(X_{\tau_S}/L_c) \leq 0) \right].
\]
We now give two examples of the distribution of $\tau_S$, and also comment briefly on the general case.

**A deterministic starting time**

When $\tau_S$ is a given fixed time,

$$\tau_S = T > 0,$$

then

$$E_x [\tau^*_S] = -\frac{1}{\nu} E_x [\ln(X_T/L_c) 1(\ln(X_T/L_c) \leq 0)].$$

We now use the following well known result.

**Lemma 3.2** If $Y \sim N(\mu, \sigma)$, then

$$E [Y 1(Y \leq 0)] = \mu \Phi(-\mu/\sigma) - \sigma \phi(-\mu/\sigma).$$

Since

$$\ln(X_T/L_c) = \ln(x/L_c) + \nu T + \sigma W_T \sim N(\ln(x/L_c) + \nu T, \sigma \sqrt{T})$$

when $X_0 = x$ we get

$$E_x [\tau^*_S] = \frac{1}{\nu} \left[ \sigma \sqrt{T} \varphi \left( -\frac{\ln(x/L_c) + \nu T}{\sigma \sqrt{T}} \right) - (\ln(x/L_c) + \nu T) \Phi \left( -\frac{\ln(x/L_c) + \nu T}{\sigma \sqrt{T}} \right) \right].$$

Hence, the mean of the time until the option is exercised is

$$E_x [\tau^*] = T + \frac{1}{\nu} \left[ \sigma \sqrt{T} \varphi \left( -\frac{\ln(x/L_c) + \nu T}{\sigma \sqrt{T}} \right) - (\ln(x/L_c) + \nu T) \Phi \left( -\frac{\ln(x/L_c) + \nu T}{\sigma \sqrt{T}} \right) \right].$$

**An exponentially distributed starting time**

In this case we need to calculate

$$-E [Y 1(Y \leq 0)]$$

where $Y \sim \text{AL} \left( \ln(x/L_c), \frac{\sigma}{\sqrt{\gamma}}, \frac{\nu^2 + 2\gamma \sigma^2}{\sqrt{2} \gamma \sigma} \right)$.

We have to distinguish between the two cases

(a) $\ln(x/L_c) \geq 0 \iff x \geq L_c$, and

(b) $\ln(x/L_c) < 0 \iff x < L_c$.

In case (a) we use the fact that if $h \geq 0$ and $a > 0$ then

$$\int_{-\infty}^{0} ye^{a(y-h)}dy = -\frac{e^{-ah}}{a^2},$$

and in case (b) that if $h < 0$ and $a, b > 0$ then

$$\int_{-\infty}^{h} ye^{a(y-h)}dy + \int_{h}^{0} ye^{-b(y-h)}dy = \frac{h}{a} - \frac{1}{a^2} + \frac{h}{b} + \frac{e^{bh} - 1}{b^2}.$$
In both cases we have

\[ h = \ln(x/L_c) \quad \text{and} \quad a = \frac{2\gamma}{\sqrt{\nu^2 + 2\gamma^2} - \nu}, \]

and in case (b) we additionally have

\[ b = \frac{\sqrt{\nu^2 + 2\gamma^2} - \nu}{\sigma^2}. \]

Using these results together with Equation (3) we get the following expected times until the option is optimally exercised.

(a) When \( x \geq L_c \):

\[
E_x[\tau_S^*] = \frac{\nu^2 + \gamma^2 - \nu \sqrt{\nu^2 + 2\gamma^2}}{2\nu^2} \left(\frac{x}{L_c}\right)^{-2\gamma/(\sqrt{\nu^2 + 2\gamma^2} - \nu)}.
\]

(b) When \( x < L_c \):

\[
E_x[\tau_S^*] = \left(\frac{\sqrt{\nu^2 + 2\gamma^2} - \nu}{2\gamma}\right) \ln\left(\frac{x}{L_c}\right) - \left(\frac{\sqrt{\nu^2 + 2\gamma^2} - \nu}{4\gamma^2}\right) + \frac{\sigma^2}{\sqrt{\nu^2 + 2\gamma^2} - \nu} \cdot \ln\left(\frac{x}{L_c}\right) + \frac{\sigma^4}{\left(\sqrt{\nu^2 + 2\gamma^2} - \nu\right)^2} \cdot \left[\left(\frac{x}{L_c}\right)^{\frac{\sqrt{\nu^2 + 2\gamma^2} - \nu}{\sigma^2}} - 1\right].
\]

To get the mean time until the option is optimally exercised we simply add \( E_x[\tau_S^*] = 1/\gamma \):

\[
E_x[\tau^*] = 1/\gamma + E_x[\tau_S^*]
\]

The general case

In general, when \( \tau_S \) is any random variable, we can use Lemma 3.2 above. Since

\[
\ln(X_{\tau_S}/L_c)|\tau_S \sim N(\ln(x/L_c) + \mu_{\tau_S}, \sigma_{\sqrt{\tau_S}})
\]

when \( X_0 = x \), it follows that

\[
E_x[\ln(X_{\tau_S}/L_c)1(\ln(X_{\tau_S}/L_c) \leq 0)] =
E_x\left[\ln(x/L_c) + \mu_{\tau_S}\Phi\left(-\frac{\ln(x/L_c) + \mu_{\tau_S}}{\sigma_{\sqrt{\tau_S}}}\right) - \sigma_{\sqrt{\tau_S}}\varphi\left(-\frac{\ln(x/L_c) + \mu_{\tau_S}}{\sigma_{\sqrt{\tau_S}}}\right)\right].
\]

If analytical methods are not working, the right-hand side can be calculated using simulation techniques.
3.4 An abandonment option

We start by describing the standard American version of this example. At time \( t = 0 \) we pay a sunk cost for the right to invest in a project at any future time. There is also a possibility to abandon the right to carry out the project, and in this case we get the recovery amount \( K \). Hence we want to find

\[
V(x) = \sup_{\tau} E^Q \left[ e^{-r\tau} \max(X_\tau, K) \right],
\]

where \( X_t \) is the value of the project if it is initiated at time \( t \). Note that since the cost for the investment is paid for at the start, it does not enter into the optimal timing problem.

Under the dynamics given at the beginning of this Section, the optimal value is given by

\[
V(x) = \begin{cases} 
K & \text{when } x \in [0, L_1] \\
K \left[ \frac{-\beta_2}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_1} + \frac{\beta_1}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_2} \right] & \text{when } x \in (L_1, L_2) \\
x & \text{when } x \in [L_2, \infty).
\end{cases}
\]

where

\[
L_1 = K \cdot \frac{\beta_2}{\beta_2 - 1} \left( -\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2} \right)^{(1-\beta_1)/(\beta_1 - \beta_2)}
\]

and

\[
L_2 = K \cdot \frac{\beta_2}{\beta_2 - 1} \left( -\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2} \right)^{-\beta_1/(\beta_1 - \beta_2)}.
\]

A proof of this is given in Appendix B. See also Yu [17]. Let us now turn to the problem of valuing the random start version of this option. We start by writing the optimal value of the standard American perpetuate option as

\[
V(x) = \begin{cases} 
K \mathbf{1}(x \leq L_1) \\
+K \left[ \frac{-\beta_2}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_1} + \frac{\beta_1}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_2} \right] \mathbf{1}(L_1 < x < L_2) \\
+K \mathbf{1}(x \geq L_2)
\end{cases}
\]

\[
= \begin{cases} 
K \mathbf{1}(x \leq L_1) \\
+K \left[ \frac{-\beta_2}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_1} + \frac{\beta_1}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_2} \right] \mathbf{1}(x < L_2) - \mathbf{1}(x \leq L_1)
\end{cases}
\]

Again we can use Proposition 3.1 to get the value of the random start version.
of the perpetuate option.

\[
\text{Value} = E_x^Q \left[ e^{-r\tau_s} \max(X_{\tau_s}, K) \right] \\
= \gamma K J \left( \gamma + r, \frac{1}{\sigma} \ln \left( \frac{L_1}{x} \right), -\frac{r - \delta - \sigma^2/2}{\sigma} \right) \\
+ \gamma K \frac{1}{\beta_1 - \beta_2} \left( -\beta_2 \left( \frac{x}{L_1} \right)^{\beta_1} J \left( \gamma, \frac{1}{\sigma} \ln \left( \frac{L_2}{x} \right), -\sqrt{\frac{1}{2} - \frac{r - \delta}{\sigma^2}} + \frac{2r}{\sigma^2} \right) - \beta_1 \left( \frac{x}{L_1} \right)^{\beta_2} \left[ J \left( \gamma, \frac{1}{\sigma} \ln \left( \frac{L_2}{x} \right), \sqrt{\frac{1}{2} - \frac{r - \delta}{\sigma^2}} + \frac{2r}{\sigma^2} \right) \right] - \gamma x \gamma + \delta - \gamma x J \left( \gamma + \delta, \frac{1}{\sigma} \ln \left( \frac{L_2}{x} \right), -\frac{r - \delta + \sigma^2/2}{\sigma} \right) \right).
\]

Figure 2: Gain function (solid curve), value of the standard abandonment option (dotted curve) and value of the random start abandonment option (dashed curve) all with \( K = 100 \). The parameter values are \( r = 0.01 \), \( \delta = 0.02 \), \( \sigma = 0.15 \) and \( \gamma = 0.1 \).

See Figure 2 for an example of the value of the standard and the random start abandonment option respectively.
To calculate the mean time until the random start abandonment option is exercised we use the fact that the expected time until a Brownian motion with drift $\mu$ per unit of time, volatility $\sigma$ and started at $x$ exits from the interval $(a,b)$ is given by

$$m(x; a, b) = \frac{b - x}{\mu} - \frac{b - a}{\mu} \cdot \frac{e^{-2ax/\sigma^2} - e^{-2bx/\sigma^2}}{e^{-2ax/\sigma^2} - e^{-2bx/\sigma^2}} \text{ for } a \leq x \leq b$$

(see e.g. Dominé [5]). We will now sketch how the expected time until the abandonment option is optimally exercise can be calculated. If we again let

$$\tau^*_c = \inf\{t \geq 0 \mid X_t \notin (L_1, L_2)\}$$

be the optimal stopping time of the standard version of the perpetuate American option under consideration, and let

$$\tau^*_s = \inf\{t \geq \tau^*_c \mid X_t \notin (L_1, L_2)\}$$

be the optimal stopping time of the random start version of the option. Again the relation (3) holds between these random variables. In the GBM case we have here

$$X_t \notin (L_1, L_2) \iff \ln X_t \notin (\ln L_1, \ln L_2),$$

and we also have

$$\ln X_t = \ln x + \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t = \ln x + \nu t + \sigma W_t.$$
With $L_2 = L_c$ this is consistent with the expression for the expectation in the optimal investment problem above.

Again we use
$$\tau^*_S|X_{\tau^*_S} = x \quad d = \tau^*_c|X_0 = x,$$

and get
$$E_x[\tau^*_S] = E_x[E_x[\tau^*_S|X_{\tau^*_S}]]$$
$$= E_x[m(\ln X_{\tau^*_S}; \ln L_1, \ln L_2)\mathbf{1}(L_1 \leq X_{\tau^*_S} \leq L_2)]$$
$$= \frac{1}{\nu} \left( \ln L_2 + \frac{1}{1 - (L_1/L_2)^{-2\nu/\sigma^2}} \ln \frac{L_1}{L_2} \right) P(L_1 \leq X_{\tau^*_S} \leq L_2)$$
$$- \frac{1}{\nu} E \left[ \left( \ln X_{\tau^*_S} + \frac{(X_{\tau^*_S}/L_2)^{-2\nu/\sigma^2} \ln L_1}{1 - (L_1/L_2)^{-2\nu/\sigma^2}} \right) \mathbf{1}(L_1 \leq X_{\tau^*_S} \leq L_2) \right].$$

We stop here, but using the fact that $\ln X_{\tau^*_S}$ has a known distribution makes it possible to explicitly evaluate the expression in the right-hand side.

**Remark 3.4** A more realistic model is perhaps to consider the payoff function
$$G(x) = \max(K, x - I).$$

In this model the investment can be terminated for a payoff of $K$, or initiated at a cost of $I$ – in this case paid at the time the project is undertaken. The $\beta$-values are the same (since they are determined by the dynamics of the underlying diffusion) as above, but the matching condition at the level at which we choose to initiate the project is different from the one above. It does not seem to exist an analytical solution in this case, so we have to use some numerical method to get the value of the standard American perpetual option.

## 4 Extensions

In this section we briefly comment on some possible ways of extending the model used so far.

### 4.1 Lévy processes with negative jumps

Instead of assuming a geometric Brownian motion, as we did above, we can assume a more general model driven by a Lévy process $Z$ which is assumed to have finite exponential moments and only negative jumps. In this case the solution to the standard American call option is known and has the same form as in the GBM case (see Mordecki [13] for details). More explicitly we assume that for $t \geq 0$ we have
$$X_t = X_0 e^{Z_t}.$$
under $Q$ (we focus on the valuation problem here), and as in the GBM case we will use Proposition 2.2 to calculate the value of the random start option when $t < \tau_S$. In the Lévy process case we get for $u \geq t$

$$E^Q[X_u^a 1(X_u \leq b) | \mathcal{F}_t] = X_t^a E^Q\left[e^{a(X_u - Z_u)} 1(X_u \leq b)\right]$$

$$= X_t^a E^Q\left[e^{a(Z_u - t) 1(Z_u - t \leq \ln\left(b/X_t\right))}\right]$$

$$= X_t^a \int_{-\infty}^{\ln(b/X_t)} e^{az} dF_{u-t}(z),$$

where

$$F_t(z) = Q(Z_t \leq z).$$

To continue we need to be able to calculate this expression, and then proceed to prove a new version of Proposition 3.1. Even under the assumption of a constant intensity it seems hard to get explicit expressions for the value of any interesting random start options, and we will have to use numerical methods.

### 4.2 A more general model

If we move away from the Markovian case, then the value of an American perpetual option with gain function $G : \mathbb{R} \to \mathbb{R}_+$ is given by

$$U_t = \text{ess sup}_{\nu \in S_t} E^Q\left[e^{-r(\nu-t)G(X_\nu)} \mid \mathcal{F}_t\right].$$

Again $S_t$ is the set of stopping times greater than or equal to $t$. In this case the value of the random start American perpetual option is given by

$$\text{Value} = U_t 1(\tau_S \leq t) + E^Q\left[e^{-r(\tau_S-t)U_{\tau_S}} \mid \mathcal{F}_t\right] (1 - 1(\tau_S \leq t))$$

$$= \begin{cases} 
U_t & \text{on } \{\tau_S \leq t\} \\
E^Q\left[e^{-r(\tau_S-t)U_{\tau_S}} \mid \mathcal{F}_t\right] & \text{on } \{\tau_S > t\}
\end{cases}$$

To get explicit expressions could be hard, but the important point to make is that we do not need to make the assumption of a Markovian model; the same principle holds for random start options in the general case.

### 4.3 A more general random time $\tau_S$

Instead of assuming that $\tau_S$ is independent of the driving process(es) under both $P$ and $Q$ we can use constructions that are used in credit risk models. Let $H_t = 1(\tau_S \leq t)$ and define $\mathcal{H}_t = \sigma(H_u, 0 \leq u \leq t)$. One approach is to assume that the full information available at time $t \geq 0$ is given by the $\sigma$-algebra $\mathcal{H}_t$, which in turn is assumed to be decomposed according to $\mathcal{F}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Here $\mathcal{F}_t$ represents all information up to and including time $t$ in excess of knowing if the random time $\tau_S$ has occurred or not (this information is given by $\mathcal{H}_t$). In these
type of models it is assumed that $\tau_S$ is not an $(\mathcal{F}_t)$-stopping time (it is obviously a $(\mathcal{G}_t)$-stopping time). In credit risk modelling this is known as the reduced form approach (see e.g. Jeanblanc et al [8] for more on reduced form modelling). If we are only interested in the value of the random start option, then it is quite straightforward to use this approach. If we want to use properties of $\tau_S$ under $P$, e.g. to calculate the mean time until an option is exercised, then we need to extend the reduced form models to also take care of the properties of $\tau_S$ under $P$.

5 Conclusions

We have considered a model in which an American option cannot be exercised until a random time has occurred. The main application we have in mind is when an irreversible investment should be done (an example of a timing option), but where we have to wait for a permit before the investment can be done. The value of this optionalty is calculated in two cases, and we also determine the expected time until this random option is optimally exercised.

A Proof of Proposition 3.1

We use the following two results:

Lemma A.1 Define for $k > 0$ and $L, M \in \mathbb{R}$

$$J(k, L, M) = \int_0^\infty \Phi \left( M \sqrt{x} + \frac{L}{\sqrt{x}} \right) e^{-kx} \, dx,$$

where $\Phi$ is the distribution function of a standard normal distributed random variable. Then

$$J(k, L, M) = \begin{cases} \frac{1}{2} e^{-L(M-\sqrt{M^2+2k})} \left( \frac{M}{\sqrt{M^2+2k}} + 1 \right) & \text{if } L < 0 \\ \frac{1}{k} + \frac{1}{2} e^{-L(M+\sqrt{M^2+2k})} \left( \frac{M}{\sqrt{M^2+2k}} - 1 \right) & \text{if } L \geq 0 \end{cases}$$

For a proof, see p. 19-20 in Armerin & Song [1].

Lemma A.2 If $X$ is the geometric Brownian motion

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

and $a, b \in \mathbb{R}$ are two constants, then for $0 \leq t < u$ it holds that

$$E[ X_u^a 1(X_u \leq b) | \mathcal{F}_t ] = x^a e^{a(\mu+(a-1)\sigma^2/2)(u-t)} \Phi(D(u-t)),$$

where

$$D(z) = \frac{\ln(b/x) - (\mu - \sigma^2/2) z}{\sigma \sqrt{z}} - a \sigma \sqrt{z}$$

$$= \frac{1}{\sigma} \ln \left( \frac{b}{x} \right) + \frac{1}{2} \left( \frac{\sigma}{\mu} - a \sigma \right) \sqrt{z}.$$
Proof. Using
\[ X_u = X_t e^{(\mu - \sigma^2/2)(u-t) + \sigma(W_u - W_t)} \]
we get
\[ 1(X_u \leq b) = 1 \left( \frac{W_u - W_t}{\sqrt{u-t}} \leq \frac{\ln(b/X_t) - (\mu - \sigma^2/2)(u-t)}{\sigma \sqrt{u-t}} \right). \]
Since
\[ \frac{W_u - W_t}{\sqrt{u-t}} \sim N(0,1), \]
and letting \( d(z) = D(z) + a\sigma \sqrt{z} \), we get
\[ E[X^a_u 1(X_u \leq b) | \mathcal{F}_t] = X_t^a e^{a(\mu - \sigma^2/2)(u-t)} E \left[ e^{a\sigma(W_u - W_t)} 1(X_u \leq b) | \mathcal{F}_t \right] \]
\[ = X_t^a e^{a(\mu - \sigma^2/2)(u-t)} \int_{-\infty}^{d(u-t)} e^{a\sigma \sqrt{u-t} z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \]
\[ = X_t^a e^{a(\mu - \sigma^2/2)(u-t)} \int_{-\infty}^{d(u-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(z-a\sigma \sqrt{u-t})^2 - a^2 \sigma^2 (u-t)]} dz \]
\[ = X_t^a e^{a(\mu + (a-1)\sigma^2/2)(u-t)} \Phi(d(u-t) - a\sigma \sqrt{u-t}) \]
\[ = X_t^a e^{a(\mu + (a-1)\sigma^2/2)(u-t)} \Phi(D(u-t)). \]

We now present the proof of Proposition 3.1.

Proof. For general \( a, b \in \mathbb{R} \) we have
\[ E^Q \left[ e^{-r(t-s)} X^a_{t-s} 1(X_{t-s} \leq b) \right] = \{ \text{Use Proposition 2.2 with } f(x) = x^a 1(x \leq b) \} \]
\[ = \int_t^\infty E^Q [X^a_u 1(X_u \leq b) | \mathcal{F}_t] \gamma e^{-(r+\gamma)(u-t)} du \]
\[ = \{ \text{Use Lemma A.2} \} \]
\[ = \int_t^\infty X_t^a e^{a(r-\delta + (a-1)\sigma^2/2)(u-t)} \Phi(D(u-t)) \gamma e^{-(r+\gamma)(u-t)} du \]
\[ = \{ \text{Use Lemma A.1} \} \]
\[ = \gamma X_t^a J \left( r + \gamma - a(r - \delta + (a-1)\sigma^2/2), \frac{1}{\sigma} \ln \frac{b}{X_t}, \frac{r - \delta}{\sigma} - a\sigma \right). \]

With \( a = \beta_t \) we get
\[ r + \gamma - \beta_t(r - \delta + (\beta_t - 1)\sigma^2/2) = \gamma \]
and
\[ \frac{\sigma}{2} - \frac{r - \delta}{\sigma} - \beta_i \sigma = -\text{sgn}(\beta_i) \sqrt{\left[ \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right]^2 + \frac{2r}{\sigma^2}}, \]
and the proof is complete. \(\square\)

**B  The value of the abandonment option**

We want to solve the problem
\[ V(x) = \sup_{\tau} E^Q_x \left[ e^{-r\tau} \max(K, X_\tau) \right], \]
where \(X\) is a geometric Brownian motion with dynamics given by
\[ dX_t = (r - \delta)X_t dt + \sigma X_t dW^Q_t. \]
Here \(W^Q\) is a \(Q\)-Wiener process and we assume that \(r > 0, \sigma > 0\) and \(\delta \in \mathbb{R}\).
We also want to find, if it exists, an optimal stopping time \(\tau^*\) of this problem.
To find a candidate optimal solution \(V\) (which we then verify is the optimal solution), we assume that there exists two levels \(L_1\) and \(L_2\) satisfying
\[ 0 < L_1 < K < L_2 < \infty \]
and such that the interval \((L_1, L_2)\) is the continuation region. The candidate for the optimal stopping time is then
\[ \hat{\tau} = \inf \{ t \geq 0 | X_t = L_1 \text{ or } X_t = L_2 \}. \]
For \(x \in (L_1, L_2)\) we want the function \(\hat{V}\) to satisfy
\[ \frac{1}{2} \sigma^2 x^2 \hat{V}''(x) + (r - \delta)x \hat{V}'(x) - r \hat{V}(x) = 0. \]
We also introduce the value matching and smooth pasting conditions
\[ \begin{align*}
\hat{V}(L_1) &= K \\
\hat{V}(L_2) &= L_2 \\
\hat{V}'(L_1) &= 0 \\
\hat{V}'(L_2) &= 1.
\end{align*} \]
The solution to the ODE is given by
\[ \hat{V}(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2}, \]
where \(\beta_1, \beta_2\) solves Equation (2) with explicit solutions
\[ \beta_1 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left[ \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right]^2 + \frac{2r}{\sigma^2}} > 1. \]
and
\[
\beta_2 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0.
\]
and \(A_1, A_2 \in \mathbb{R}\) are two constants to be determined. The value and smooth pasting conditions now becomes
\[
\begin{align*}
A_1 L_1^{\beta_1} + A_2 L_1^{\beta_2} &= K \\
A_1 L_2^{\beta_1} + A_2 L_2^{\beta_2} &= L_2 \\
A_1 \beta_1 L_1^{\beta_1} + A_2 \beta_2 L_1^{\beta_2} &= 0 \\
A_1 \beta_1 L_2^{\beta_1} + A_2 \beta_2 L_2^{\beta_2} &= 1
\end{align*}
\]
We now multiply the third equation above with \(L_1\), the fourth with \(L_2\) and introduce
\[
u_1 = A_1 L_1^{\beta_1}, \quad \nu_2 = A_2 L_1^{\beta_2}, \quad \nu_1 = A_1 L_2^{\beta_1} \quad \text{and} \quad \nu_2 = A_2 L_2^{\beta_2}.
\]
Using the system of equations we get
\[
u_1 = K \frac{\beta_2}{\beta_2 - \beta_1}, \quad \nu_2 = -K \frac{\beta_1}{\beta_2 - \beta_1}, \quad \nu_1 = L_2 \frac{1 - \beta_2}{\beta_1 - \beta_2} \quad \text{and} \quad \nu_2 = L_2 \frac{\beta_1 - 1}{\beta_1 - \beta_2}.
\]
Now
\[
-\frac{\beta_2}{\beta_1} = \frac{u_1}{u_2} = \frac{A_1}{A_2} L_1^{\beta_1 - \beta_2}
\]
and
\[
\frac{1 - \beta_2}{\beta_1 - 1} = \frac{v_1}{v_2} = \frac{A_1}{A_2} L_2^{\beta_1 - \beta_2}.
\]
It follows that
\[
-\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2} = \left(\frac{L_1}{L_2}\right)^{\beta_1 - \beta_2}
\]
and
\[
L_1 = \left(\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2}\right)^{1/\beta_2} \quad \text{and} \quad L_2 = k L_2.
\]
We also have
\[
\frac{L_2}{K} \frac{\beta_2 - 1}{\beta_2} = \frac{v_1}{u_1} = \left(\frac{L_2}{L_1}\right)^{\beta_1} = k^{-\beta_1} \quad \iff \quad L_2 = K k^{-\beta_1} \frac{\beta_2}{\beta_2 - 1}
\]
and
\[
L_1 = K k^{1-\beta_1} \frac{\beta_2}{\beta_2 - 1}.
\]
Finally we get
\[
A_1 = KL_1^{-\beta_1} \frac{\beta_2}{\beta_2 - \beta_1}
\]
\[
A_2 = -KL_1^{-\beta_2} \frac{\beta_1}{\beta_2 - \beta_1}.
\]
Hence, we can write our candidate solution as
\[
\hat{V}(x) = K \left[ \frac{-\beta_2}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_1} + \frac{\beta_1}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_2} \right] =: k_1 x^{\beta_1} + k_2 x^{\beta_2}
\]
when \( x \in (L_1, L_2) \), and otherwise
\[
\hat{V}(x) = \begin{cases} 
    K, & x \in [0, L_1] \\
    x, & x \in [L_2, \infty).
\end{cases}
\]

Now we have a candidate to the optimal value function \( V \) and to the optimal stopping time \( \tau^* \), and we need to prove that they are the optimal value function and optimal stopping time respectively. To do this, we will use the following observation from Mordecki [13]. We formulate it under the pricing measure \( Q \), but this is only because it fits with our application, and for a general gains function \( G \).

**Observation B.1** If a function \( \hat{V} \) and a stopping time \( \hat{\tau} \) fulfills

\[
\begin{align*}
    (i) & \quad \hat{V}(x) = E^Q_x \left[ e^{-r \hat{\tau}} G(X_{\hat{\tau}}) \right] \\
    (ii) & \quad \hat{V}(x) \geq E^Q_x \left[ e^{-r \tau} G(X_{\tau}) \right] \quad \text{for every stopping time } \tau,
\end{align*}
\]

then
\[
V = \hat{V} \quad \text{and} \quad \tau^* = \hat{\tau}.
\]

**Lemma B.2** Assume that \( G(x) \geq 0 \) for every \( x \in \mathbb{R} \). Sufficient conditions for (ii) in the Observation above to hold are

- \( \hat{V}(x) \geq G(x) \) for every \( x \in \mathbb{R} \), and
- \( e^{-rt} \hat{V}(X_t) \) is a \( Q \)-supermartingale.

**Proof.** Let \( \tau \) be a stopping time. Since \( e^{-rt} \hat{V}(X_t) \) is a supermartingale by assumption, for any \( n \in \mathbb{Z}_+ \) we have
\[
E^Q_x \left[ e^{-r(\tau \wedge n)} \hat{V}(X_{\tau \wedge n}) \right] \leq \hat{V}(x).
\]

It follows that
\[
\liminf_{n \to \infty} E^Q_x \left[ e^{-r(\tau \wedge n)} \hat{V}(X_{\tau \wedge n}) \right] \leq \hat{V}(x),
\]

and using Fatou’s lemma (since \( G \) is non-negative and \( \hat{V} \geq G \), the process \( e^{-rt} \hat{V}(X_t) \) is also non-negative) we get
\[
E^Q_x \left[ e^{-r\tau} \hat{V}(X_{\tau}) \right] \leq \hat{V}(x).
\]

Using \( \hat{V}(x) \geq G(x) \) for every \( x \in \mathbb{R} \) we finally get
\[
E^Q_x \left[ e^{-r\tau} G(X_{\tau}) \right] \leq \hat{V}(x),
\]

which is condition (ii). \( \square \)
Theorem B.3 The optimal value and the optimal stopping time to optimal stopping problem above are

\[ V(x) = \begin{cases} 
K & \text{when } x \in [0, L_1] \\
K \left( \frac{x}{L_1} \right)^{\beta_1} + \frac{\beta_1}{\beta_1 - \beta_2} \left( \frac{x}{L_1} \right)^{\beta_2} & \text{when } x \in (L_1, L_2) \\
x & \text{when } x \in [L_2, \infty) 
\end{cases} \]

and

\[ \tau^* = \inf \{ t \geq 0 \mid X_t = L_1 \text{ or } X_t = L_2 \} \]

respectively, with \( L_1, L_2, \beta_1, \beta_2 \) as above.

Proof. To prove the Theorem we use Observation B.1 with \( \hat{\tau} \) and \( \hat{V} \) as above. We start by noting that \( Q(\hat{\tau} < \infty) = 1 \) and that \( M_i \) are non-negative \( Q \)-martingales. It follows that

\[
E_Q \left[ e^{-rt \max(K, X_{\hat{\tau}})} \right] = E_Q \left[ e^{-rt \hat{V}(X_{\hat{\tau}})} \right] = E_Q \left[ e^{-rt (k_1 X_{\hat{\tau}}^{\beta_1} + k_2 X_{\hat{\tau}}^{\beta_2})} \right] = E_Q \left[ k_1 M_1^t + k_2 M_2^t \right].
\]

Since for \( i = 1, 2 \) and every integer \( n \) we have

\[
0 \leq M_i^{\wedge n} \leq L_2^{\beta_i}
\]

we can use bounded convergence to get

\[
E_Q \left[ e^{-rt \max(K, X_{\hat{\tau}})} \right] = \lim_{n \to \infty} E_Q \left[ k_1 M_1^{\hat{\tau} \wedge n} + k_2 M_2^{\hat{\tau} \wedge n} \right] = k_1 M_1^{\hat{\tau}} + k_2 M_2^{\hat{\tau}} = \hat{V}(x).
\]

This shows that \( \hat{V} \) and \( \hat{\tau} \) satisfies condition (A). To prove that condition (B) is satisfied we start by noting that

\[
\hat{V}(x) \geq \max(K, x) = G(x).
\]

We need to show that

\[
e^{-rt \hat{V}(X_t)}
\]

is a \( Q \)-supermartingale. To do this we begin by defining the function

\[
F(x) = \begin{cases} 
K x^{-\beta_1} & \text{if } x \in (0, L_1] \\
A_1 + A_2 x^{\beta_2 - \beta_1} & \text{if } x \in (L_1, L_2) \\
x^{1-\beta_1} & \text{if } x \in [L_2, \infty).
\end{cases}
\]

The function \( F \) is decreasing and concave, and we have

\[
\hat{V}(x) = x^{\beta_1} F(x) \quad \text{when } x \in (0, \infty).
\]
Now take $0 \leq s \leq t$:

$$
E^Q \left[ e^{-rt} \hat{V}(X_t) \Big| \mathcal{F}_s \right] = E^Q \left[ e^{-rt} X_t^{\beta_1} F(X_t) \Big| \mathcal{F}_s \right] \\
= M_t^1 E^Q \left[ \frac{M^1_t}{M^2_t} F(X_t) \Big| \mathcal{F}_s \right] \\
= M_t^1 E^1 \left[ F(X_t) \big| \mathcal{F}_s \right] \\
\leq M_t^1 F \left( E^1 \left[ X_t \big| \mathcal{F}_s \right] \right) \\
= M_t^1 F \left( E^Q \left[ M_t^1 X_t \big| \mathcal{F}_s \right] \right) \\
\leq M_t^1 F(X_s) \\
= e^{-rs} X_s^{\beta_1} F(X_s) \\
= e^{-rs} \hat{V}(X_s)
$$

Here the measure which implies the expectation operator $E^1$ is the one defined by the Radon-Nikodym derivative $M_t^1$ with respect to $Q$ on $\mathcal{F}_t$. The first inequality above follows from Jensen’s inequality (since $F$ is concave), and the second from the facts that $(M_t^1 X_t) = \left( e^{-rt} X_t^{\beta_1+1} \right)$ is a $Q$-submartingale and that $F$ is decreasing.

\[\square\]

References


