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Operations on Étale Sheaves of Sets

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ABSTRACT. Rydh showed in 2011 that any unramified morphism f of algebraic spaces (algebraic stacks) has a canonical and universal factorization through an algebraic space (algebraic stack) called the *étale envelope* of f , where the first morphism is a closed immersion and the second is étale. We show that when f is étale then the étale envelope can be described by applying the left adjoint of the pullback of f to the constant sheaf defined by a pointed set with two elements. When f is a monomorphism locally of finite type we have a similar construction using the direct image with proper support.

SAMMANFATTNING. Rydh visade 2011 att varje oramifierad morfi f av algebraiska rum (algebraiska stackar) har en kanonisk och universell faktorisering genom ett algebraiskt rum (algebraisk stack) som han kallar den *étala omslutningen* av f , där den första morfin är en sluten immersion och den andra är étale. Vi visar att då f är étale så kan den étala omslutningen beskrivas genom att applicera vänsteradjunkten till tillbakadragningen av f på den konstanta kärven som definieras av en punkterad mängd med två element. Då f är en monomorfi, lokalt av ändlig typ så har vi en liknande beskrivning i termer av framtryckning med propert stöd.

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Introduction

There are cases when the Zariski topology is too coarse to work in. For example if we want to mimic results that are true in the Euclidean topology like the implicit function theorem or cohomology. Hence it may be convenient to work in finer topologies like the étale topology which has properties more like the Euclidean topology. The étale topology is an example of a Grothendieck topology and was defined by A. Grothendieck who developed it together with M. Artin and J.-L. Verdier. The aim was to define étale cohomology in order to prove the Weil conjectures [Wei49].

Given a category \mathcal{C} we may define a Grothendieck topology on \mathcal{C} by assigning a collection of coverings $\{U_i \rightarrow U\}$ for each object U in \mathcal{C} . A category with a Grothendieck topology is called a site. An example of a site is the big étale site $S_{\text{ét}}$ on a scheme S , where the underlying category is (Sch/S) and a covering of an S -scheme U is a jointly surjective family $\{U_i \rightarrow U\}$ of étale S -morphisms. Given a site \mathcal{S} with underlying category \mathcal{C} , we may consider sheaves on \mathcal{S} . That is, functors

$$\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow (\text{Set})$$

satisfying a certain gluing condition for each covering $\{U_i \rightarrow U\}$. Every S -scheme X is a sheaf on $S_{\text{ét}}$ when identifying X with the contravariant functor $\text{h}_X = \text{Hom}_{(\text{Sch}/S)}(-, X)$.

If $R \rightrightarrows X$ are étale S -morphisms such that the induced map

$$\text{Hom}_{(\text{Sch}/S)}(T, R) \rightarrow \text{Hom}_{(\text{Sch}/S)}(T, X) \times \text{Hom}_{(\text{Sch}/S)}(T, X)$$

is injective for every S -scheme T , and gives an equivalence relation \sim on the set $\text{Hom}_{(\text{Sch}/S)}(T, X)$, then we may form the presheaf quotient $T \mapsto X(T)/\sim$. The sheafification of this presheaf is an algebraic space over S and is denoted X/R . This generalizes the concept of schemes.

The small site $S_{\text{ét}}$ on a scheme (or algebraic space) S has underlying category $(\text{ét}/S)$ (or $\text{ét}(S)$), i.e., the category of étale schemes (algebraic spaces) over S , and coverings as in $S_{\text{ét}}$. Given a sheaf \mathcal{F} on $S_{\text{ét}}$ we may construct its espace étalé $\mathcal{F}^{\text{ét}}$ which is an étale algebraic space over S . This gives an equivalence of categories between sheaves on the small étale site $S_{\text{ét}}$ and étale algebraic spaces over S . In particular, every sheaf on the small étale site $S_{\text{ét}}$ of an algebraic space S is representable by an étale algebraic space over S . The espace étalé has the following analogue in classical topology: given a topological space B and a sheaf of sets \mathcal{G} on B , the espace étalé of \mathcal{G} is a topological space E together with a local homeomorphism $\pi: E \rightarrow B$ such that \mathcal{G} is the sheaf of sections of π (see e.g. [MLM94, Section II.5]).

Every morphism $f: T \rightarrow S$ of schemes (algebraic spaces), gives rise to a morphism $T_{\text{ét}} \rightarrow S_{\text{ét}}$ of sites. Hence we may consider push-forwards $f_*: \text{Sh}(T_{\text{ét}}) \rightarrow \text{Sh}(S_{\text{ét}})$ and pullbacks $f^*: \text{Sh}(S_{\text{ét}}) \rightarrow \text{Sh}(T_{\text{ét}})$. We have that $f^*\mathcal{F}$ is just the restriction of the fiber product $T \times_S \mathcal{F}^{\text{ét}}$ to the small étale site. In certain cases f^* has a left adjoint denoted by $f_!$. For example, in case $f: T \rightarrow S$ is an object in $(\text{ét}/S)$

we get that $f_! \mathcal{F}$ is the sheaf given by

$$U \mapsto \coprod_{\varphi} \mathcal{F}(U)$$

for every S -scheme $\psi: U \rightarrow S$ where the disjoint union is over all S -morphisms $U \rightarrow T$. In case, \mathcal{F} is a sheaf of *pointed* sets, we get that $f_! \mathcal{F}$ is the sheafification of the presheaf

$$U \mapsto \bigvee_{\varphi} \mathcal{F}(U).$$

Rydh shows in [Ryd11] that any unramified morphisms $X \rightarrow Y$ of algebraic spaces (algebraic stacks) factors as $X \hookrightarrow E_{X/Y} \rightarrow Y$ where the first morphism is a closed immersion and the second morphism is étale. We show that in the case when f is a monomorphism, we get that the restriction $E_{X/Y}$ to the small étale site is naturally isomorphic to the sheaf $f_c \{0, 1\}_X$, where $\{0, 1\}_X$ denotes the constant sheaf on the small étale site on X and f_c is the direct image with proper support. If f is étale then $E_{X/Y, \text{ét}} = f_! \{0, 1\}_X$, where $f_!$ is the left adjoint of the pullback. Hence we have the following conjecture:

CONJECTURE. *Let X and Y be algebraic spaces and let $f: X \rightarrow Y$ be a morphism locally of finite type. There exists a functor*

$$f_{\#}: \text{Sh}_*(X_{\text{ét}}) \rightarrow \text{Sh}_*(Y_{\text{ét}})$$

of sheaves of pointed sets such that:

- (1) *if f is unramified, we have $E_{X/Y} = f_{\#} \{0, 1\}_X$;*
- (2) *if f is étale we have $f_{\#} = f_!$;*
- (3) *if f is a monomorphism we have $f_{\#} = f_c$.*

Preliminaries

By a *ring* we always mean a commutative ring with unity. All rings are assumed to be Noetherian and all schemes are assumed to be locally Noetherian. A morphism of schemes is called *proper* if it is of finite type, separated, and universally closed. A morphism $f: X \rightarrow Y$ of schemes is called *finite* if there is an open covering $Y = \bigcup V_i$ of Y by affine open subschemes V_i such that for every i we have $f^{-1}(V_i)$ is affine and the induced homomorphism $\mathcal{O}_Y(V_i) \rightarrow \mathcal{O}_X(f^{-1}(V_i))$ is finite. Or equivalently (see for example [GW10, 12.9]), for every open affine subscheme $V \subseteq Y$, the inverse image $f^{-1}(V)$ is affine and $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$ finite. In particular, every finite morphism is by definition *affine*. A morphism is called *quasi-finite* if it is of finite type and the fiber over each point consists only of finitely many points.

THEOREM 0.0.1 (Zariski's Main Theorem [Mil80, I.1.8]). *Let $f: X \rightarrow Y$ be a morphism of schemes and assume that Y is quasi-compact. The following are equivalent:*

- (1) *f is quasi-finite and separated;*
- (2) *f factors as $X \xrightarrow{\alpha} \overline{X} \xrightarrow{\beta} Y$ where α is an open immersion and β is finite.*

LEMMA 0.0.2 ([GW10, 12.89]). *Let $f: X \rightarrow Y$ be a morphism. The following are equivalent:*

- (1) *f is finite;*
- (2) *f is quasi-finite and proper;*
- (3) *f is affine and proper.*

For locally Noetherian schemes we have the following topological property:

LEMMA 0.0.3. *Let X be a locally Noetherian scheme and let $V \subseteq X$ be a subset. Then the following are equivalent:*

- (1) V is clopen (open and closed) in X ;
- (2) V is a union of connected components of X .

PROOF. (1) \Rightarrow (2): Let $V \subseteq X$ be a clopen subset intersecting a connected component C of X . Then $C \cap V$ and $(C \setminus V)$ are both open. Thus $C = (C \cap V) \amalg (C \setminus V)$ and we conclude that $C \cap V = C$ since C is connected. Hence we see that a clopen subset is always a union of connected components.

(2) \Rightarrow (1): A connected component is always closed since the closure of a connected subset is connected. We will show that every connected component of X is open. Since X is locally Noetherian it is locally connected (see for example [Sta, Tag 04MF]). But X is locally connected if and only if every connected component of X is open [Bou98, I.11.6.11]. Hence we get that every connected component of X is clopen (open and closed). Now if W is a union of connected components (i.e. clopen subsets) then so is its complement $X \setminus W$ and hence both W and $X \setminus W$ are both open, and hence also closed. Thus W is clopen. \square

The following lemma is trivial but will be useful later on.

LEMMA 0.0.4. *Suppose that we have a cartesian square*

$$\begin{array}{ccc} T \times_S U & \xrightarrow{\text{pr}_1} & U \\ \downarrow & & \downarrow f \\ T & \longrightarrow & S \end{array}$$

in the category (Sch) of schemes such that there is a morphism $s: S \rightarrow U$ satisfying $f \circ s = \text{id}_S$. Then T is the fiber product of s and pr_1 .

CHAPTER 1

Étale morphisms

An étale morphism is the algebraic analogue of a local homeomorphism. For example, a morphism of nonsingular varieties over an algebraically closed field is étale at a point if and only if it induces an isomorphism of the tangent spaces. The main references to this chapter are [Mil80] and [AK70].

1.1. Flat morphisms

Recall that a ring homomorphism $A \rightarrow B$ is called *flat*, if B is flat when considered as an A -module, i.e., if the functor $- \otimes_A B$ is exact. It is called *faithfully flat* if $- \otimes_A B$ is faithful and exact.

PROPOSITION 1.1.1. *Let $\varphi: A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

- (1) φ is flat;
- (2) For every ideal $I \subseteq A$, the map $I \otimes B \rightarrow B; a \otimes b \mapsto \varphi(a)b$ is injective.

PROOF. If φ is flat then clearly $I \otimes B \rightarrow B$ is injective since $I \rightarrow A$ injective implies that $I \otimes B \rightarrow A \otimes B = B$ is injective. For the converse, see [Mil80, I.2.2]. \square

PROPOSITION 1.1.2. *Let $\varphi: A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

- (1) φ is flat;
- (2) For every $\mathfrak{m} \in \text{Spm } B$, the induced homomorphism $A_{\varphi^{-1}(\mathfrak{m})} \rightarrow B_{\mathfrak{m}}$ is flat.

DEFINITION 1.1.3. Let $f: X \rightarrow Y$ be a morphism of schemes. Then we say that f is *flat at $x \in X$* if the induced map $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat. We say that f is *flat* if it is flat at every $x \in X$.

REMARK 1.1.4. Proposition 1.1.2 implies that a morphism is flat if and only if it is flat at all closed points.

REMARK 1.1.5. A flat ring homomorphism induces a flat morphism of spectra.

PROPOSITION 1.1.6 ([Mil80, I.2.5]). *Let $A \rightarrow B$ be a ring homomorphism that makes B a flat A -algebra. Take $b \in B$ and suppose that the image of b in $B/\mathfrak{m}B$ is not a zero-divisor for every maximal ideal \mathfrak{m} of A . Then $B/(b)$ is a flat A -algebra.*

EXAMPLE 1.1.7. Let A be a ring and let $f \in A[T_1, \dots, T_n]$ be non-zero. Let $V \subseteq \text{Spec } A[T_1, \dots, T_n]$ be the closed subscheme given by the ideal (f) , i.e.,

$$V \cong \text{Spec } A[T_1, \dots, T_n]/(f).$$

Suppose that the image of f in $(A/\mathfrak{m})[T_1, \dots, T_n]$ is non-zero for every maximal ideal \mathfrak{m} of A , or equivalently, that the ideal generated by the coefficients of f is A . Then the morphism $V \rightarrow \text{Spec } A$ induced by the morphism $\varphi: A \rightarrow A[T_1, \dots, T_n]/(f)$ is

flat by Proposition 1.1.6. The converse is also true since if the coefficients of f is contained in a maximal ideal \mathfrak{m} of A , then the homomorphism

$$\begin{aligned} \mathfrak{m} \otimes_A A[T_1, \dots, T_n]/(f) &\rightarrow A[T_1, \dots, T_n]/(f) \\ a \otimes g &\mapsto \varphi(a)g \end{aligned}$$

is not injective. Indeed, f may be written as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$$

and the non-zero element

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha \otimes T^\alpha$$

in $\mathfrak{m} \otimes_A A[T_1, \dots, T_n]/(f)$ will be mapped to zero. Hence by Proposition 1.1.1, φ is not flat.

PROPOSITION 1.1.8 ([Mil80, I.2.7], [AK70, V.1.9]). *Let $\varphi: A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

- (1) φ is faithfully flat;
- (2) φ is injective and $B/\varphi(A)$ is flat over A ;
- (3) a sequence $M' \rightarrow M \rightarrow M''$ of A -modules is exact if and only if the sequence $M' \otimes_A B \rightarrow M \otimes_A B \rightarrow M'' \otimes_A B$ is exact;
- (4) φ is flat and the induced morphism $\text{Spec } B \rightarrow \text{Spec } A$ is surjective;
- (5) φ is flat and for every maximal ideal $\mathfrak{m} \subset A$, we have $\varphi(\mathfrak{m})B \neq B$.

Hence the following definition agrees with the definition for rings.

DEFINITION 1.1.9. Let $f: X \rightarrow Y$ be a morphism of schemes. Then we say that f is *faithfully flat* if it is flat and surjective.

REMARK 1.1.10. Proposition 1.1.8 implies that a morphism $\text{Spec } B \rightarrow \text{Spec } A$ is faithfully flat if and only if the ring homomorphism $A \rightarrow B$ is faithfully flat.

EXAMPLE 1.1.11. For a scheme X , the projection $X \amalg \cdots \amalg X \rightarrow X$ is certainly faithfully flat.

LEMMA 1.1.12. *Let (A, \mathfrak{m}_A) and (B, \mathfrak{m}_B) be local rings. Then any flat local homomorphism $\varphi: A \rightarrow B$ is faithfully flat.*

PROOF. Since φ is local we have $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ and hence $\mathfrak{m}_A B \neq B$. Hence the Lemma follows from Proposition 1.1.8. \square

LEMMA 1.1.13. *A composition of flat morphisms is flat and a base change of a flat morphism is flat.*

PROOF. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be flat morphisms. Take $x \in X$ and put $y = f(x)$ and $z = g(y)$. Flatness of $g \circ f$ follows from the fact that if M is an \mathcal{O}_z -module, then

$$(M \otimes_{\mathcal{O}_z} \mathcal{O}_y) \otimes_{\mathcal{O}_y} \mathcal{O}_x \cong M \otimes_{\mathcal{O}_z} \mathcal{O}_x.$$

To show that a base change of a flat map is flat, let $f: X \rightarrow Y$ be a flat morphism and let $f': Y' \rightarrow Y$ be a morphism. Take any $x \in X$ and any $y' \in Y'$ such that $f(x) = f'(y') =: y \in Y$. We must show that the induced homomorphism $\mathcal{O}_{y'} \rightarrow \mathcal{O}_{y'} \otimes_{\mathcal{O}_y} \mathcal{O}_x$ is flat. But again, this follows trivially since

$$M \otimes_{\mathcal{O}_{y'}} (\mathcal{O}_{y'} \otimes_{\mathcal{O}_y} \mathcal{O}_x) \cong M \otimes_{\mathcal{O}_y} \mathcal{O}_x. \quad \square$$

Here are some topological properties of flat morphisms.

THEOREM 1.1.14 ([Mil80, I.2.12]). *Any flat morphism that is locally of finite type is open.*

COROLLARY 1.1.15 ([Mil80, I.3.10]). *Any closed immersion which is flat is an open immersion.*

PROPOSITION 1.1.16 ([Gro65, 2.3.12]). *If $f: X \rightarrow Y$ is a flat surjective quasi-compact morphism of schemes then Y has the quotient topology induced by f .*

THEOREM 1.1.17. *Let $f: X \rightarrow Y$ be a morphism of schemes. Then the set*

$$\text{flat}(f) = \{x \in X : f \text{ is flat at } x\}$$

is open in X .

PROOF. See [AK70, V.5.5] □

1.2. Unramified morphisms

DEFINITION 1.2.1. Let k be a field and \bar{k} its algebraic closure. A k -algebra A is called *separable* if the Jacobson radical of $A \otimes_k \bar{k}$ is zero.

DEFINITION 1.2.2. Let $f: X \rightarrow Y$ be a morphism of schemes which is locally of finite type. Then we say that f is *unramified at $x \in X$* if $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_x$ and $\kappa(x)$ is a finite separable field extension of $\kappa(y)$, where $y = f(x)$. We say that f is *unramified* if it is unramified at every $x \in X$.

DEFINITION 1.2.3. A *geometric point* of a scheme X is a morphism $\bar{x}: \text{Spec } \Omega \rightarrow X$ where Ω is a separably closed field. If $Y \rightarrow X$ is a morphism then the *geometric fiber* over a geometric point \bar{x} is the fiber product $Y \times_X \text{Spec } \Omega$.

PROPOSITION 1.2.4 ([Mil80, I.3.2]). *Let $f: X \rightarrow Y$ be a morphism which is locally of finite type. The following are equivalent:*

- (1) f is unramified;
- (2) for all $y \in Y$, the projection $X_y \rightarrow \text{Spec } \kappa(y)$ is unramified;
- (3) for all geometric points $\bar{y}: \text{Spec } \Omega \rightarrow Y$, the projection $X_{\bar{y}} \rightarrow \text{Spec } \Omega$ is unramified;
- (4) for every $y \in Y$, there is a covering of X_y by spectra of finite separable $\kappa(y)$ -algebras;
- (5) for every $y \in Y$, we have $X_y \cong \coprod \text{Spec } k_i$, where the k_i are finite separable field extensions of $\kappa(y)$.

LEMMA 1.2.5. *A composition of unramified morphisms is unramified and a base change of an unramified morphism is unramified.*

PROOF. The composition part is trivial. To show the second part, let $X \rightarrow Y$ be unramified and $Z \rightarrow Y$ any morphism. By Proposition 1.2.4 it is enough to show that $Z \times_Y X \rightarrow Z$ is unramified after base change to a geometric point. But a geometric point in Z gives a geometric point in Y and $\text{Spec } \Omega \times_Z (Z \times_Y X) = \text{Spec } \Omega \times_Y X$ and we already know that $\text{Spec } \Omega \times_Y X \rightarrow \text{Spec } \Omega$ is unramified. □

PROPOSITION 1.2.6 ([AK70, VI.3.3], [Mil80, I.3.5]). *Let X and Y be schemes, x a point in X , and $f: X \rightarrow Y$ a morphism locally of finite type. Let $\Omega_{X/Y}$ denote the sheaf of relative differentials of X over Y . The following are equivalent:*

- (1) f is unramified at x ;
- (2) we have $(\Omega_{X/Y})_x = 0$;
- (3) the diagonal $\Delta_{X/Y}$ is an open immersion in a neighborhood of x .

Note that $(\Omega_{X/Y})_x = \Omega_{\mathcal{O}_x/\mathcal{O}_y}$.

PROOF. (1) \Rightarrow (2): By base change, we may assume that $Y = \text{Spec } \kappa(y)$ and $X = X_y$ (see [Har77, II.8.10]). The fact that f is unramified at x implies that $\{x\}$ is open in $f^{-1}(f(x))$ [AK70, VI.2.3], and hence we may assume that $X =$

$\text{Spec } \kappa(x)$. Hence we need only show that $\Omega_{\kappa(x)/\kappa(y)} = 0$. But this is clear since $\kappa(x)$ is a finite separable extension of $\kappa(y)$.

(2) \Rightarrow (3): The diagonal $\Delta_{X/Y}: X \rightarrow X \times_Y X$ is locally closed and hence we may choose an open subscheme U of $X \times_Y X$, containing $\Delta_{X/Y}(X)$, such that $X \rightarrow U$ is a closed immersion. Denote this map $i: X \rightarrow U$ and let $\mathcal{J} = i_*\mathcal{O}_X$. By definition we have $\Omega_{X/Y} = \Delta_{X/Y}^*(\mathcal{J}/\mathcal{J}^2)$. Hence $0 = (\Omega_{X/Y})_x \cong (\mathcal{J}/\mathcal{J}^2)_{i(x)}$ and by Nakayama's lemma we have that $\mathcal{J}_{i(x)} = 0$. Hence there is an open neighborhood $V \subseteq U$ of $i(x)$ such that $\mathcal{J}|_V = 0$. Thus, $\Delta_{X/Y}|_V = i|_V$ is an open immersion.

(3) \Rightarrow (1): By 1.2.4 we may assume that $Y = \text{Spec } k$ where k is an algebraically closed field (we may choose $Y = \text{Spec } \kappa(y)$ where $y = f(x)$ and then change base to Spec of the algebraic closure of $\kappa(y)$). Since unramified is a local property, we may assume that $X = \text{Spec } A$ is affine and that $\Delta_{X/Y}$ is an open immersion. Let $z \in X$ be a closed point. Then Hilbert's nullstellensatz implies that $\kappa(z) = k$. Let $\varphi: X \rightarrow X \times_Y X$ be the morphism induced by the identity morphism on X and the constant morphism $X \rightarrow X$ with value z . Since the diagonal is open, so is $\varphi^{-1}(\Delta_{X/Y}(X)) = \{z\}$. Hence every closed point of A is open, i.e., every prime ideal is maximal. Thus, A is Artinian and hence we may assume that $A = \mathcal{O}_{X,x}$ with maximal ideal \mathfrak{m} and $\kappa(x) = k$ since x is a closed point. Hence we get that $A \otimes_k A$ has a unique maximal ideal $\mathfrak{m} \otimes A + A \otimes \mathfrak{m}$ and since $\Delta_{X/Y}: \text{Spec } A \rightarrow \text{Spec } (A \otimes_k A)$ is an open immersion we have that $A \otimes_k A \cong A$. But $\dim_k(A \otimes_k A) = \dim_k(A) \cdot \dim_k(A)$ and hence we conclude that $A \cong k$. This implies (1). \square

COROLLARY 1.2.7. *Let $f: X \rightarrow Y$ be a morphism. The following are equivalent:*

- (1) f is unramified;
- (2) we have $\Omega_{X/Y} = 0$;
- (3) the diagonal $\Delta_{X/Y}: X \rightarrow X \times_Y X$ is an open immersion.

COROLLARY 1.2.8. *Let $f: X \rightarrow Y$ be a morphism of schemes. Then the set*

$$\text{unram}(f) = \{x \in X : f \text{ is unramified at } x\}$$

is open in X .

PROPOSITION 1.2.9. *Any section of an unramified morphism is an open immersion.*

PROOF. If $f: X \rightarrow Y$ is unramified then the diagonal $\Delta: X \rightarrow X \times_Y X$ is an open immersion. Given a section $s: Y \rightarrow X$ of f , we have that Y is the fiber product of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \Delta \\ X = Y \times_Y X & \xrightarrow{s \times \text{id}_X} & X \times_Y X \end{array}$$

and the projections $Y \rightarrow X$ both coincide with s . Thus s is obtained by base change from Δ which is an open immersion, and hence s is an open immersion. \square

1.3. Étale morphisms

DEFINITION 1.3.1. Let $f: X \rightarrow Y$ be a morphism of schemes. Then we say that f is étale at $x \in X$ if it is flat and unramified at x . We say that f is étale if it is étale at every $x \in X$.

REMARK 1.3.2. Note that Theorem 1.1.14 implies that every étale morphism is open as a map between topological spaces.

LEMMA 1.3.3. *Let $f: X \rightarrow Y$ be a morphism of schemes. Then the set*

$$\text{étale}(f) = \{x \in X : f \text{ is étale at } x\}$$

is open in X .

PROOF. This follows from Proposition 1.2.6 and Theorem 1.1.17. \square

EXAMPLE 1.3.4. *Let X and Y be nonsingular varieties over an algebraically closed field k , and let $f: X \rightarrow Y$ be a morphism of schemes. Then f is étale at $x \in X$ if and only if it induces an isomorphism $T_f: T_{X,x} \rightarrow T_{Y,f(x)}$ of the tangent spaces.*

PROOF. Indeed, let $x \in X$ be a closed point and put $y = f(x)$. Then $\kappa(y) \cong \kappa(x) \cong k$. Suppose that f is étale and put $\mathcal{O}_x = \mathcal{O}_{X,x}$ and $\mathcal{O}_y = \mathcal{O}_{Y,y}$. We have homomorphisms $k \rightarrow \mathcal{O}_x \rightarrow \kappa(x)$ and $k \rightarrow \mathcal{O}_y \rightarrow \kappa(y)$, which yield exact sequences

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow \Omega_{\mathcal{O}_x/k} \otimes_{\mathcal{O}_x} \kappa(x) \rightarrow \Omega_{\kappa(x)/k} = 0,$$

and

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \Omega_{\mathcal{O}_y/k} \otimes_{\mathcal{O}_y} \kappa(y) \rightarrow \Omega_{\kappa(y)/k} = 0$$

[Mat86, 25.2]. The first map in each of the sequences is an isomorphism [Har77, 8.7]. We also have homomorphisms $k \rightarrow \mathcal{O}_y \rightarrow \mathcal{O}_x$, where the last one is faithfully flat since it is a local homomorphism and f is flat. We get an exact sequence

$$\Omega_{\mathcal{O}_y/k} \otimes_{\mathcal{O}_y} \mathcal{O}_x \rightarrow \Omega_{\mathcal{O}_x/k} \rightarrow \Omega_{\mathcal{O}_x/\mathcal{O}_y} \rightarrow 0$$

[Mat86, 25.1], where the first map is an isomorphism by [AK70, VI.4.9]. These may all be viewed as \mathcal{O}_x -modules. If we tensor with $\kappa(x)$ we get that

$$\Omega_{\mathcal{O}_y/k} \otimes_{\mathcal{O}_y} \kappa(x) \cong \Omega_{\mathcal{O}_x/k} \otimes_{\mathcal{O}_x} \kappa(x),$$

and hence

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \cong \mathfrak{m}_x/\mathfrak{m}_x^2.$$

Since the cotangent spaces are isomorphic and so are the duals.

Conversely, if $T_f: T_{X,x} \rightarrow T_{Y,y}$ is an isomorphism, then so is the induced map $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$. Let $d = \dim(\mathfrak{m}_y/\mathfrak{m}_y^2)$. Then \mathfrak{m}_y can be generated by d elements, t_1, \dots, t_d [AM69, 11.22]. The ring $\mathcal{O}_x/(t_1, \dots, t_d)$ is flat over $\mathcal{O}_y/(t_1, \dots, t_d) = \kappa(y) \cong k$ and by [Har77, 10.3.A], $\mathcal{O}_x/(t_1, \dots, t_i)$ is flat over $\mathcal{O}_y/(t_1, \dots, t_i)$ for each $i = d, d-1, \dots, 0$. Hence \mathcal{O}_x is flat over \mathcal{O}_y . Since $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$ is an isomorphism we get from the exact sequence

$$\Omega_{\mathcal{O}_y/k} \otimes_{\mathcal{O}_y} \kappa(y) \rightarrow \Omega_{\mathcal{O}_x/k} \otimes_{\mathcal{O}_x} \kappa(x) \rightarrow \Omega_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} \kappa(x) \rightarrow 0,$$

that $\Omega_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} \kappa(x) \cong \Omega_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} \kappa(x) = 0$. Hence, by Nakayama's lemma, we conclude that $(\Omega_{X/Y})_x = \Omega_{\mathcal{O}_x/\mathcal{O}_y} = 0$, and by Proposition 1.2.6, f is unramified at x . \square

LEMMA 1.3.5. *A composition of étale morphisms is étale and a base change of an étale morphism is étale.*

PROOF. Follows from Lemma 1.1.13 and Lemma 1.2.5. \square

DEFINITION 1.3.6. A morphism of schemes is called *smooth* if it is flat, locally of finite presentation, and if the geometric fibers are regular.

REMARK 1.3.7. A morphism is étale if and only if it is smooth and quasi-finite.

1.4. Local structure of étale morphisms

Let A be a ring and $p(T) \in A[T]$ a monic polynomial. Then $A[T]/(p)$ is a finitely generated free A -module, and hence flat. Suppose that $b \in A[T]/(p)$ is such that the formal derivative $p'(T)$ is invertible in $(A[T]/(p))_b$.

DEFINITION 1.4.1. The morphism of spectra

$$\mathrm{Spec}(A[T]/(p))_b \rightarrow \mathrm{Spec} A$$

induced by the canonical homomorphism $\varphi: A \rightarrow (A[T]/(p))_b$ is called *standard étale*.

A *standard étale morphism is étale*. Indeed, it is flat since $A[T]/(p)$ is a free A -module and $(A[T]/(p))_b$ is a flat $A[T]/(p)$ -module. Now put $B = A[T]/(p)$. To show that $\mathrm{Spec} B_b \rightarrow \mathrm{Spec} A$ is unramified, it is enough to prove that the B_b -module $\Omega_{B_b/A}$ is zero. We have that $\Omega_{B/A}$ is the B -module generated by dT and the relation $p'(T)dT = 0$ (see e.g. [Mat86, p. 195]). That is $\Omega_{B/A}$ is isomorphic to $A[T]/(p', p)$ as a B -module. But then $\Omega_{B_b/A} \cong (\Omega_{B/A})_b \cong (A[T]/(p', p))_b \cong (A[T]/(p))_b/(p')_b = 0$ since p' is invertible in B_b . Hence $\mathrm{Spec} B_b \rightarrow \mathrm{Spec} A$ is unramified and thus étale.

THEOREM 1.4.2 (Local structure theorem, [Mil80, I.3.14, I.3.16]). *Let $f: X \rightarrow Y$ be a morphism of schemes. The following are equivalent:*

- (1) f is étale at x ;
- (2) There exists open affine sets $U \ni x$ and $V \ni f(x)$ such that $f(U) \subseteq V$ and $f|_U: U \rightarrow V$ is standard étale;
- (3) There exists open affine sets $U = \mathrm{Spec} B \ni x$ and $V = \mathrm{Spec} A \ni f(x)$, such that $B = A[T_1, \dots, T_n]/(p_1, \dots, p_n)$ where $\det(\partial p_i/\partial T_j)$ is invertible in B , and $f|_U: U \rightarrow V$ is induced by the canonical homomorphism $A \rightarrow B$.

PROOF. (1) \Rightarrow (2): By Lemma 1.3.3, f is étale in a neighborhood of x . Now see [Mil80, I.3.14].

(2) \Rightarrow (3): This follows since $(A[T]/p)_b \cong A[T, S]/(p, bS - 1)$ and

$$\begin{pmatrix} \partial p/\partial T & \partial p/\partial S \\ \partial(bS - 1)/\partial T & \partial(bS - 1)/\partial S \end{pmatrix} = \begin{pmatrix} p'(T) & 0 \\ b' & b \end{pmatrix}$$

is invertible.

Since we have already showed that every standard étale morphism is étale, it is enough to show that (3) implies (2) to finish the proof.

(3) \Rightarrow (2): Since B is generated as an A -algebra by the elements T_1, \dots, T_n , we have that $\Omega_{B/A}$ is generated as a B -module by the elements dT_1, \dots, dT_n and the relations

$$(1.4.0.1) \quad dp_i = \sum_{j=1}^n \frac{\partial p_i}{\partial T_j} dT_j = 0, \quad 1 \leq i \leq n.$$

Indeed, the derivation $d: B \rightarrow \Omega_{B/A}$ is surjective and every element in B may be written as a polynomial $f(T_1, \dots, T_n)$. By the Leibniz rule we have

$$df(T_1, \dots, T_n) = \sum_{i=1}^n \frac{\partial f}{\partial T_i} dT_i.$$

Since the image of $\det(\partial p_i/\partial T_j)$ in B is a unit, there is a unique solution to (1.4.0.1), namely $dT_1 = \dots = dT_n = 0$. Hence $\mathrm{Spec} B \rightarrow \mathrm{Spec} A$ is unramified by Proposition 1.2.6.

To show that B is flat as an A -module, one may use Proposition 1.1.6 and induction on n . We have that $A[T_1, \dots, T_n]$ is a free A -module and hence flat over

A. The idea is to show inductively that $A[T_1, \dots, T_n]/(p_1, \dots, p_i)$ is flat over A as i ranges from 0 to n . This is done in [Mum99, p. 221]. \square

EXAMPLE 1.4.3. Let n be a positive integer and $X = \text{Spec } \mathbb{Z}[T]/(T^n - 1)$. Consider the morphism $f: X \rightarrow \text{Spec } \mathbb{Z}$ given by the canonical homomorphism $\mathbb{Z} \hookrightarrow \mathbb{Z}[T]/(T^n - 1)$. It is clear that f is étale in the open subscheme $D(n) = X \setminus V((n))$ since $\partial(T^n - 1)/\partial T = nT^{n-1}$ and $\mathcal{O}_X(D(n)) = (\mathbb{Z}[T]/(T^n - 1))_n$. That is, nT^{n-1} has an inverse $n^{-1}T$ in $(\mathbb{Z}[T]/(T^n - 1))_n$.

EXAMPLE 1.4.4 (Artin-Schreier cover). Let k be a field of non-zero characteristic p and take $f \in k[T]$. The morphism

$$\text{Spec } k[T, x]/(x^p - x - f) \rightarrow \text{Spec } k[T]$$

is étale since $\partial(x^p - x - f)/\partial x = px^{p-1} - 1 = -1$. If p does not divide the degree of f then this covering is non-trivial.

1.5. Henselian rings

For a scheme X , we denote by $\text{ClOp}(X)$ the collection of clopen subsets of X .

DEFINITION 1.5.1. Let X be a scheme and X_0 a closed subscheme. The pair (X, X_0) is called a *Henselian pair* if for every finite morphism $X' \rightarrow X$, the induced map $\text{ClOp}(X') \rightarrow \text{ClOp}(X' \times_X X_0)$ is bijective.

DEFINITION 1.5.2. A local ring (A, \mathfrak{m}) is called *Henselian* if $(\text{Spec } A, \text{Spec } A/\mathfrak{m})$ is a Henselian pair.

LEMMA 1.5.3. *There is a bijective correspondence*

$$\text{ClOp}(\text{Spec } A) \simeq \{\text{idempotents of } A\}.$$

PROOF. If $e \in A$ is idempotent and $\mathfrak{p} \in \text{Spec } A$, then $e \in \mathfrak{p}$ if and only if $1 - e \notin \mathfrak{p}$. Hence we get that $V(e) = D(1 - e)$ is clopen with complement $D(e) = V(1 - e)$.

Conversely, if $U \subseteq \text{Spec } A$ is clopen, then $U \cup (\text{Spec } A \setminus U)$ is an open cover of $\text{Spec } A$ and hence, by the sheaf property, there is a unique element

$$a \in \mathcal{O}_{\text{Spec } A}(\text{Spec } A) = A,$$

such that $a|_U = 1$ and $a|_{\text{Spec } A \setminus U} = 0$. Hence $(1 - a)|_U = 0$ and $(1 - a)|_{\text{Spec } A \setminus U} = 1$. Thus $a(a - 1) = 0$ since the restrictions to U and $\text{Spec } A \setminus U$ are zero and hence a is idempotent. We get that $U = D(a)$. \square

If f is a polynomial with coefficients in a local ring A with maximal ideal \mathfrak{m} , then we use the notation \bar{f} for its image in $(A/\mathfrak{m})[x]$.

THEOREM 1.5.4 ([Mil80, I.4.2]). *Let (A, \mathfrak{m}) be a local ring, $X = \text{Spec } A$, and let x be the closed point in X . The following are equivalent:*

- (1) A is Henselian;
- (2) every finite A -algebra B is a direct product of local rings $B = \prod B_i$;
- (3) if $f: Y \rightarrow X$ is a quasi-finite and separated morphism, then

$$Y = Y_0 \amalg Y_1 \amalg \dots \amalg Y_n,$$

where $x \notin f(Y_0)$ and for $i \geq 1$, $Y_i = \text{Spec } B_i$ is finite over X where each B_i is a local ring;

- (4) if $f: Y \rightarrow X$ is an étale morphism then every morphism $\gamma: \text{Spec } \kappa(x) \rightarrow Y$, such that $f \circ \gamma(\text{Spec } \kappa(x)) = x$, factors through a section $s: X \rightarrow Y$ of f ;
- (5) if $f \in A[x]$ is a monic polynomial such that \bar{f} factors as $\bar{f} = g_0 \bar{h}_0$, with g_0 and h_0 coprime, then f factors as gh , where $\bar{g} = g_0$ and $\bar{h} = h_0$.

PROOF. (1) \Rightarrow (2): Let $f: \text{Spec } B \rightarrow \text{Spec } A$ be finite. We have

$$\text{Spec } B \times_A \text{Spec } \kappa(x) = \text{Spec } (B \otimes_A \kappa(x)).$$

There is a bijective correspondence between idempotents of B and idempotents of $B \otimes_A \kappa(x) \cong B/\mathfrak{m}_x B$. If B is not local, then there exists a non-trivial idempotent $\bar{e} \in B/\mathfrak{m}_x B$, and hence \bar{e} lifts to some non-trivial idempotent $e \in B$. Hence $B \cong eB \times (1-e)B$ where $eB \neq 0$ and $(1-e)B \neq 0$. Iterating this process yields the desired splitting.

(2) \Rightarrow (3): Let $f: Y \rightarrow X$ be quasi-finite and separated. According to Theorem 0.0.1, f factors as

$$Y \xrightarrow{f'} Y' \xrightarrow{g} X$$

where f' is an open immersion and g is finite. Hence $Y' = \text{Spec } B$ for some finite A -algebra B , and by (2), $B = \prod B_i$. Each B_i is of the form $B_i = \mathcal{O}_{Y', y'}$ for some closed point $y' \in Y'$. Let $Y_1 = \coprod \text{Spec } \mathcal{O}_{Y', y}$ where the disjoint union is over all closed points y of Y' that are contained in Y . Thus Y_1 is clopen in Y' and hence also clopen in Y . Put $Y_0 = Y \setminus Y_1$. Then we have $Y = Y_0 \coprod Y_1$ and it is clear that Y_0 contains no closed points of Y' . Since Y_1 is finite over X we get that all points in the fiber over x are closed. Hence they are also closed in Y' since the preimage of x in Y' is closed. Thus $x \notin f(Y_0)$.

(3) \Rightarrow (4): Suppose that $f: Y \rightarrow X$ is étale and we have a morphism

$$\text{Spec } \kappa(x) \rightarrow Y$$

with image $y \in Y$ such that $f(y) = x$. Then we have embeddings $\kappa(x) \hookrightarrow \kappa(y) \hookrightarrow \kappa(x)$ and hence $\kappa(y) = \kappa(x)$. Since $\mathcal{O}_{Y, y}$ is a flat A -module, it is free [Mil80, I.2.9]. But f is étale and hence we have that $\mathfrak{m}_y = \mathfrak{m}_x \mathcal{O}_{Y, y}$ and $\kappa(x) = \kappa(y) = \mathcal{O}_{Y, y} \otimes_A \kappa(x)$. That is, $\mathcal{O}_{Y, y}$ has rank 1, i.e., $\mathcal{O}_{Y, y} \cong A$. By (3) we may assume that $Y = \text{Spec } B$ where B is a local ring. That is, $B = \mathcal{O}_{Y, y} \cong A$. Hence (4) holds.

(4) \Rightarrow (5): See the proof of (d) \Rightarrow (d') \Rightarrow (d) in [Mil80, I.4.2].

(5) \Rightarrow (1): It is enough to show that for every finite A -algebra B , the homomorphism

$$B \rightarrow B \otimes_A (A/\mathfrak{m}) \cong B/\mathfrak{m}B$$

gives a bijection of idempotents. But this follows immediately from (5) since every nilpotent in $B/\mathfrak{m}B$ lifts to a unique idempotent in B . \square

REMARK 1.5.5. One may actually replace *étale* with *smooth* in Theorem 1.5.4 (4) (see [Gro67, Corollaire 17.16.3]).

EXAMPLE 1.5.6. Any complete local ring is Henselian [Sta, Tag 04GM].

Representable functors

2.1. Definitions and examples

Let \mathcal{C} be a category. A functor $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow (\text{Set})$ is called *representable* if it is isomorphic to the functor $h_X = \text{Hom}_{\mathcal{C}}(-, X)$ for some object X in \mathcal{C} . We also say that X *represents* the functor \mathcal{F} . Note that for a morphism $\varphi: Y \rightarrow Z$ in \mathcal{C} , the map $\varphi^*: h_X(Z) \rightarrow h_X(Y)$ is given by sending a morphism $\psi: Z \rightarrow X$ to the morphism $\psi \circ \varphi: Y \rightarrow X$. Furthermore, we have a natural transformation

$$h_{\varphi}: h_Y \rightarrow h_Z$$

defined by sending a morphism $\beta: W \rightarrow Y$ to the composition $\varphi \circ \beta: W \rightarrow Z$.

REMARK 2.1.1. A functor $F: \mathcal{C}^{\text{op}} \rightarrow (\text{Set})$ is representable if and only if it has a *universal object*, that is, if there exists a pair (X, ξ) , where X is an object in \mathcal{C} and $\xi \in FX$, such that for any $Y \in \mathcal{C}$ and any element $\eta \in FY$, there exists a unique $f \in \text{Hom}(Y, X)$ such that $f^*(\xi) = \eta$.

EXAMPLE 2.1.2. Let A be a ring, let $f_1, \dots, f_m \in A[T_1, \dots, T_n]$ be polynomials, and put $R = A[T_1, \dots, T_n]/(f_1, \dots, f_m)$. Let S be a scheme over $\text{Spec } A$ and put $X = \text{Spec } R$. We have

$$\begin{aligned} \text{Hom}_{(\text{Sch})}(S, X) &\simeq \text{Hom}_{(A\text{-alg})}(R, \Gamma(S, \mathcal{O}_S)) \\ &\simeq \{s \in \Gamma(S, \mathcal{O}_S)^n : f_1(s) = \dots = f_n(s) = 0\}, \end{aligned}$$

where the last isomorphism is given by sending an A -algebra homomorphism φ to the tuple $(\varphi(T_1), \dots, \varphi(T_n))$ (clearly $f_i(\varphi(T_1), \dots, \varphi(T_n)) = \varphi(f_i(T_1, \dots, T_n))$ for all $1 \leq i \leq n$). Hence we see that the functor $(\text{Sch}/A) \rightarrow (\text{Set})$ that sends a scheme S over $\text{Spec } A$ to the set

$$\{s \in \Gamma(S, \mathcal{O}_S)^n : f_1(s) = \dots = f_n(s) = 0\}$$

is represented by $\text{Spec}(A[T_1, \dots, T_n]/(f_1, \dots, f_n))$.

EXAMPLE 2.1.3 (Affine n -space). In particular, the functor

$$(\text{Sch})^{\text{op}} \rightarrow (\text{Set})$$

that sends a scheme S to the set $\Gamma(S, \mathcal{O}_S)^n$ is represented by $\text{Spec}(\mathbb{Z}[T_1, \dots, T_n])$. Indeed, we have bijections

$$\text{Hom}_{(\text{Sch})}(S, \mathbb{A}^n) \simeq \text{Hom}_{(\text{Ring})}(\mathbb{Z}[T_1, \dots, T_n], \Gamma(S, \mathcal{O}_S)) \simeq \Gamma(S, \mathcal{O}_S)^n,$$

which are natural in S .

EXAMPLE 2.1.4 ($\mathbb{G}_m = \text{Spec } \mathbb{Z}[T, T^{-1}]$). As another special case of Example 2.1.2, we get that the functor $(\text{Sch})^{\text{op}} \rightarrow (\text{Set})$ that sends a scheme S to the set $\Gamma(S, \mathcal{O}_S)^*$ of units in $\Gamma(S, \mathcal{O}_S)$ is represented by \mathbb{G}_m . This follows from the isomorphism $\mathbb{Z}[T, T^{-1}] \cong \mathbb{Z}[T, X]/(TX - 1)$.

EXAMPLE 2.1.5. (The Grassmannian) Consider the functor

$$G_{k,n}: (\text{Sch})^{\text{op}} \rightarrow (\text{Set})$$

defined by

$$G_{k,n}(X) = \{\mathcal{F} \subseteq \mathcal{O}_X^{\oplus n} : \mathcal{O}_X^{\oplus n}/\mathcal{F} \text{ is locally free of rank } n - k\}$$

and which takes a morphism $f: Y \rightarrow X$ to the map $G_{k,n}(f): G_{k,n}(X) \rightarrow G_{k,n}(Y)$ which takes \mathcal{F} to the pullback $f^*\mathcal{F}$. To see that $f^*\mathcal{F} \in G_{k,n}(Y)$ note first that $i: \mathcal{F} \hookrightarrow \mathcal{O}_X^{\oplus n}$ gives a morphism $f^*\mathcal{F} \rightarrow f^*(\mathcal{O}_X^{\oplus n}) = \mathcal{O}_Y^{\oplus n}$ which is injective since i is injective and $\mathcal{O}_X^{\oplus n}/i(\mathcal{F})$ is locally free [GW10, 8.10]. For any subset $I \subseteq \{1, \dots, n\}$ we may define a subfunctor $G_I \subseteq G_{k,n}$ by

$$G_I(X) = \{\mathcal{F} \in G_{k,n}(X) : \mathcal{O}_X^{\oplus I} \hookrightarrow \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X^{\oplus n}/\mathcal{F} \text{ is an isomorphism}\},$$

where the morphism $\mathcal{O}_X^{\oplus I} \hookrightarrow \mathcal{O}_X^{\oplus n}$ is induced by the inclusion $I \hookrightarrow \{1, \dots, n\}$ and by $\mathcal{O}_X^{\oplus I}$ we mean $\mathcal{O}_X \oplus \dots \oplus \mathcal{O}_X$ with one component for each index in I .

For every $\mathcal{F} \in G_I(X)$ we have a morphism $\mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X^{\oplus I}$ with kernel \mathcal{F} , and conversely, for every retraction $\tau: \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X^{\oplus I}$ of the inclusion $\mathcal{O}_X^{\oplus I} \hookrightarrow \mathcal{O}_X^{\oplus n}$, we get that $\ker(\tau) \in G_I(X)$. Hence there is a bijection between the set of retractions $r: \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X^{\oplus I}$ of the inclusion $\mathcal{O}_X^{\oplus I} \hookrightarrow \mathcal{O}_X^{\oplus n}$ and elements of $G_I(X)$. It is not hard to see that this bijection is functorial in X . Such a retraction r must be the identity on the indices in I and hence r is completely determined by its values on the index set $I^c = \{1, \dots, n\} \setminus I$. Hence we conclude that we have a functorial bijection

$$G_I(X) \simeq \text{Hom}_{(\mathcal{O}_X\text{-mod})}(\mathcal{O}_X^{\oplus I^c}, \mathcal{O}_X^{\oplus I}).$$

But we also have natural bijections

$$\text{Hom}_{(\mathcal{O}_X\text{-mod})}(\mathcal{O}_X^{\oplus I^c}, \mathcal{O}_X^{\oplus I}) \simeq \text{Hom}_{(\text{Set})}(I^c \times I, \Gamma(X, \mathcal{O}_X)) \simeq \Gamma(X, \mathcal{O}_X)^{k(n-k)}$$

(see [GW10, 7.4.6]), and by Example 2.1.3 we conclude that there is a natural bijection

$$G_I(X) \simeq \text{Hom}_{(\text{Sch})}(X, \mathbb{A}^{k(n-k)}).$$

That is, the functor G_I is represented by the affine scheme $\mathbb{A}^{k(n-k)}$. This may be used to show that the functor $G_{k,n}$ is representable (see [GW10, Proposition 8.14]). In particular, one may show that $G_{1,n+1}$ is represented by the projective space $\mathbb{P}_{\mathbb{Z}}^n$.

2.2. The Yoneda embedding

LEMMA 2.2.1 (Yoneda's lemma). *For any object X in \mathcal{C} , the map*

$$\begin{aligned} \alpha_{X,\mathcal{F}}: \text{Hom}(\mathfrak{h}_X, \mathcal{F}) &\xrightarrow{\sim} \mathcal{F}(X) \\ \tau &\mapsto \tau_X(\text{id}_X), \end{aligned}$$

is a bijection which is natural in X and \mathcal{F} .

PROOF. The first part follows from the fact that any natural transformation $\tau: \mathfrak{h}_X \rightarrow \mathcal{F}$ is completely determined by the image of $\text{id}_X \in \mathfrak{h}_X(X)$ in $\mathcal{F}(X)$. Indeed, consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_X(X) & \xrightarrow{\tau_X} & \mathcal{F}(X) \\ f^* \downarrow & & \downarrow f^* \\ \mathfrak{h}_X(Y) & \xrightarrow{\tau_Y} & \mathcal{F}(Y) \end{array}$$

induced by a morphism $f: Y \rightarrow X$. We have $\tau_Y(f) = \tau_Y(f^*(\text{id}_X)) = f^*(\tau_X(\text{id}_X))$. This proves the first part.

Let $f: Y \rightarrow X$ be a morphism in \mathcal{C} , and let $h_f: h_Y \rightarrow h_X$ be the induced natural transformation. To prove naturality in X , we need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(h_X, \mathcal{F}) & \xrightarrow{\alpha_{X, \mathcal{F}}} & \mathcal{F}(X) \\ f^\# \downarrow & & \downarrow f^* \\ \mathrm{Hom}(h_Y, \mathcal{F}) & \xrightarrow{\alpha_{Y, \mathcal{F}}} & \mathcal{F}(Y) \end{array}$$

where $f^\#$ is the map defined by taking $h_X \rightarrow \mathcal{F}$ to the composition $h_Y \rightarrow h_X \rightarrow \mathcal{F}$. Let $\tau: h_X \rightarrow \mathcal{F}$ be a natural transformation. Then

$$\begin{aligned} f^* \circ \alpha_{X, \mathcal{F}}(\tau) &= f^*(\tau_X(\mathrm{id}_X)) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \mathrm{id}_Y) \\ &= (\tau \circ h_f)_Y(\mathrm{id}_Y) \\ &= \alpha_{Y, \mathcal{F}} \circ f^\#(\tau). \end{aligned}$$

Naturality in \mathcal{F} is trivial since if $\eta: \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of functors, then $\mathrm{Hom}(h_X, \mathcal{F}) \rightarrow \mathrm{Hom}(h_X, \mathcal{G})$ is just given by composition with η and by definition we have $(\eta \circ \tau)_X(\mathrm{id}_X) = \eta_X(\tau_X(\mathrm{id}_X))$. \square

Hence we have a functor $\mathrm{Hom}(h_{(-)}, \mathcal{F}): \mathcal{C}^{\mathrm{op}} \rightarrow (\mathrm{Set})$; $X \mapsto \mathrm{Hom}(h_X, \mathcal{F})$ and Lemma 2.2.1 says that there is an isomorphism of functors

$$\mathrm{Hom}(h_{(-)}, \mathcal{F}) \cong \mathcal{F}.$$

REMARK 2.2.2. Note that Yoneda's lemma implies that any map $\mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ given by a morphism $Y \rightarrow X$ is exactly the map given by left composition by $h_Y \rightarrow h_X$.

A morphism $f: X \rightarrow Y$ in a category \mathcal{C} gives a natural transformation $h_X \rightarrow h_Y$ by composing with f . Thus, the assignment $X \mapsto h_X$ is a functor $\mathcal{C} \rightarrow \mathrm{PreSh}(\mathcal{C})$ from \mathcal{C} to the category $\mathrm{PreSh}(\mathcal{C})$ of functors $\mathcal{C}^{\mathrm{op}} \rightarrow (\mathrm{Set})$ (that is, the category of *presheaves* on \mathcal{C}). Yoneda's lemma implies that

$$\mathrm{Hom}_{\mathrm{PreSh}(\mathcal{C})}(h_X, h_Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

i.e., the functor $X \mapsto h_X$ is fully faithful.

DEFINITION 2.2.3. The embedding $\mathcal{C} \rightarrow \mathrm{PreSh}(\mathcal{C})$; $X \mapsto h_X$ is called the *Yoneda embedding*.

REMARK 2.2.4. Yoneda's lemma implies that there is an equivalence of categories between \mathcal{C} and the category of representable functors $F: \mathcal{C}^{\mathrm{op}} \rightarrow (\mathrm{Set})$ given by sending an object X to h_X . Hence if Y is an object in \mathcal{C} , we write

$$X(Y) = h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

REMARK 2.2.5. Note that two objects X and Y in a category \mathcal{C} are isomorphic if and only if h_X and h_Y are isomorphic as functors

$$(\mathcal{C}_{X, Y})^{\mathrm{op}} \rightarrow (\mathrm{Set})$$

where $\mathcal{C}_{X, Y}$ is the full subcategory of \mathcal{C} with only two objects X and Y .

REMARK 2.2.6. Note that since $\mathrm{id}: S \rightarrow S$ is the final object in Sch/S , we have that h_S is the final object in the category $\mathrm{PreSh}(\mathrm{Sch}/S)$. Indeed, every S -scheme

X comes with a morphism $f: X \rightarrow S$. Morphisms $\varphi: X \rightarrow S$ are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & S \\ & \searrow f & \swarrow \text{id} \\ & & S \end{array}$$

Obviously, we must have $f = \varphi$ and hence $h_S(X)$ consists of a single point. Thus, if \mathcal{F} is a presheaf on (Sch/S) then there is a unique map $\mathcal{F}(X) \rightarrow h_S(X)$ for each S -scheme X . This gives the unique natural transformation $\mathcal{F} \rightarrow h_S$.

CHAPTER 3

Sheaves of sets

In the following chapter we will discuss sheaves on sites, which is a generalization of the concept of a sheaf on a topological space. The definition is very similar, keeping in mind that the fiber product $U_i \times_U U_j$ in the category $\text{Open}(X)$ of open subsets of a topological space X , with morphisms given by inclusions, is just the intersection $U_i \cap U_j$ taken in U .

3.1. Grothendieck topologies and sites

The main references to the following section are [FGI⁺05, Mil80, Mil21, Tam94].

DEFINITION 3.1.1. Let \mathcal{C} be a category with fiber products. A *Grothendieck topology* on \mathcal{C} is defined by the following data: for each object U in \mathcal{C} we have a collection $\text{Cov}(U)$ of *coverings* of U . A covering is a set of arrows $\{U_i \rightarrow U\}_{i \in I}$. The coverings satisfy the following axioms:

- (1) if $V \rightarrow U$ is an isomorphism then $\{V \rightarrow U\}$ is a covering;
- (2) if $\{U_i \rightarrow U\}$ is a covering and $V \rightarrow U$ is a morphism, then $\{V \times_U U_i \rightarrow V\}$ is a covering;
- (3) if $\{U_i \rightarrow U\}$ is a covering and for every index i we have a covering $\{V_{ij} \rightarrow U_i\}$, then $\{V_{ij} \rightarrow U_i \rightarrow U\}$ is a covering of U .

A category together with a Grothendieck topology is called a *site*. If \mathcal{S} is a site then the underlying category is denoted by $\text{Cat}(\mathcal{S})$.

For any family of maps $\{\phi_i: U_i \rightarrow U\}$ between spaces of any kind, we say $\{\phi_i: U_i \rightarrow U\}$ is *jointly surjective* if $U = \bigcup \phi_i(U_i)$.

REMARK 3.1.2. We will sometimes also use the following notation, as in [Mil80, II.1]: Let \mathbf{E} be a class of morphisms of schemes such that

- (1) every isomorphism is in \mathbf{E} ,
- (2) any composition of morphisms in \mathbf{E} is in \mathbf{E} , and
- (3) any base change of a morphism in \mathbf{E} is in \mathbf{E} .

Let S be a scheme and \mathbf{E} a class of morphisms as above. Let \mathcal{C}/S be a full subcategory of Sch/S which is closed under taking fiber products and such that for any $U \rightarrow S$ in \mathcal{C}/S and any \mathbf{E} -morphism $U' \rightarrow U$, the composition $U' \rightarrow U \rightarrow S$ is in \mathcal{C}/S . Then we get a Grothendieck topology on \mathcal{C}/S by taking as coverings: all collections $\{\phi_i: U_i \rightarrow U\}$ of \mathbf{E} -morphisms over S such that $U = \bigcup \phi_i(U_i)$. The resulting site will be denoted by $S_{\mathbf{E}}$ or $(\mathcal{C}/S)_{\mathbf{E}}$.

EXAMPLE 3.1.3 (Small classical topology on a topological space X). Consider the category $\text{Open}(X)$ of open subsets of a topological space X , where the morphisms are given by inclusions. A covering of an open subset $U \subseteq X$ is a jointly surjective family $\{U_i \rightarrow U\}$.

EXAMPLE 3.1.4 (The big classical topology on (Top)). Consider the category (Top) of topological spaces. A covering of a topological space U is a jointly surjective family of open embeddings $U_i \rightarrow U$.

EXAMPLE 3.1.5 (Small Zariski site on X). Consider the category $\text{ZarOp}(X)$ of Zariski-open subsets of a scheme X with morphisms that are inclusion maps. A covering is a jointly surjective family $\{U_i \rightarrow U\}$. This site is denoted X_{zar} .

EXAMPLE 3.1.6 (Big Zariski site on S). Consider the category (Sch/S) of schemes over S . A covering of a scheme $U \rightarrow S$ is a jointly surjective family of open immersions $U_i \rightarrow U$ over S . The corresponding site is denoted by S_{zar} .

EXAMPLE 3.1.7 (Small étale site on X). Let $(\text{ét}/X)$ be the category whose objects are étale morphisms $U \rightarrow X$ and whose arrows are X -morphisms $V \rightarrow U$ of schemes. The coverings are jointly surjective families of morphisms $\{U_i \rightarrow U\}$ in $(\text{ét}/X)$ (i.e., étale X -morphisms). By Lemma 1.3.5, the property of being étale is stable under base change and composition and hence this defines a Grothendieck topology on $(\text{ét}/X)$. The corresponding site is denoted $X_{\text{ét}}$.

EXAMPLE 3.1.8 (Big étale site over S). The site with underlying category (Sch/S) and coverings which are jointly surjective families $\{U_i \rightarrow U\}$ of étale S -morphisms is denoted by $S_{\text{ét}}$.

EXAMPLE 3.1.9 (Big flat site on S). Consider the category (Sch/S) with coverings which are jointly surjective families $\{U_i \rightarrow U\}$ of flat S -morphisms which are locally of finite presentation. Note that this implies that the induced map $\coprod U_i \rightarrow U$ is flat and surjective, i.e., faithfully flat. This site is denoted by S_{fl} .

DEFINITION 3.1.10. A morphism $X \rightarrow Y$ of schemes is called an *fpqc morphism* if it is faithfully flat and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X .

EXAMPLE 3.1.11 (Big fpqc site on S). The site with underlying category (Sch/S) and coverings which are jointly surjective families $\{U_i \rightarrow U\}$ of S -morphisms such that the induced map $\coprod U_i \rightarrow U$ is fpqc is denoted by S_{fpqc} .

REMARK 3.1.12. For a scheme S , we have *continuous morphisms* (see Definition 4.1.1)

$$\begin{array}{ccccccc} S_{\text{fpqc}} & \longrightarrow & S_{\text{fl}} & \longrightarrow & S_{\text{ét}} & \longrightarrow & S_{\text{zar}} \\ & & & & \downarrow & & \downarrow \\ & & & & S_{\text{ét}} & \longrightarrow & S_{\text{zar}} \end{array}$$

induced by the identity morphism $S \rightarrow S$.

PROPOSITION 3.1.13 ([FGI⁺05, Proposition 2.33]). *Let $f: X \rightarrow Y$ be a surjective morphism of schemes. The following are equivalent:*

- (1) every quasi-compact open subset of Y is the image of a quasi-compact open subset of X ;
- (2) there is a covering $Y = \bigcup V_i$ of Y by open affine subschemes V_i such that each V_i is the image of a quasi-compact open subset of X ;
- (3) for every $x \in X$, there is an open neighborhood U of x , such that the restriction $f: U \rightarrow f(U)$ is quasi-compact and $f(U)$ is open in Y ;
- (4) for every $x \in X$ there is a quasi-compact open neighborhood U of x such that $f(U)$ is open in Y and affine.

3.2. Sheaves of sets

DEFINITION 3.2.1. A *presheaf* (of sets) on a site \mathcal{S} is a functor $\mathcal{F}: \text{Cat}(\mathcal{S})^{\text{op}} \rightarrow (\text{Set})$. A presheaf \mathcal{F} is called *separated* if for every covering $\{U_i \rightarrow U\}$, the map

$$\mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$$

is injective.

A presheaf \mathcal{F} is called a *sheaf* if the diagram

$$(3.2.0.2) \quad \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} \prod_{(j,l) \in I \times I} \mathcal{F}(U_j \times_U U_l)$$

is an equalizer diagram for every covering $\{U_i \rightarrow U\}_{i \in I}$ in \mathcal{S} , where the parallel arrows are defined as follows:

$$\text{pr}_k: \prod_i \mathcal{F}(U_i) \rightarrow \prod_{j,l} \mathcal{F}(U_j \times_U U_l), \quad k \in \{1, 2\}$$

sends $(a_i)_i$ to the element with component at index (j, l) equal to $\text{pr}_1^* a_j$ if $k = 1$ and $\text{pr}_2^* a_l$ if $k = 2$.

Given a presheaf \mathcal{F} and a morphism $U \rightarrow V$, we call the induced map $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ a *restriction map*.

REMARK 3.2.2. To say that (3.2.0.2) is an equalizer diagram, or that $\mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_i)$ is an equalizer of the diagram $\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j)$ is to say that for each arrow $A \rightarrow \prod \mathcal{F}(U_i)$ such that the composites $h: A \rightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j)$ coincide, there is a unique arrow $h': A \rightarrow \mathcal{F}(U)$ such that h is the composition of h' with the arrow $\mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_i)$.

To say that there always exists such an arrow h' is to say that $\mathcal{F}(U)$ maps surjectively onto the subset of $\prod \mathcal{F}(U_i)$ consisting of all elements whose images under the two maps to $\prod \mathcal{F}(U_i \times_U U_j)$ coincide.

To say that such a map h' (whenever it exists) is unique, is to say that the map $\mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_i)$ is injective, i.e., that \mathcal{F} is separated.

REMARK 3.2.3. Note that if we have morphisms

$$\begin{array}{ccc} & Y & \\ & \alpha \downarrow & \\ Z & \xrightarrow{\beta} & X \end{array}$$

in some category \mathcal{C} with fiber products, then

$$\text{h}_{Z \times_X Y} = \text{h}_Z \times_{\text{h}_X} \text{h}_Y$$

in the category $\text{PreSh}(\mathcal{C})$. Indeed, let \mathcal{F} be a presheaf and suppose that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \text{h}_Y \\ \downarrow & & \downarrow \\ \text{h}_Z & \longrightarrow & \text{h}_X \end{array}$$

Let W be an object in \mathcal{C} . An element in $\mathcal{F}(W)$ will be mapped to some $\varphi: W \rightarrow Y$ and some $\psi: W \rightarrow Z$ such that $\alpha \circ \varphi = \beta \circ \psi$. But this gives a unique map $\gamma: W \rightarrow Z \times_X Y$ such that $\varphi = \text{pr}_Y \circ \gamma$ and $\psi = \text{pr}_Z \circ \gamma$. Hence we conclude that every $\mathcal{F}(W) \rightarrow \text{h}_Y(W)$ and $\mathcal{F}(W) \rightarrow \text{h}_Z(W)$ factors uniquely through $\text{h}_{Z \times_X Y}(W)$. That is, $\text{h}_{Z \times_X Y} = \text{h}_Z \times_{\text{h}_X} \text{h}_Y$.

REMARK 3.2.4. To define a presheaf we do not need a topology on the category, and hence we may not only speak of presheaves on sites, but also *presheaves on categories*.

A morphism of sheaves is just a natural transformation of presheaves. Given a site \mathcal{S} , we get a category of sheaves on \mathcal{S} , i.e., a category where the objects are sheaves on \mathcal{S} and the morphisms are morphisms of sheaves on \mathcal{S} .

DEFINITION 3.2.5. Given a site \mathcal{S} , let $\text{PreSh}(\mathcal{S})$ denote the category of presheaves on \mathcal{S} and let $\text{Sh}(\mathcal{S})$ denote the category of sheaves on \mathcal{S} .

EXAMPLE 3.2.6 (Sheaf on $(\mathcal{C}/S)_{\text{E}}$ of a set M). Let M be a set and S a scheme. For every S -scheme X , define

$$M(X) = M^{\pi_0(X)} = \text{Hom}_{(\text{Set})}(\pi_0(X), M),$$

where $\pi_0(X)$ denotes the set of connected components of X . A morphism $f: Y \rightarrow X$ of S -schemes maps a connected component of Y into a connected component of X , and hence f defines a map $\sigma: \pi_0(Y) \rightarrow \pi_0(X)$. Hence we get a map

$$\text{Hom}_{(\text{Set})}(\pi_0(X), M) \rightarrow \text{Hom}_{(\text{Set})}(\pi_0(Y), M).$$

This defines a presheaf and it is not hard to see that this presheaf is also a sheaf. This sheaf will be denoted by M_S or just M .

EXAMPLE 3.2.7. A representable functor $(\text{Top})^{\text{op}} \rightarrow (\text{Set})$ is a sheaf in the big classical topology (defined in Example 3.1.4). Indeed, consider the functor $\text{Hom}_{(\text{Top})}(-, X)$ where X is a topological space. Let $\bigcup U_i$ be an open covering of U and suppose that we have continuous maps $f_i: U_i \rightarrow X$ for each $i \in I$, such that f_i and f_j agree on $U_i \cap U_j$ for each $i, j \in I$. Then there is a unique continuous map $f: U \rightarrow X$ such that $f|_{U_i} = f_i$.

EXAMPLE 3.2.8. Any representable presheaf \mathcal{F} on S_{Zar} is a sheaf. Indeed, let X be an S -scheme and let $\{U_i \rightarrow U\}$ be a covering in S_{Zar} . The fiber product $U_i \times_U U_j$ may be identified with the intersection $U_i \cap U_j$ in U . It is a well known fact that if $U = \bigcup U_i$ and we have S -morphisms $f_i: U_i \rightarrow X$ that agree on each intersection $U_i \cap U_j$, then there is a unique S -morphism $f: U \rightarrow X$ such that $f|_{U_i} = f_i$.

LEMMA 3.2.9 ([FGI⁺05, 2.60]). A presheaf \mathcal{F} on S_{Fpqc} is a sheaf if and only if it satisfies the following two conditions:

- (1) \mathcal{F} satisfies the sheaf condition for Zariski open coverings;
- (2) for any cover $\{V \rightarrow U\}$ in S_{Fpqc} with U and V affine, we have that

$$\mathcal{F}(U) \rightarrow \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is an equalizer diagram.

PROOF. It is clear that the two conditions are necessary for \mathcal{F} to be a sheaf on S_{Fpqc} . For the converse, suppose that $\{U_i \rightarrow U\}_{i \in I}$ is a covering in S_{Fpqc} , with U and each U_i affine. If \mathcal{F} satisfies (1) then $\mathcal{F}(\coprod U_i) = \prod \mathcal{F}(U_i)$ since $\{U_j \rightarrow \coprod U_i\}_j$ is a Zariski open covering. We have that

$$\left(\coprod U_i\right) \times_U \left(\coprod U_i\right) = \coprod (U_i \times_U U_j),$$

and hence if the index set I is finite then $\coprod U_i$ is affine, and hence the upper row in the diagram

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \prod \mathcal{F}(U_i) & \rightrightarrows & \prod \mathcal{F}(U_i \times_U U_j) \\ & & \parallel & & \parallel \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(\coprod U_i) & \longrightarrow & \mathcal{F}((\coprod U_i) \times_U (\coprod U_i)) \end{array}$$

is an equalizer diagram by (2) (this is the diagram arising from the cover $\{\coprod U_i \rightarrow U\}$).

Now let $\{g: V' \rightarrow V\}$ be any covering in S_{Fpqc} . By Proposition 3.1.13 there is a covering $V' = \bigcup V'_i$ of open quasi-compact subschemes such that the image $V_i := g(V'_i)$ is open and affine for each i . Hence we may write each V'_i as a finite

union of open subschemes $V'_{ik} \subseteq V'_i$. The V'_i 's form an open affine covering of V' . Now consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{F}(V) & \xrightarrow{\quad \gamma \quad} & \mathcal{F}(V') & \xrightarrow{\quad \cong \quad} & \mathcal{F}(V' \times_V V') \\
\downarrow \alpha & & \downarrow \delta & & \downarrow \\
\prod_i \mathcal{F}(V_i) & \xrightarrow{\quad \beta \quad} & \prod_{i,k} \mathcal{F}(V'_{ik}) & \xrightarrow{\quad \cong \quad} & \prod_{i,k,l} \mathcal{F}(V'_{ik} \times_V V'_{il}) \\
\Downarrow & & \Downarrow & & \\
\prod_{i,j} \mathcal{F}(V_i \cap V_j) & \longrightarrow & \prod_{i,j,k,l} \mathcal{F}(V'_{ik} \cap V'_{jl}) & &
\end{array}$$

The first two columns are equalizer diagrams by (1) and the second row is an equalizer diagram by (2). The maps α and β are injective and hence so is γ . Thus \mathcal{F} is a separated presheaf on S_{Fpqc} and hence the bottom row is injective. Now take an element $s \in \mathcal{F}(V')$ and suppose that s maps to the same element via the two maps to $\mathcal{F}(V' \times_V V')$. Then $\delta(s)$ maps to the same element via the two maps to $\prod_i \prod_{k,l} \mathcal{F}(V'_{ik} \times_V V'_{il})$, which implies that $\delta(s) \in \text{im}(\beta)$. Let $t \in \prod_i \mathcal{F}(V_i)$ be the element such that $\beta(t) = \delta(s)$. We have that $\delta(s)$ maps to the same element in $\prod_{i,j} \prod_{k,l} \mathcal{F}(V'_{ik} \cap V'_{jl})$ and since the bottom row is injective t must map to the same element via the two maps to $\prod_{i,j} \mathcal{F}(V_i \cap V_j)$. Thus $t \in \text{im}(\alpha)$ and since δ is injective, we see that $s \in \text{im}(\gamma)$. Thus the top row is an equalizer diagram and hence \mathcal{F} is a sheaf on S_{Fpqc} . \square

3.3. Sieves and elementary topoi

Let X be a set and A a subset of X . Then A is completely determined by a characteristic map $\chi_A: X \rightarrow \{0, 1\}$, where $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = 1$ if $x \notin A$. This gives a pullback diagram

$$\begin{array}{ccc}
A & \longrightarrow & \{0\} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \{0, 1\}
\end{array}$$

One says that the inclusion (monomorphism) $\{0\} \rightarrow \{0, 1\}$ is a "subobject classifier". In general, if \mathcal{C} is a category with terminal object 1 , then a *subobject classifier* for \mathcal{C} is a monomorphism $1 \rightarrow \Omega$ such that for every monomorphism $A \rightarrow X$ in \mathcal{C} , there is a unique pullback square

$$\begin{array}{ccc}
A & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Omega
\end{array}$$

(see [ML98, p. 105]).

A category \mathcal{C} is called an *elementary topos*, if it satisfies the following three properties:

- (1) \mathcal{C} has all finite limits;
- (2) \mathcal{C} has a subobject classifier;
- (3) \mathcal{C} is cartesian closed (see [ML98, p. 97]).

The category $\text{Sh}(\mathcal{S})$ of sheaves on a site \mathcal{S} is an elementary topos [MLM94, III.7.4]. The subobject classifier is defined in terms of *sieves*.

DEFINITION 3.3.1. Let U be an object in a category \mathcal{C} . Then a subfunctor of

$$h_U: \mathcal{C}^{\text{op}} \rightarrow (\text{Set})$$

is called a *sieve*.

The subobject classifier Ω in $\text{Sh}(\mathcal{S})$ is defined by taking $\Omega(U)$ to be the set of all so called "closed sieves" on U for any object U in \mathcal{S} . Let \mathcal{F} be a sheaf on \mathcal{S} . Every subsheaf \mathcal{G} of \mathcal{F} is determined by its *characteristic morphism* $\chi_{\mathcal{G}}: \mathcal{F} \rightarrow \Omega$. See [MLM94, Section III.7] for details. We will only use the fact that distinct subsheaves gives distinct characteristic morphisms.

3.4. Epimorphisms

Recall that an arrow $X \rightarrow Y$ between objects in a category \mathcal{C} is called an *epimorphism* if whenever we have two arrows $f, g: Y \rightrightarrows Z$ such that the compositions $X \rightarrow Y \rightrightarrows Z$ agree, we have $f = g$. Equivalently we can say that the map $\text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$ is injective for every object Z . Recall also that a diagram

$$W \begin{array}{c} \xrightarrow{\beta} \\ \rightrightarrows X \rightarrow Y \\ \xleftarrow{\alpha} \end{array}$$

is a *coequalizer diagram* ($X \rightarrow Y$ is a coequalizer of $W \rightrightarrows X$) if every morphism $X \rightarrow V$ such that the two maps $W \rightrightarrows X \rightarrow V$ coincide factors uniquely through $X \rightarrow Y$. Now, if we have a pair of morphisms $\alpha, \beta: W \rightarrow X$, then every morphism $X \rightarrow V$ such that the maps

$$W \begin{array}{c} \xrightarrow{\beta} \\ \rightrightarrows X \rightarrow V \\ \xleftarrow{\alpha} \end{array}$$

coincide factors through $X \rightarrow Y$ if and only if $\text{Hom}_{\mathcal{C}}(Y, V)$ maps surjectively onto the subset $\{f \in \text{Hom}(X, V) : f \circ \alpha = f \circ \beta\}$. Hence we conclude that

$$W \begin{array}{c} \xrightarrow{\beta} \\ \rightrightarrows X \rightarrow Y \\ \xleftarrow{\alpha} \end{array}$$

is a coequalizer diagram in a small category \mathcal{C} if and only if the diagram

$$\text{Hom}_{\mathcal{C}}(Y, V) \rightarrow \text{Hom}_{\mathcal{C}}(X, V) \rightrightarrows \text{Hom}_{\mathcal{C}}(W, V)$$

is an equalizer diagram in (Set) for every object V in \mathcal{C} . This means in particular that $\text{Hom}_{\mathcal{C}}(Y, V) \rightarrow \text{Hom}_{\mathcal{C}}(X, V)$ is injective for every $Y \rightarrow V$ and thus that $X \rightarrow Y$ is an epimorphism.

DEFINITION 3.4.1. A morphism $X \rightarrow Y$ in a category \mathcal{C} with fiber products is called an *effective epimorphism* if the following diagram is a coequalizer diagram:

$$X \times_Y X \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \rightrightarrows X \rightarrow Y \\ \xleftarrow{\text{pr}_2} \end{array}$$

We say that $X \rightarrow Y$ is a *universally effective epimorphism* if for every morphism $Y' \rightarrow Y$, the morphism $X \times_Y Y' \rightarrow Y'$ is an effective epimorphism.

REMARK 3.4.2. Thus, to say that any presheaf h_X on S_{fpqc} (given by an S -scheme X) is a sheaf, (it is a sheaf in the Zariski topology) is to say that any fpqc morphism $V \rightarrow U$, with U and V affine, is an effective epimorphism.

DEFINITION 3.4.3. For an object X in a category \mathcal{C} , let h^X be the covariant functor

$$\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow (\text{Set}).$$

There is also a covariant version of Yoneda's lemma which states that for any covariant functor $F: \mathcal{C} \rightarrow (\mathbf{Set})$ there is a bijection

$$\mathrm{Hom}(\mathbf{h}^X, F) \simeq F(X).$$

In particular, we have

$$\mathrm{Hom}(\mathbf{h}^X, \mathbf{h}^Y) \simeq \mathrm{Hom}(Y, X).$$

Note that the functor $X \mapsto \mathbf{h}^X$ is contravariant, since a morphism $Y \rightarrow X$ gives a morphism $\mathbf{h}^X \rightarrow \mathbf{h}^Y$ by precomposing with $Y \rightarrow X$.

LEMMA 3.4.4. *Let X, Y , and W be objects in a category \mathcal{C} . Then the diagram*

$$W \rightrightarrows X \rightarrow Y$$

is a coequalizer diagram in \mathcal{C} if and only if the induced diagram

$$\mathbf{h}^Y \rightarrow \mathbf{h}^X \rightrightarrows \mathbf{h}^W$$

is an equalizer diagram in the category $\mathrm{Func}(\mathcal{C}, (\mathbf{Set}))$ of functors $\mathcal{C} \rightarrow (\mathbf{Set})$.

PROOF. The "if part" is an easy consequence of the covariant Yoneda lemma. To prove the converse, suppose that we have a natural transformation of functors $F \rightarrow \mathbf{h}^X$ such that the compositions $F \rightarrow \mathbf{h}^X \rightrightarrows \mathbf{h}^W$ coincide. By the discussion above, we know that for every object Z in \mathcal{C} , $F(Z) \rightarrow \mathbf{h}^X(Z)$ factors through $\mathbf{h}^Y(Z) \rightarrow \mathbf{h}^X(Z)$. If $Z \rightarrow Z'$ is a morphism then we get a diagram

$$\begin{array}{ccccc} F(Z) & \longrightarrow & \mathbf{h}^Y(Z) & \hookrightarrow & \mathbf{h}^X(Z) \\ \downarrow & & \downarrow & & \downarrow \\ F(Z') & \longrightarrow & \mathbf{h}^Y(Z') & \hookrightarrow & \mathbf{h}^X(Z') \end{array}$$

where the right square commutes and the big square commutes. Since the map $\mathbf{h}^Y(Z') \rightarrow \mathbf{h}^X(Z')$ is injective by the discussion above, we get that the left square is commutative and hence $F \rightarrow \mathbf{h}^X$ factors through $\mathbf{h}^Y \rightarrow \mathbf{h}^X$. \square

LEMMA 3.4.5. *Let \mathcal{S} be a site on which representable presheaves are sheaves. If $\{V \rightarrow U\}$ is a covering in \mathcal{S} then the induced morphism $\mathbf{h}_V \rightarrow \mathbf{h}_U$ is an effective epimorphism in the category $\mathrm{Sh}(\mathcal{S})$.*

PROOF. Let \mathcal{F} be a sheaf on \mathcal{S} . Then the diagram

$$\mathcal{F}(U) \rightarrow \mathcal{F}(V) \begin{array}{c} \xrightarrow{\mathrm{pr}_1^*} \\ \rightrightarrows \\ \xrightarrow{\mathrm{pr}_2^*} \end{array} \mathcal{F}(V \times_U V)$$

is an equalizer diagram in (\mathbf{Set}) and by Yoneda's lemma and Remark 3.2.3, this is isomorphic to

$$\mathrm{Hom}(\mathbf{h}_U, \mathcal{F}) \rightarrow \mathrm{Hom}(\mathbf{h}_V, \mathcal{F}) \begin{array}{c} \xrightarrow{\mathrm{pr}_1^*} \\ \rightrightarrows \\ \xrightarrow{\mathrm{pr}_2^*} \end{array} \mathrm{Hom}(\mathbf{h}_V \times_{\mathbf{h}_U} \mathbf{h}_V, \mathcal{F}).$$

If we have a morphism $\tau: \mathbf{h}_V \rightarrow \mathcal{F}$ such that the composites $\mathbf{h}_V \times_{\mathbf{h}_U} \mathbf{h}_V \rightrightarrows \mathbf{h}_V \rightarrow \mathcal{F}$ coincides, then $\mathrm{pr}_1^*(\tau) = \mathrm{pr}_2^*(\tau)$ and hence τ is in the image of

$$\mathrm{Hom}(\mathbf{h}_U, \mathcal{F}) \hookrightarrow \mathrm{Hom}(\mathbf{h}_V, \mathcal{F}).$$

That is, there is a unique morphism $\mathbf{h}_U \rightarrow \mathcal{F}$ such that $\mathbf{h}_V \rightarrow \mathcal{F}$ factors through $\mathbf{h}_U \rightarrow \mathbf{h}_V$. \square

DEFINITION 3.4.6. Let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on a site \mathcal{S} . Then we say that α is *locally surjective* if for every object U in \mathcal{S} , the following is true: for every $y \in \mathcal{G}(U)$, there is a covering $\{\varphi_i: U_i \rightarrow U\}$ such that for every index i we have that $\varphi_i^*(y) \in \mathcal{G}(U_i)$ is in the image of $\alpha_{U_i}: \mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i)$.

A *subsheaf* of a sheaf \mathcal{F} is a subfunctor/subpresheaf of \mathcal{F} which is a sheaf.

LEMMA 3.4.7. *Let \mathcal{F} be a sheaf on a site \mathcal{S} and let $\mathcal{G} \subseteq \mathcal{F}$ be a subpresheaf. Then the following are equivalent:*

- (1) \mathcal{G} is a sheaf;
- (2) for every object U in \mathcal{S} and every element $x \in \mathcal{F}(U)$, the following holds: for every covering $\{\varphi_i: U_i \rightarrow U\}$, if $\varphi_i^*(x) \in \mathcal{G}(U_i)$ for every i , then $x \in \mathcal{G}(U)$.

PROOF. (2) \Rightarrow (1) : $\mathcal{G}(U) \rightarrow \prod \mathcal{G}(U_i)$ is injective since $\mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_i)$ is injective. But (2) says exactly that $\mathcal{G}(U) \rightarrow \prod \mathcal{G}(U_i)$ is surjective onto the set of points $y \in \prod \mathcal{G}(U_i)$ such that $p_1(y) = p_2(y)$ where p_1, p_2 are the morphisms $\prod \mathcal{G}(U_i) \rightrightarrows \prod \mathcal{G}(U_i \times_U U_j)$. Hence \mathcal{G} is a sheaf.

(1) \Rightarrow (2): This is clear from the definition of a sheaf. \square

LEMMA 3.4.8. *Let \mathcal{S} be a site. For a morphism $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ in $\text{Sh}(\mathcal{S})$, the following are equivalent:*

- (1) α is an epimorphism;
- (2) α is locally surjective.

PROOF. Define a presheaf \mathcal{G}' by defining $\mathcal{G}'(U)$ to be the set of $x \in \mathcal{G}(U)$ such that there exists a covering $\{\varphi_i: U_i \rightarrow U\}$ such that $\varphi_i^*(x) \in \text{im}(\alpha_{U_i})$ for all i . Then \mathcal{G}' is a sheaf by Lemma 3.4.7. Consider the characteristic morphisms

$$\chi_{\mathcal{G}}, \chi_{\mathcal{G}'}: \mathcal{G} \rightrightarrows \Omega.$$

Then since $\alpha_U(\mathcal{F}(U)) \subseteq \mathcal{G}'(U)$, we have $\chi_{\mathcal{G}} \circ \alpha = \chi_{\mathcal{G}'} \circ \alpha$ and if α is an epimorphism, then we have $\chi_{\mathcal{G}'} = \chi_{\mathcal{G}}$, i.e., $\mathcal{G} = \mathcal{G}'$ and hence α is locally surjective. Another way to say this is that since α factors through the monomorphism $i: \mathcal{G}' \rightarrow \mathcal{G}$ and α is an epimorphism, we have that i is also an epimorphism. But in a topos, every monomorphism which is also an epimorphism is an isomorphism [MLM94, IV.2.2].

Conversely, suppose that α is locally surjective and let $f, g: \mathcal{G} \rightrightarrows \mathcal{H}$ be two morphisms of sheaves such that the compositions $\mathcal{F} \rightarrow \mathcal{G} \rightrightarrows \mathcal{H}$ agree. Then for every object $U \in \mathcal{S}$ the maps $\mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightrightarrows \mathcal{H}(U)$ agree. For every $y \in \mathcal{G}(U)$ there is a covering $\{\varphi_i: U_i \rightarrow U\}$ such that $\varphi_i^*(y) \in \text{im}(\alpha_{U_i})$ for every index i . Thus we have a diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \begin{array}{c} \xrightarrow{f_U} \\ \xrightarrow{g_U} \end{array} & \mathcal{H}(U) \\ \Pi \varphi_i^* \downarrow & & \downarrow \Pi \varphi_i^* \\ \prod \mathcal{G}(U_i) & \begin{array}{c} \xrightarrow{\prod f_{U_i}} \\ \xrightarrow{\prod g_{U_i}} \end{array} & \prod \mathcal{H}(U_i) \end{array}$$

where $(\prod \varphi_i^*)(f_U(y)) = (\prod f_{U_i})(\prod \varphi_i^*(y)) = (\prod g_{U_i})(\prod \varphi_i^*(y)) = (\prod \varphi_i^*)(g_U(y))$. But $(\prod \varphi_i^*): \mathcal{H}(U) \rightarrow \prod \mathcal{H}(U_i)$ is injective since \mathcal{H} is a sheaf, and hence we have that $f_U(y) = g_U(y)$. Thus $f = g$ and α is an epimorphism. \square

LEMMA 3.4.9. *Let \mathcal{S} be a site. Any epimorphism $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ in $\text{Sh}(\mathcal{S})$ is effective.*

PROOF. If α is an epimorphism, then it is locally surjective. That is, for every object U in site and every element $x \in \mathcal{G}(U)$, there is a covering $\{\varphi_i: U_i \rightarrow U\}$ such that $\varphi_i^*(x) \in \text{im}(\alpha_{U_i})$ for all i . To show that α is effective, let \mathcal{H} be a sheaf and suppose that we have a morphism $\mathcal{F} \rightarrow \mathcal{H}$. To say that the compositions

$$\prod \mathcal{F}(U_i) \times_{\prod \mathcal{G}(U_i)} \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i) \rightarrow \prod \mathcal{H}(U_i)$$

coincide is to say that whenever two elements $a, b \in \prod \mathcal{F}(U_i)$ map to the same element in $\prod \mathcal{G}(U_i)$, they also map to the same element in $\prod \mathcal{H}(U_i)$. That is,

every preimage of $(\varphi_i^*(x))_i \in \prod \mathcal{G}(U_i)$ in $\prod \mathcal{F}(U_i)$ will map to the same element $z \in \prod \mathcal{H}(U_i)$. But this argument also holds if we replace $\prod U_i$ by $\prod U_i \times_U U_j$ and since the two images of $(\varphi_i^*(x))_i$ in $\prod \mathcal{G}(U_i \times_U U_j)$ coincide, we get that the two images of z in $\mathcal{H}(U_i \times_U U_j)$ coincide. Since \mathcal{H} is a sheaf, this means that z has a (unique) preimage $w \in \mathcal{H}(U)$. Define a map $\gamma_U: \mathcal{G}(U) \rightarrow \mathcal{H}(U)$ by $\gamma_U(x) = w$. Since α is a natural transformation, these γ_U will patch together to a natural transformation $\gamma: \mathcal{G} \rightarrow \mathcal{H}$ such that $\mathcal{F} \rightarrow \mathcal{H}$ factors through γ . Since $\mathcal{F} \rightarrow \mathcal{G}$ is an epimorphism, γ is the only morphism with this property. Hence $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \rightarrow \mathcal{G}$ is a coequalizer diagram. \square

COROLLARY 3.4.10. *Any morphism $\tau: \mathcal{F} \rightarrow \mathcal{G}$ of sheaves which is an epimorphism and a monomorphism is an isomorphism.*

PROOF. Lemma 3.4.9 implies that

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \rightarrow \mathcal{G}$$

is a coequalizer diagram. But τ is a monomorphism, and hence the two projections $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F}$ coincide. This implies that the identity $\text{id}_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}$ factors through $\tau: \mathcal{F} \rightarrow \mathcal{G}$. Hence we have a morphism $\eta: \mathcal{G} \rightarrow \mathcal{F}$ such that $\eta \circ \tau = \text{id}_{\mathcal{F}}$. Thus $\tau = \tau \circ \eta \circ \tau$ and since τ is an epimorphism, we get that $\tau \circ \eta = \text{id}_{\mathcal{G}}$. \square

3.5. Examples of sheaves

To show that a representable presheaf on S_{Fqc} (or $S_{\text{Fl}}, S_{\text{ét}}, X_{\text{ét}}$) is a sheaf, we will use the following lemma.

LEMMA 3.5.1. *Let $\varphi: A \rightarrow B$ be a faithfully flat ring homomorphism. Then the following diagram is exact:*

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & B \otimes_A B \\ & & & & & & b \mapsto b \otimes 1 - 1 \otimes b. \end{array}$$

PROOF. We have that φ is injective by Proposition 1.1.8. It is clear that $\varphi(A)$ is contained in the kernel of ψ and hence we need only prove the reverse inclusion.

Again, by Proposition 1.1.8, it is enough to check that the sequence

$$B \xrightarrow{\varphi \otimes \text{id}_B} B \otimes_A B \xrightarrow{\psi \otimes \text{id}_B} B \otimes_A B \otimes_A B$$

is exact. Define a map $r: B \otimes_A B \rightarrow B$; $r(b \otimes b') = bb'$. Now take $b \otimes b' \in \ker(\psi \otimes \text{id}_B)$. Then $b \otimes 1 \otimes b' = 1 \otimes b \otimes b'$ since $(b \otimes 1 - 1 \otimes b) \otimes b' = b \otimes 1 \otimes b' - 1 \otimes b \otimes b'$. Hence we have that

$$b \otimes b' = (\text{id}_B \otimes r)(b \otimes 1 \otimes b') = (\text{id}_B \otimes r)(1 \otimes b \otimes b') = 1 \otimes bb',$$

which is clearly in the image of $\varphi \otimes bb'$. This finishes the proof. \square

If \mathcal{S} is a site and X is an object in \mathcal{S} , then we may define a Grothendieck topology on $(\text{Cat}(\mathcal{S})/X)$ by taking the covers to be collections of commutative diagrams (*morphisms over X*)

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ & \searrow & \swarrow \\ & & X \end{array}$$

such that $\{U_i \rightarrow U\}$ is a cover in \mathcal{S} . It is not hard to see that this defines a Grothendieck topology on $(\text{Cat}(\mathcal{S})/X)$ and we denote the resulting site by (\mathcal{S}/X) .

EXAMPLE 3.5.2. The site $X_{\text{ét}}$ is the same as the site $((\text{Spec } \mathbb{Z})_{\text{ét}}/X)$ since $\text{Spec } \mathbb{Z}$ is the final object in (Sch) .

PROPOSITION 3.5.3. *Let \mathcal{S} be a site on which every representable presheaf is a sheaf and let X be an object in \mathcal{S} . Then every representable presheaf on (\mathcal{S}/X) is a sheaf on (\mathcal{S}/X) .*

PROOF. Let $\{\varphi_i: U_i \rightarrow U\}$ be a covering in \mathcal{S}/X . That is, U and each U_i comes with a morphism p_U and p_{U_i} (respectively) to X such that for each i , $U_i \rightarrow U$ is a morphism over X . Let Y be an object in (\mathcal{S}/X) . For objects V, W in (\mathcal{S}/X) , we write $\text{Hom}_X(V, W)$ for the set of morphisms $V \rightarrow W$ in $\text{Cat}(\mathcal{S}/X)$ (i.e., morphisms over X) and we write $\text{h}_W(V)$ for the set of morphisms $V \rightarrow W$ in $\text{Cat}(\mathcal{S})$. We need to show that for any object $p_Z: Z \rightarrow X$ in (\mathcal{S}/X)

$$\text{Hom}_X(U, Z) \longrightarrow \prod_i \text{Hom}_X(U_i, Z) \rightrightarrows \prod_{i,j} \text{Hom}_X(U_i \times_U U_j, Z)$$

is an equalizer diagram. We have a commutative diagram

$$\begin{array}{ccccc} \text{Hom}_X(U, Z) & \longrightarrow & \prod_i \text{Hom}_X(U_i, Z) & \rightrightarrows & \prod_{i,j} \text{Hom}_X(U_i \times_U U_j, Z) \\ \downarrow & & \downarrow & & \downarrow \\ \text{h}_Z(U) & \hookrightarrow & \prod_i \text{h}_Z(U_i) & \rightrightarrows & \prod_{i,j} \text{h}_X(U_i \times_U U_j) \end{array}$$

from which it is clear that $\text{Hom}_X(U, Z) \rightarrow \prod \text{Hom}_X(U_i, Z)$ is injective. If an element $\alpha \in \prod \text{Hom}_X(U_i, Z)$ is mapped to the same element via the two maps

$$\prod \text{Hom}_X(U_i, Z) \rightrightarrows \prod \text{Hom}_X(U_i \times_U U_j, Z)$$

then the image of α in $\prod \text{h}_Z(U_i)$ is mapped to the same element via the two maps $\prod \text{h}_Z(U_i \times_U U_j)$ and hence has a preimage $\gamma \in \text{h}_Z(U)$. For each i we have that $p_Z \circ \gamma \circ \varphi_i = p_{U_i} = p_U \circ \varphi_i$. But h_X is a sheaf and hence the map $\text{h}_X(U) \rightarrow \prod \text{h}_X(U_i)$ is injective. That is, $p_Z \circ \gamma = p_U$ and hence γ is a morphism over X . \square

EXAMPLE 3.5.4 (Sheaf defined by an S -scheme X). Consider the category (Sch/S) and the presheaf h_X given by an S -scheme X . This is a sheaf on S_{fpqc} [FGI⁺05, Theorem 2.55], i.e., every representable presheaf on S_{fpqc} is a sheaf. Indeed, h_X satisfies the sheaf condition for Zariski open coverings by Example 3.2.8.

By Proposition 3.5.3, we may assume that h_X is just a presheaf $(\text{Sch})^{\text{op}} \rightarrow (\text{Set})$, i.e., we don't need to worry about any base scheme. Let $\{V \rightarrow U\}$ be a fpqc-covering with $V = \text{Spec } B$ and $U = \text{Spec } A$.

We first show that h_X satisfies the property (2) of Lemma 3.2.9 in case X is affine. Suppose that $X = \text{Spec } R$. From the exact sequence of Lemma 3.5.1 and left exactness of the functor $\text{Hom}(R, -)$, we have an exact sequence

$$0 \rightarrow \text{Hom}(R, A) \rightarrow \text{Hom}(R, B) \rightarrow \text{Hom}(R, B \otimes_A B).$$

But this sequence is isomorphic to a sequence

$$0 \rightarrow \text{h}_X(U) \rightarrow \text{h}_X(V) \rightarrow \text{h}_X(V \times_U V),$$

which is exactly the sheaf condition for h_X .

If X is any scheme, write $X = \bigcup X_i$ as a union of affine open subschemes. Suppose that $f, g: U \rightarrow X$ are maps such that the compositions $V \rightarrow U \rightrightarrows X$ are equal. Since $V \rightarrow U$ is surjective, this implies that f and g are equal as maps of topological spaces. Let U_i be the preimages of the X_i under any of these maps and let V_i be the preimages of the U_i in V . Then for each i the maps $V_i \rightarrow U_i \rightrightarrows X_i$ are equal and since V_i, U_i , and X_i are affine, the previous part implies that the maps $U_i \rightrightarrows X_i$ are equal as morphisms of schemes. Hence h_X is separated.

Suppose that $f \in \text{h}_X(V)$ is a morphism such that the compositions

$$V \times_U V \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \rightrightarrows \\ \xrightarrow{\text{pr}_2} \end{array} V \rightarrow X$$

coincide. This implies that if $a_1, a_2 \in V$ maps to the same element in U , then they map to the same element in X . Since $V \rightarrow U$ is surjective, this implies that g factors through U as a function of sets. Proposition 1.1.16 says that U has the quotient topology induced by the map $V \rightarrow U$ and hence g factors through U as a continuous map. Denote the resulting map by $g: U \rightarrow X$.

Now define sets $U_i = g^{-1}(X_i)$ and $V_i = f^{-1}(X_i)$ for all i . Then the morphisms

$$V_i \times_U V_i \rightrightarrows V_i \xrightarrow{f|_{V_i}} X_i$$

are equal and since X_i is affine, $f|_{V_i}$ factors uniquely through some morphism $g_i: U_i \rightarrow X_i$. Since h_X is separated, we have that g_i and g_j agree on the intersection $U_i \cap U_j$. Hence we may glue the g_i to a map $U \rightarrow X$ through which f factors. This proves the fact that h_X is a sheaf on S_{Fpqc} (and hence also on $S_{\text{Fl}}, S_{\text{Ét}}$, and $S_{\text{ét}}$).

EXAMPLE 3.5.5 (The structure sheaf on S_{Fpqc}). Let $\mathcal{O}_{S_{\text{Fpqc}}}$ be the presheaf on S_{Fpqc} defined by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for any $U \rightarrow S$ in S_{Fpqc} . It is clear that $\mathcal{O}_{S_{\text{Fpqc}}}$ satisfies the sheaf condition for Zariski open coverings and hence it remains to check the second condition of Lemma 3.2.9. If $\{V \rightarrow U\}$ is an fpqc covering with $U = \text{Spec } A$ and $V = \text{Spec } B$, it is in particular flat and surjective, i.e., the corresponding homomorphism $A \rightarrow B$ is faithfully flat. Condition (2) of Lemma 3.2.9 now follows from Lemma 3.5.1 since $V \times_U V = \text{Spec}(B \otimes_A B)$ and taking global sections of U, V , and $V \times_U V$ gives exactly A, B , and $B \otimes_A B$. Hence $\mathcal{O}_{S_{\text{Fpqc}}}$ is a sheaf on S_{Fpqc} . This example is actually a special case of the next example.

EXAMPLE 3.5.6 (Sheaf defined by a quasi-coherent \mathcal{O}_S -module). Let \mathcal{F} be a coherent \mathcal{O}_S -module. Then we have a presheaf $\mathcal{F}_{\text{Fpqc}}$ on S_{Fpqc} defined by $U \mapsto \Gamma(U, \varphi^* \mathcal{F})$ for every morphism $\varphi: U \rightarrow S$ and the obvious restriction maps. We have that $\mathcal{F}_{\text{Fpqc}}$ satisfies the sheaf condition for Zariski coverings since $\varphi^* \mathcal{F}$ is a quasi-coherent \mathcal{O}_U -module [Har77, II.5.8]. To show that $\mathcal{F}_{\text{Fpqc}}$ is a sheaf on S_{Fpqc} , it remains to show that

$$\mathcal{F}_{\text{Fpqc}}(U) \rightarrow \mathcal{F}_{\text{Fpqc}}(V) \rightrightarrows \mathcal{F}_{\text{Fpqc}}(V \times_U V)$$

is an equalizer diagram for each fpqc cover $\{f: V \rightarrow U\}$ with U and V affine.

We have a commutative diagram

$$\begin{array}{ccc} V \times_U V & \xrightarrow{\text{pr}_1} & V \\ \downarrow \text{pr}_2 & & \downarrow f \\ V & \xrightarrow{f} & U \end{array} \quad \begin{array}{c} \nearrow \theta \\ \searrow \varphi \end{array} \quad \begin{array}{c} \\ S \end{array}$$

Since $\theta = \varphi \circ f$, it follows that $\theta^* \mathcal{F} = (\varphi \circ f)^* \mathcal{F} = f^*(\varphi^* \mathcal{F})$. If $U = \text{Spec } A$ and $V = \text{Spec } B$, then $\varphi^* \mathcal{F} = M^\sim$ for some A -module M . Hence

$$\theta^* \mathcal{F} = f^*(M^\sim) \cong (M \otimes_A B)^\sim.$$

Similarly we get that $(\varphi \circ f \circ \text{pr}_i)^* \mathcal{F} \cong (M \otimes_A B \otimes_A B)^\sim$. Let $\alpha: A \rightarrow B$ be the homomorphism corresponding to f . Then we only need to show that the following diagram is exact:

$$0 \rightarrow M \xrightarrow{\text{id}_M \otimes \alpha} M \otimes_A B \xrightarrow{\text{id}_M \otimes \psi} M \otimes_A B \otimes_A B,$$

where ψ is as in Lemma 3.5.1, that is, $\psi = e_1 - e_2$ where e_1 and e_2 are the ring homomorphisms corresponding to the projections $\text{pr}_1, \text{pr}_2: V \times_U V \rightrightarrows V$ respectively.

As in the proof of Lemma 3.5.1, let $r: B \otimes_A B \rightarrow B$ be the map $b \otimes b' \mapsto bb'$. It is clear that the sequence is exact at M and hence, by Proposition 1.1.8 it is enough to check that the sequence is exact after applying the functor $- \otimes_A B$ to it. Take

$m \otimes b \otimes b' \in \ker(\text{id}_M \otimes \psi \otimes \text{id}_B) \subseteq M \otimes_A B \otimes_A B$. Then $m \otimes b \otimes 1 \otimes b' = m \otimes 1 \otimes b \otimes b'$ and applying $\text{id}_M \otimes \text{id}_B \otimes r$ to both sides of the equality we get that $m \otimes b \otimes b' = m \otimes 1 \otimes b b'$ which is in the image of $\text{id}_M \otimes \alpha \otimes \text{id}_B: M \otimes_A B \rightarrow M \otimes_A B \otimes_A B$. This proves that the sequence is exact and hence $\mathcal{F}_{\text{Fppc}}$ is a sheaf on S_{Fppc} .

3.6. Stalks

Similarly as for sheaves of topological spaces we may define the stalk of a sheaf at a point. In this case, a "point" will be a *geometric point* $\text{Spec } \Omega \rightarrow X$ and the colimit will be over *étale neighborhoods*.

DEFINITION 3.6.1. An *étale neighborhood* of a geometric point $\bar{x}: \text{Spec } \Omega \rightarrow X$ is a commutative diagram

$$\begin{array}{ccc} \text{Spec } \Omega & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

where $U \rightarrow X$ is étale.

DEFINITION 3.6.2. Let X be a scheme and \mathcal{F} a presheaf on $X_{\text{ét}}$. The *stalk* of \mathcal{F} at a geometric point $\bar{x}: \text{Spec } \Omega \rightarrow X$ is defined as

$$\mathcal{F}_{\bar{x}} = \varinjlim \mathcal{F}(U),$$

where the colimit is over all étale neighborhoods U of \bar{x} .

PROPOSITION 3.6.3. Let $\tau: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on $X_{\text{ét}}$. The following are equivalent:

- (1) τ is an isomorphism;
- (2) $\tau_{\bar{x}}: \mathcal{F}_{\bar{x}} \rightarrow \mathcal{G}_{\bar{x}}$ is a bijection for each geometric point \bar{x} in X .

PROOF. (1) \Rightarrow (2): We have that τ is an isomorphism if and only if $\tau_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a bijection for each étale $U \rightarrow X$ and hence (2) holds.

(2) \Rightarrow (1): By Corollary 3.4.10, it is enough to show that τ is an epimorphism and a monomorphism. If $\tau_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for each étale $U \rightarrow X$, then τ is a monomorphism. Hence it is enough to show that τ is locally surjective and that $\tau_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for each étale $U \rightarrow X$.

Locally surjective: Take any $s \in \mathcal{G}(U)$. Let \bar{x} be a geometric point in U and let $s_{\bar{x}}$ be the image of s in $\mathcal{G}_{\bar{x}}$. Since $\tau_{\bar{x}}$ is surjective, there is a $t_{\bar{x}} \in \mathcal{F}_{\bar{x}}$ mapping to $s_{\bar{x}}$. Let $\varphi: V \rightarrow U$ be an étale neighborhood of \bar{x} and let $t \in \mathcal{F}(V)$ be an element with image $t_{\bar{x}}$ in $\mathcal{F}_{\bar{x}}$. Then t and $\varphi^*(s)$ both have image $t_{\bar{x}}$ in $\mathcal{F}_{\bar{x}}$. This implies that we may choose V such that $\varphi^*(s) = \tau_V(t)$.

Injectivity: Suppose that $t_1, t_2 \in \mathcal{F}(U)$ satisfies $\tau_U(t_1) = \tau_U(t_2)$. Let \bar{x} be a geometric point in U . Let $s'_{\bar{x}}$ be the image of $\tau_U(t_1) = \tau_U(t_2)$ in $\mathcal{G}_{\bar{x}}$ and let $t'_{\bar{x}}$ be its preimage in $\mathcal{F}_{\bar{x}}$. Then t_1 and t_2 must have image $t'_{\bar{x}}$ in $\mathcal{F}_{\bar{x}}$ and hence there is an étale neighborhood $\psi: W_{\bar{x}} \rightarrow X$ of \bar{x} , such that $\psi^*(t_1) = \psi^*(t_2)$. Now we may choose geometric points \bar{x} such that the $W_{\bar{x}}$'s form an étale covering of U and since \mathcal{F} is a sheaf, we conclude that $t_1 = t_2$. \square

EXAMPLE 3.6.4. Let M be a set and X a scheme, and consider the sheaf M_X on $X_{\text{ét}}$. Let $\bar{x}: \text{Spec } \Omega \rightarrow X$ be a geometric point in X . Then we have

$$(M_X)_{\bar{x}} = M.$$

Indeed, for every étale neighborhood $U \rightarrow X$ of \bar{x} we have a morphism

$$M^{\pi_0(U)} \rightarrow M^{\pi_0(\text{Spec } \Omega)} = M,$$

and hence we have a map $\varphi: (M_X)\bar{x} \rightarrow M$. To see that φ is a bijection, let a be an element in M and let $U \rightarrow X$ be any étale neighborhood of \bar{x} . Let U' be the connected component in U containing \bar{x} . Then $U' \rightarrow U$ is étale and the map $\pi_0(\text{Spec } \Omega) \rightarrow \pi_0(U')$ is clearly a bijection since it is just a point going to a point. Thus

$$M^{\pi_0(U')} \rightarrow M^{\pi_0(\text{Spec } \Omega)} = M$$

is bijective. Thus $\varphi: (M_X)\bar{x} \rightarrow M$ is a bijection.

3.7. Sheafification of a sheaf

DEFINITION 3.7.1. Let \mathcal{F} be a presheaf on a site \mathcal{S} . A *sheafification* of \mathcal{F} is a sheaf \mathcal{F}^a together with a morphism $\mathcal{F} \rightarrow \mathcal{F}^a$, such that any morphism $\mathcal{F} \rightarrow \mathcal{G}$ from \mathcal{F} to a sheaf \mathcal{G} on \mathcal{S} factors uniquely through $\mathcal{F} \rightarrow \mathcal{F}^a$, i.e.,

$$\text{Hom}_{\text{Sh}(\mathcal{S})}(\mathcal{F}^a, \mathcal{G}) \cong \text{Hom}_{\text{PreSh}(\mathcal{S})}(\mathcal{F}, \mathcal{G}).$$

Note that if \mathcal{F} and \mathcal{G} are presheaves and there exists sheafifications of \mathcal{F} and \mathcal{G} , then a morphism $\mathcal{F} \rightarrow \mathcal{G}$ of presheaves gives a morphism $\mathcal{F} \rightarrow \mathcal{G}^a$ which factors uniquely through a morphism $\mathcal{F}^a \rightarrow \mathcal{G}^a$. Hence, if we find a consistent way to assign to each presheaf \mathcal{F} a specific sheafification $\mathcal{F} \rightarrow \mathcal{F}^a$, then this assignment $\mathcal{F} \mapsto \mathcal{F}^a$ becomes a functor, which is a left adjoint of the forgetful functor from sheaves to presheaves.

PROPOSITION 3.7.2. *Given a presheaf \mathcal{F} on a site \mathcal{S} , there exists a sheafification $\mathcal{F} \rightarrow \mathcal{F}^a$, which is unique up to canonical isomorphism.*

PROOF. For each object U in $\text{Cat}(\mathcal{S})$, define a relation \sim on $\mathcal{F}(U)$ by saying that $a \sim b$ if there exists a covering $\{\varphi_i: U_i \rightarrow U\}_{i \in I}$ such that $\varphi_i^*(a) = \varphi_i^*(b)$ for all $i \in I$. It is clear that this is an equivalence relation. Let $f: V \rightarrow U$ be a morphism in $\text{Cat}(\mathcal{S})$. Then $\{V \times_U U_i \rightarrow V\}$ is a covering and we get, for each i , a commutative square

$$\begin{array}{ccc} \mathcal{F}(V \times_U U_i) & \longleftarrow & \mathcal{F}(U_i) \\ \uparrow & & \uparrow \\ \mathcal{F}(V) & \xleftarrow{f^*} & \mathcal{F}(U). \end{array}$$

Hence it is clear that $f^*(a) \sim f^*(b)$ whenever $a \sim b$ in $\mathcal{F}(U)$ and we get a well-defined map $\mathcal{F}(U)/\sim \rightarrow \mathcal{F}(V)/\sim$. This implies that $U \mapsto \mathcal{F}(U)/\sim$ gives a presheaf on \mathcal{S} and it is clear from its definition that this presheaf is separated. Denote this presheaf by \mathcal{F}^+ and the canonical morphism $\mathcal{F} \rightarrow \mathcal{F}^+$ by α . If $\tau: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism to a sheaf \mathcal{G} , we get for each i a commutative square

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\tau_U} & \mathcal{G}(U) \\ \varphi_i^* \downarrow & & \downarrow \varphi_i^* \\ \mathcal{F}(U_i) & \xrightarrow{\tau_{U_i}} & \mathcal{G}(U_i) \end{array}$$

If $\varphi_i^*(a) = \varphi_i^*(b)$ then $\mathcal{G}\varphi_i(\tau_U(a)) = \mathcal{G}\varphi_i(\tau_U(b))$, and it follows from the definition of a sheaf that $\tau_U(a) = \tau_U(b)$. Hence $\mathcal{F} \rightarrow \mathcal{G}$ factors uniquely through $\alpha: \mathcal{F} \rightarrow \mathcal{F}^+$.

Now let \mathcal{U} be the collection of pairs $(\{U_i \rightarrow U\}, \{s_i\})$ where $\{U_i \rightarrow U\}$ is a covering and $s_i \in \mathcal{F}^+(U_i)$ is such that $\text{pr}_1^*(s_i) = \text{pr}_2^*(s_j) \in \mathcal{F}^+(U_i \times_U U_j)$ for all indices i, j . We define a relation on \mathcal{U} by saying that $(\{U_i \rightarrow U\}, \{s_i\}) \sim (\{V_j \rightarrow U\}, \{t_j\})$ if, for all i, j , the elements s_i and t_j have the same image in $\mathcal{F}^+(U_i \times_U V_j)$ under the restriction maps corresponding to the two projections. To see that this is an equivalence relation we only need to check transitivity. This follows from

the fact that \mathcal{F}^+ is separated. Indeed, if $(\{V_j \rightarrow U\}, \{t_j\}) \sim (\{W_k \rightarrow U\}, \{z_k\})$, consider the following diagram

$$\begin{array}{ccccccc}
& & \mathcal{F}^+(U_i) & \longrightarrow & \mathcal{F}^+(U_i \times_U V_j) & & \\
& \nearrow & & & \searrow & & \\
\mathcal{F}^+(U) & \longrightarrow & \mathcal{F}^+(V_j) & & \mathcal{F}^+(U_i \times_U W_k) & \longrightarrow & \mathcal{F}(U_i \times_U V_j \times_U W_k) \\
& \searrow & & & \nearrow & & \\
& & \mathcal{F}^+(W_k) & \longrightarrow & \mathcal{F}^+(V_j \times_U W_k) & &
\end{array}$$

Since \mathcal{F}^+ is separated, it follows that all arrows are injective and since s_i and z_k pulls back to the same element in $\mathcal{F}^+(U_i \times_U V_j \times_U W_k)$, they must agree already in $\mathcal{F}^+(U_i \times_U W_k)$. Hence \sim is an equivalence relation.

Define $\mathcal{F}^a(U) = \mathcal{U} / \sim$. If $V \rightarrow U$ is a morphism, we define a map $\mathcal{U} \rightarrow \mathcal{V}$ by sending a pair $(\{U_i \rightarrow U\}, \{s_i\})$ to the pair $(\{U_i \times_U V \rightarrow V\}, \{\text{pr}_1^*(s_i)\})$, where $\text{pr}_1: U_i \times_U V \rightarrow U_i$ is the projection. We have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}^+(U_i) & \longrightarrow & \mathcal{F}^+(U_i \times_U V_j) & \longleftarrow & \mathcal{F}^+(V_j) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}^+(U_i \times_U V) & \longrightarrow & \mathcal{F}^+(U_i \times_U V_j \times_U V) & \longleftarrow & \mathcal{F}^+(V_j \times_U V)
\end{array}$$

from which it follows that two pairs are equivalent in \mathcal{U} only if their images in \mathcal{V} are equivalent. Hence we have a well-defined map $\mathcal{F}^a(U) \rightarrow \mathcal{F}^a(V)$ and we conclude that \mathcal{F}^a is a presheaf. We have a map $\mathcal{F}^+(U) \rightarrow \mathcal{F}^a(U)$ defined by sending $s \in \mathcal{F}^+(U)$ to the pair $\{U \xrightarrow{\text{id}} U, \{s\}\}$. Hence we have a morphism $\mathcal{F} \rightarrow \mathcal{F}^a$.

To show that \mathcal{F}^a is separated, let $\{\varphi_i: U_i \rightarrow U\}$ be any covering. The map $\mathcal{F}^a(U) \rightarrow \mathcal{F}^a(U_i)$ is defined by sending the class represented by $(\{V_j \rightarrow U\}, \{t_j\})$ to the class represented by

$$(\{V_j \times_U U_i \rightarrow U_i\}, \{\text{pr}_1^*(t_j)\}).$$

Suppose that $(\{V'_k \rightarrow U\}, \{t'_k\})$ is such that

$$(\{V_j \times_U U_i \rightarrow U_i\}, \{\text{pr}_1^*(t_j)\}) \sim (\{V'_k \times_U U_i \rightarrow U_i\}, \{\text{pr}_1^*(t'_k)\})$$

for all indices i . Then $\text{pr}_1^*(t_j)$ and $\text{pr}_1^*(t'_k)$ will pull back to the same element in $t_{jk} \in \mathcal{F}^+(V_j \times_U V'_k \times_U U_i)$ for all indices j, k . But then both $\text{pr}_1^*(t_j) \in \mathcal{F}^+(V_j \times_U V'_k)$ and $\text{pr}_2^*(t'_k) \in \mathcal{F}^+(V_j \times_U V'_k)$ will pull back to t_{jk} in $\mathcal{F}^+(V_j \times_U V'_k \times_U U_i)$ and since \mathcal{F}^+ is separated, it follows that

$$\text{pr}_1^*(t_j) = \text{pr}_2^*(t'_k) \in \mathcal{F}^+(V_j \times_U V'_k).$$

That is, $(\{V_j \rightarrow U\}, \{t_j\}) \sim (\{V'_k \rightarrow U\}, \{t'_k\})$. Thus $\mathcal{F}^a(U) \rightarrow \prod \mathcal{F}^a(U_i)$ is injective, i.e., \mathcal{F}^a is separated.

Now suppose that $s_i \in \mathcal{F}^a(U_i)$ and $s_j \in \mathcal{F}^a(U_j)$ satisfies $\text{pr}_1^*(s_i) = \text{pr}_2^*(s_j) \in \mathcal{F}^a(U_i \times_U U_j)$ for all $i, j \in I$. But s_i and s_j are represented by pairs $(\{U_{ik} \rightarrow U_i\}, \{s_{ik}\})$ and $(\{U_{jl} \rightarrow U_j\}, \{s_{jl}\})$, and thus the pairs $(\{U_{ik} \times_U U_j \rightarrow U_i \times_U U_j\}, \{\text{pr}_1^*(s_{ik})\})$ and $(\{U_i \times_U U_{jl} \rightarrow U_i \times_U U_j\}, \{\text{pr}_2^*(s_{jl})\})$ are equivalent. That is, $\text{pr}_1^*(s_{ik})$ and $\text{pr}_2^*(s_{jl})$ pulls back to the same element in

$$\mathcal{F}^+((U_{ik} \times_U U_j) \times_{U_i \times_U U_j} (U_i \times_U U_{jl})) = \mathcal{F}^+(U_{ik} \times_U U_{jl}).$$

But the pairs $(\{U_{ik} \rightarrow U_i\}, \{s_{ik}\})$ and $(\{U_{jl} \rightarrow U_j\}, \{s_{jl}\})$ gives pairs $(\{U_{ik} \rightarrow U_i \rightarrow U\}, \{s_{ik}\})$ and $(\{U_{jl} \rightarrow U_j \rightarrow U\}, \{s_{jl}\})$ in \mathcal{U} and hence s_{ik} and s_{jl} must pull back to the same element in $\mathcal{F}^+(U_{ik} \times_U U_{jl})$, namely the same element as the one $\text{pr}_1^*(s_{ik})$ and $\mathcal{F}^+ \text{pr}_{jl}(s_{jl})$ pulls back to. The element in $\mathcal{F}^a(U)$ represented by

($\{U_{ik} \rightarrow U_i \rightarrow U\}, \{s_{ik}\}$) and ($\{U_{jl} \rightarrow U_j \rightarrow U\}, \{s_{jl}\}$) will then pull back to s_i in $\mathcal{F}^a(U_i)$ and to s_j in $\mathcal{F}^a(U_j)$. This shows that $\mathcal{F}^a(U) \rightarrow \prod \mathcal{F}^a(U_i)$ is an equalizer of $\prod \mathcal{F}^a(U_i) \rightrightarrows \prod \mathcal{F}^a(U_i \times_U U_j)$ and hence \mathcal{F}^a is a sheaf.

By construction, $\mathcal{F} \rightarrow \mathcal{F}^a$ is a sheafification of \mathcal{F} , and by the universal property of the sheafification \mathcal{F}^a is unique up to isomorphism. Given the morphism $\mathcal{F} \rightarrow \mathcal{F}^a$, this isomorphism is unique, i.e., \mathcal{F}^a is unique up to canonical isomorphism. \square

LEMMA 3.7.3. *Let X be a scheme. Let \mathcal{F} be a presheaf on $X_{\text{ét}}$ and \mathcal{F}^a its sheafification. If $\bar{x}: \text{Spec } \Omega \rightarrow X$ is a geometric point then $\mathcal{F}_{\bar{x}} \simeq (\mathcal{F}^a)_{\bar{x}}$.*

PROOF. See [Mil80, II.2.14.(c)]. \square

COROLLARY 3.7.4. *Let \mathcal{P} be a presheaf on $X_{\text{ét}}$ and \mathcal{F} a sheaf on $X_{\text{ét}}$. Suppose that we have a natural transformation $\tau: \mathcal{P} \rightarrow \mathcal{F}$ such that the induced map $\tau_{\bar{x}}: \mathcal{P}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}}$ is a bijection for every geometric point \bar{x} in X . Then $\tau: \mathcal{P} \rightarrow \mathcal{F}$ is a sheafification of \mathcal{P} .*

3.8. Fiber products and pushouts

If \mathcal{F}, \mathcal{G} , and \mathcal{H} are presheaves on a site \mathcal{S} , then it is not hard to see that the fiber product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ in $\text{PreSh}(\mathcal{S})$ is the presheaf defined by

$$X \mapsto \mathcal{F}(X) \times_{\mathcal{H}(X)} \mathcal{G}(X),$$

where the fiber product is taken in (Set) . Indeed, to give a natural transformation $\tau: \mathcal{M} \rightarrow \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ of presheaves is to give (in a natural way), for every object X in \mathcal{S} , a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(X) & \longrightarrow & \mathcal{G}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \mathcal{H}(X). \end{array}$$

LEMMA 3.8.1. *If \mathcal{F}, \mathcal{G} , and \mathcal{H} are sheaves on a site \mathcal{S} , then the presheaf defined by*

$$X \mapsto \mathcal{F}(X) \times_{\mathcal{H}(X)} \mathcal{G}(X)$$

is a sheaf.

PROOF. Easy. Consider a covering $\{U_i \rightarrow U\}$ and draw a big diagram. \square

COROLLARY 3.8.2. *If $\mathcal{F} \rightarrow \mathcal{H}$ and $\mathcal{G} \rightarrow \mathcal{H}$ are morphisms of sheaves on a site \mathcal{S} , then the presheaf fiber product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is a sheaf.*

REMARK 3.8.3. It is not true in general that the presheaf pushout of sheaves is a sheaf. However, if we take the sheafification of the presheaf pushout, we get a pushout in the category of sheaves.

Operations on sheaves of sets

4.1. Morphisms of sites

Let $f: X \rightarrow Y$ be a morphism of schemes and let $U \rightarrow Y$ be an étale morphism. By Lemma 1.3.5, the projection $U \times_Y X \rightarrow X$ is étale. Furthermore, if V is an étale Y -scheme and $U \rightarrow V$ is a Y -morphism (hence étale), we get a morphism $U \times_Y X \rightarrow V$. Hence the universal property of the fiber product implies that we get an X -morphism $U \times_Y X \rightarrow V \times_Y X$, which again is étale since it is given by base change from $U \rightarrow V$. Hence we have a functor

$$\begin{aligned} f^\times: (\text{ét}/Y) &\rightarrow (\text{ét}/X) \\ U &\mapsto U \times_Y X. \end{aligned}$$

Moreover, if $\{U_i \rightarrow U\}$ is a covering then $\{U_i \times_Y X \rightarrow U \times_Y X\}$ is a covering since $U_i \times_Y X = U_i \times_U U \times_Y X$. That is, the functor f^\times takes coverings to coverings. Furthermore, if $T \rightarrow U$ is a morphism, then

$$\begin{aligned} f^\times(T \times_U U_i) &= T \times_U U_i \times_Y X \\ &= T \times_U (U \times_Y X) \times_{U \times_Y X} (U_i \times_Y X) \\ &= (T \times_Y X) \times_{U \times_Y X} (U_i \times_Y X) \\ &= f^\times(T) \times_{f^\times(U)} f^\times(U_i). \end{aligned}$$

We say that the functor f^\times is *continuous*.

DEFINITION 4.1.1. Let \mathcal{S} and \mathcal{S}' be sites. A functor $F: \text{Cat}(\mathcal{S}) \rightarrow \text{Cat}(\mathcal{S}')$ is called *continuous* if for every covering $\{U_i \rightarrow U\}$ in \mathcal{S} , F satisfies the following properties:

- (1) $\{F(U_i) \rightarrow F(U)\}$ is a covering in \mathcal{S}' ;
- (2) for any morphism $T \rightarrow U$, the morphism $F(T \times_U U_i) \rightarrow F(T) \times_{F(U)} F(U_i)$ is an isomorphism.

DEFINITION 4.1.2. Let \mathcal{S} and \mathcal{S}' be sites. A *morphism of sites* $f: \mathcal{S} \rightarrow \mathcal{S}'$ is a continuous functor $f^{\text{cat}}: \text{Cat}(\mathcal{S}') \rightarrow \text{Cat}(\mathcal{S})$.

Note that a sheaf on a site \mathcal{S} in the diagram in remark 3.1.12 will also be a sheaf on any site to the right of \mathcal{S} in the diagram.

REMARK 4.1.3. Let X be a scheme. If \mathcal{S} and \mathcal{S}' are sites on X such that every object in \mathcal{S}' may also be considered an object in \mathcal{S} (for example if $\mathcal{S}' = X_{\text{ét}}$ and $\mathcal{S} = X_{\text{ét}}$), and every covering in \mathcal{S}' is also a covering in \mathcal{S} , then we get a morphism of sites $\pi: \mathcal{S} \rightarrow \mathcal{S}'$. We call such a morphism π a *restriction morphism*. If \mathcal{F} is a sheaf on \mathcal{S} then the *direct image* $\pi_*\mathcal{F}$ (see below) is the restriction of \mathcal{F} to \mathcal{S}' .

4.2. Direct and inverse image functors

DEFINITION 4.2.1. Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites given by a continuous functor $f^{\text{cat}}: \text{Cat}(\mathcal{S}') \rightarrow \text{Cat}(\mathcal{S})$. Let \mathcal{F} be a presheaf on \mathcal{S} . Then we define a presheaf $f_p\mathcal{F}$ on \mathcal{S}' by

$$f_p\mathcal{F} = \mathcal{F} \circ f^{\text{cat}}.$$

DEFINITION 4.2.2. Let \mathcal{F} be a presheaf on $X_{\acute{e}t}$ and let $f: X \rightarrow Y$ be a morphism of schemes. Let $\bar{f}: X_{\acute{e}t} \rightarrow Y_{\acute{e}t}$ be the morphism of sites given by the functor $f^\times: \text{Cat}(Y_{\acute{e}t}) \rightarrow \text{Cat}(X_{\acute{e}t})$. We define the *direct image (push-forward)* presheaf $f_p\mathcal{F}$ on $Y_{\acute{e}t}$ by

$$f_p\mathcal{F} = \bar{f}_p\mathcal{F}.$$

That is, for every étale $V \rightarrow Y$ we have $f_p\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$.

REMARK 4.2.3. This notation may sometimes be confusing. For a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and a presheaf \mathcal{F} on \mathcal{D} , a common notation is $F^p\mathcal{F} := \mathcal{F} \circ F$. Hence we would have $f_p\mathcal{F} = (f^\times)^p\mathcal{F}$.

Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites. It is clear that f_p is a functor $\text{PreSh}(\mathcal{S}) \rightarrow \text{PreSh}(\mathcal{S}')$ and we will show that f_p has a left adjoint f^p . That is, there exists a functor $f^p: \text{PreSh}(\mathcal{S}') \rightarrow \text{PreSh}(\mathcal{S})$ such that whenever \mathcal{F} and \mathcal{G} are presheaves on \mathcal{S} and \mathcal{S}' respectively, we have a natural bijection

$$\text{Hom}_{\text{PreSh}(\mathcal{S})}(f^p\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{\text{PreSh}(\mathcal{S}')}(\mathcal{G}, f_p\mathcal{F}).$$

DEFINITION 4.2.4. Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites given by a continuous functor $f^{\text{cat}}: \text{Cat}(\mathcal{S}') \rightarrow \text{Cat}(\mathcal{S})$, and let \mathcal{G} be a presheaf on \mathcal{S}' . We define a presheaf $f^p\mathcal{G}$ on \mathcal{S} by

$$f^p\mathcal{G}(U) = \varinjlim \mathcal{G}(V),$$

for every object U in \mathcal{S} , where the colimit is over all morphisms $\varphi: U \rightarrow f^{\text{cat}}(V)$ (V an object in \mathcal{S}') in $\text{Cat}(\mathcal{S})$ and a morphism $(V, \varphi) \rightarrow (V', \varphi')$ is a morphism $V \rightarrow V'$ such that the following diagram commutes:

$$\begin{array}{ccc} & U & \\ \varphi \swarrow & & \searrow \varphi' \\ f^{\text{cat}}(V) & \longrightarrow & f^{\text{cat}}(V') \end{array}$$

Then $f^p\mathcal{G}$ is indeed a presheaf since for a morphism $\psi: U' \rightarrow U$ in \mathcal{S} , we map every morphism $\varphi: U \rightarrow f^{\text{cat}}(V)$ to the morphism $\varphi \circ \psi: U' \rightarrow f^{\text{cat}}(V)$. In this way we get a natural map $f^p\mathcal{G}(U) \rightarrow f^p\mathcal{G}(U')$.

REMARK 4.2.5. Note that we have canonical morphisms

$$\mathcal{G} \rightarrow f_p f^p\mathcal{G}, \quad f^p f_p\mathcal{F} \rightarrow \mathcal{F}$$

if \mathcal{F} and \mathcal{G} are presheaves on sites \mathcal{S} and \mathcal{S}' respectively, and $f: \mathcal{S} \rightarrow \mathcal{S}'$ is a morphism of sites. Indeed, for every object V in \mathcal{S}' we have

$$f_p f^p\mathcal{G}(V) = f^p\mathcal{G}(f^{\text{cat}}(V)) = \varinjlim \mathcal{G}(V')$$

where the colimit is over morphisms $f^{\text{cat}}(V) \rightarrow f^{\text{cat}}(V')$. One such object is of course $\text{id}: f^{\text{cat}}(V) \rightarrow f^{\text{cat}}(V)$ and hence we have a map $\mathcal{G}(V) \rightarrow f_p f^p\mathcal{G}(V)$. It is not hard to see that this map is natural in V .

Let U be an object in \mathcal{S} . Then

$$f^p f_p\mathcal{F}(U) = \varinjlim f_p\mathcal{F}(V) = \varinjlim \mathcal{F}(f^{\text{cat}}(V))$$

where the limit is over morphisms $U \rightarrow f^{\text{cat}}(V)$. Every such morphism gives a map $\mathcal{F}(f^{\text{cat}}(V)) \rightarrow \mathcal{F}(U)$ and hence we get a map $f^p f_p\mathcal{F}(U) \rightarrow \mathcal{F}(U)$. It is not hard to see that this gives a morphism $f^p f_p\mathcal{F} \rightarrow \mathcal{F}$.

PROPOSITION 4.2.6. *Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites. The functor f^p is a left adjoint of f_p .*

PROOF. A natural transformation $\tau: f^p\mathcal{G} \rightarrow \mathcal{F}$ is given by the following data: for every U in \mathcal{S} and every morphism $U \rightarrow f^{cat}(V)$, where V is an object in \mathcal{S}' , we have maps $\mathcal{G}(V) \rightarrow \mathcal{F}(U)$ that are compatible with morphisms $V \rightarrow V'$ such that the following diagram commutes

$$\begin{array}{ccc} & U & \\ \varphi \swarrow & & \searrow \varphi' \\ f^{cat}(V) & \longrightarrow & f^{cat}(V') \end{array}$$

In particular, since τ is natural in U , we get that $\mathcal{G}(V) \rightarrow \mathcal{F}(U)$ factors through $\mathcal{F}(f^{cat}(V)) \rightarrow \mathcal{F}(U)$, the pullback of $U \rightarrow f^{cat}(V)$. Hence we have a map $\mathcal{G}(V) \rightarrow \mathcal{F}(f^{cat}(V)) = f_p\mathcal{F}(V)$ for every V , which is natural in V . That is, we have a natural transformation $\eta: \mathcal{G} \rightarrow f_p\mathcal{F}$.

Conversely, suppose that we start with a natural transformation $\eta: \mathcal{G} \rightarrow f_p\mathcal{F}$. Each morphism $U \rightarrow f^{cat}(V)$ yields a map $\mathcal{F}(f^{cat}(V)) \rightarrow \mathcal{F}(U)$ and hence we get maps $\mathcal{G}(V) \rightarrow \mathcal{F}(U)$ that are compatible with morphisms $V \rightarrow V'$ as before. Hence we have a unique morphism $f^p\mathcal{G}(U) \rightarrow \mathcal{F}(U)$. It is clear that if we start with τ , constructs η , and then apply the second construction we get τ back and vice versa. Hence these two constructions are inverses of each other. It is not hard to see that the constructions are natural in \mathcal{F} and \mathcal{G} . \square

We will now see what the pullback construction looks like for a sheaf on the small étale site of a scheme. Let $f: X \rightarrow Y$ be a morphism of schemes and \mathcal{G} a presheaf on $Y_{\text{ét}}$. Let $\bar{f}: X_{\text{ét}} \rightarrow Y_{\text{ét}}$ be the morphism of sites given by the functor $f^\times: \text{Cat}(Y_{\text{ét}}) \rightarrow \text{Cat}(X_{\text{ét}})$. Let $V \rightarrow Y$ be étale. To give a morphism

$$\varphi: U \rightarrow f^\times(V) = V \times_Y X$$

is to give a commutative square

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

and if we have a Y -morphism $h: V \rightarrow V'$ such that the following diagram commutes:

$$\begin{array}{ccc} & U & \\ \varphi \swarrow & & \searrow \varphi' \\ V \times_Y X & \longrightarrow & V' \times_Y X \end{array}$$

then we have $g' = h \circ g$.

DEFINITION 4.2.7. We define the *inverse image (pullback)* presheaf $f^p\mathcal{G}$ on $X_{\text{ét}}$ by

$$f^p\mathcal{G} = \bar{f}^p\mathcal{G}.$$

That is, for $U \rightarrow X$ étale, we have $f^p\mathcal{G}(U) = \varinjlim \mathcal{G}(V)$, where the colimit is over all commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where a morphisms between two such diagrams (V, g) and (V', g') is given by a Y -morphism $h: V \rightarrow V'$ such that $g' = h \circ g$.

COROLLARY 4.2.8. *Let $f: X \rightarrow Y$ be a morphism of schemes. The functor f^p is a left adjoint of f_p .*

PROOF. Just to enlighten things, we prove this without using Proposition 4.2.6. Given a commutative square

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

(which we again denote by (V, g)) we get a morphism $U \rightarrow V \times_Y X$. If $(V, g) \rightarrow (V', g')$ is a morphism between such diagrams, we get morphisms $U \rightarrow V \times_Y X \rightarrow V' \times_Y X$. This means that if we have a morphism $\mathcal{G} \rightarrow f_p \mathcal{F}$, we get compatible maps

$$\mathcal{G}(V) \rightarrow f_p \mathcal{F}(V) = \mathcal{F}(V \times_Y X) \rightarrow \mathcal{F}(U).$$

That is, we get a morphism $f^p \mathcal{G} \rightarrow \mathcal{F}$.

Conversely, since g factors through $V \times_Y X$, any map $\mathcal{G}(V) \rightarrow \mathcal{F}(U)$ factors through $\mathcal{F}(V \times_Y X) = f_p \mathcal{F}(V)$, and hence a morphism $f^p \mathcal{G} \rightarrow \mathcal{F}$ gives a morphism $\mathcal{G} \rightarrow f_p \mathcal{F}$. These constructions are clearly inverses of each other and hence we conclude that there is a bijection

$$\mathrm{Hom}_{\mathrm{PreSh}(X_{\acute{e}t})}(f^p \mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(Y_{\acute{e}t})}(\mathcal{G}, f_p \mathcal{F})$$

which is natural in \mathcal{F} and \mathcal{G} . \square

LEMMA 4.2.9. *Let $f: \mathcal{S}' \rightarrow \mathcal{S}$ be a morphism of sites given by a continuous functor f^{cat} . If X is an object in \mathcal{S} , then*

$$f^p \mathrm{h}_X \cong \mathrm{h}_{f^{cat}(X)}.$$

In particular, $f^p \mathrm{h}_X$ is a sheaf.

PROOF. Let U be an object in \mathcal{S}' . We have $f^p \mathrm{h}_X(U) = \varinjlim \mathrm{h}_X(V)$ where the colimit is over morphisms $\varphi: U \rightarrow f^{cat}(V)$. Every morphism $V \rightarrow X$ yields a morphism $U \rightarrow f^{cat}(V) \rightarrow f^{cat}(X)$ by applying f^{cat} and composing with $U \rightarrow f^{cat}(V)$. This gives a map $\mathrm{h}_X(V) \rightarrow \mathrm{h}_{f^{cat}(X)}(U)$ which is functorial in V . Hence we get a canonical map $\alpha_U: f^p \mathrm{h}_X(U) \rightarrow \mathrm{h}_{f^{cat}(X)}(U)$. Consider a morphism $\varphi: U \rightarrow f^{cat}(X)$. Then φ is the image of the identity $X \rightarrow X$ under the morphism $\mathrm{h}_X(X) \rightarrow \mathrm{h}_{f^{cat}(X)}(U)$. Hence α_U is surjective. To see that α_U is injective, suppose that we have morphisms $V \rightarrow X$, $V' \rightarrow X$, $U \rightarrow f^{cat}(V)$, and $U \rightarrow f^{cat}(V')$ such that the compositions $\beta: U \rightarrow f^{cat}(V) \rightarrow f^{cat}(X)$ and $\gamma: U \rightarrow f^{cat}(V') \rightarrow f^{cat}(X)$ coincide. Then β is the image of the identity $X \rightarrow X$ under the map $\mathrm{h}_X(X) \rightarrow \mathrm{h}_{f^{cat}(X)}(U)$, and γ is the image of the identity $X \rightarrow X$ under the morphism $\mathrm{h}_X(X) \rightarrow \mathrm{h}_{f^{cat}(X)}(U)$. That is, β and γ will map to the same element in $f^p \mathrm{h}_X(U)$. Thus α_U is injective. The α_U 's gives an isomorphism of functors $\alpha: f^p \mathrm{h}_X \rightarrow \mathrm{h}_{f^{cat}(X)}$. \square

COROLLARY 4.2.10. *Let $Z \rightarrow Y$ be étale and let $f: X \rightarrow Y$ be any morphism of schemes. If Z represents a sheaf \mathcal{G} on $Y_{\acute{e}t}$, then $Z \times_Y X$ represents the presheaf $f^p \mathcal{G}$ on $X_{\acute{e}t}$. In particular, $f^p \mathcal{G}$ is a sheaf.*

We will now consider the restrictions of the operations f_p and f^p to the categories of sheaves.

LEMMA 4.2.11. *Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites given by a continuous functor f^{cat} . If \mathcal{F} is a sheaf on \mathcal{S} then $f_p \mathcal{F}$ is a sheaf on \mathcal{S}' . In particular, if \mathcal{F} is a sheaf on $X_{\acute{e}t}$ and $f: X \rightarrow Y$ is a morphism of schemes, then $f_p \mathcal{F}$ is a sheaf on $Y_{\acute{e}t}$.*

PROOF. This is obvious since f^{cat} takes coverings to coverings and commutes with fiber products. \square

It is not true in general for a morphism $f: \mathcal{S} \rightarrow \mathcal{S}'$ of sites that $f^p\mathcal{G}$ is a sheaf when \mathcal{G} is a sheaf. Hence we make the following definitions for sheaves:

DEFINITION 4.2.12. Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites and let \mathcal{F} be a sheaf on \mathcal{S} . Then we define the *direct image (push-forward)* $f_*\mathcal{F}$ to be the sheaf $f_*\mathcal{F} = f_p\mathcal{F}$.

DEFINITION 4.2.13. Let $f: \mathcal{S} \rightarrow \mathcal{S}'$ be a morphism of sites and let \mathcal{G} be a sheaf on \mathcal{S}' . Then we define the *inverse image (pullback)* $f^*\mathcal{G}$ to be the sheafification of the presheaf $f^p\mathcal{G}$, i.e., $f^*\mathcal{G} = (f^p\mathcal{G})^a$.

REMARK 4.2.14. Note that the names *direct image (push-forward)* and *inverse image (pullback)* will easily lead to confusion since the direct image (push-forward) along a morphism of sites is really the inverse image (pullback) along the corresponding continuous functor and vice versa.

When $f: X \rightarrow Y$ is a morphism of schemes we will also write $f_*\mathcal{F} = \bar{f}_*\mathcal{F}$ and $f^*\mathcal{G} = \bar{f}^*\mathcal{G}$ where $\bar{f}: X_{\acute{e}t} \rightarrow Y_{\acute{e}t}$ is the morphism of sites induced by the functor f^\times , and \mathcal{F} and \mathcal{G} are sheaves on $X_{\acute{e}t}$ and $Y_{\acute{e}t}$ respectively.

REMARK 4.2.15. From the universal property of the sheafification of a presheaf, it follows that f_* and f^* are adjoints and we have

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{S})}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{S}')}(\mathcal{G}, f_*\mathcal{F}).$$

EXAMPLE 4.2.16 (The sheaf \mathbb{G}_m). Let \mathbb{G}_m be the presheaf on S_{Fpqc} defined by $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$ for each S -scheme U , i.e., $\mathbb{G}_m(U)$ is the multiplicative group of $\Gamma(U, \mathcal{O}_U)$. We will show that \mathbb{G}_m is representable, and hence a sheaf on S_{Fpqc} . We have

$$\begin{aligned} \Gamma(U, \mathcal{O}_U)^* &\simeq \mathrm{Hom}_{(\mathrm{Ring})}(\mathbb{Z}[T, T^{-1}], \Gamma(U, \mathcal{O}_U)) \\ &\simeq \mathrm{Hom}_{(\mathrm{Sch})}(U, \mathrm{Spec} \mathbb{Z}[T, T^{-1}]) \\ &\simeq \mathrm{Hom}_{(\mathrm{Sch}/S)}(U, \mathrm{Spec} \mathbb{Z}[T, T^{-1}] \times_{\mathrm{Spec} \mathbb{Z}} S) \end{aligned}$$

and hence \mathbb{G}_m is represented on (Sch/S) by the scheme $\mathrm{Spec} \mathbb{Z}[T, T^{-1}] \times_{\mathrm{Spec} \mathbb{Z}} S$ and hence \mathbb{G}_m is a sheaf on S_{Fpqc} .

EXAMPLE 4.2.17 (Sheaf μ_n of n :th roots of unity in \mathcal{O}_U). Let μ_n be the presheaf on S_{Fpqc} defined by $\mu_n(U) = \{x \in \Gamma(U, \mathcal{O}_U) : x^n = 1\}$ for each S -scheme U . We have that

$$\begin{aligned} \mu_n(U) &= \{x \in \Gamma(U, \mathcal{O}_U) : x^n = 1\} \\ &\simeq \mathrm{Hom}_{(\mathrm{Ring})}(\mathbb{Z}[T]/(T^n - 1), \Gamma(U, \mathcal{O}_U)) \\ &\simeq \mathrm{Hom}_{(\mathrm{Sch}/S)}(U, \mathrm{Spec}(\mathbb{Z}[T]/(T^n - 1)) \times_{\mathrm{Spec} \mathbb{Z}} S) \end{aligned}$$

and hence μ_n is represented on (Sch/S) by the scheme

$$\mathrm{Spec}(\mathbb{Z}[T]/(T^n - 1)) \times_{\mathrm{Spec} \mathbb{Z}} S.$$

In particular, μ_n is a sheaf on S_{Fpqc} .

4.3. The functor $j_!$ of an open immersion

DEFINITION 4.3.1. Let $j: U \rightarrow X$ be an open immersion of schemes and let \mathcal{F} be a presheaf on $U_{\acute{e}t}$. For any étale morphism $f: V \rightarrow X$ define

$$j_{p!}\mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & \text{if } f(V) \subseteq U; \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that for any presheaf \mathcal{G} on $X_{\acute{e}t}$, we have a bijection

$$\mathrm{Hom}_{\mathrm{PreSh}(X_{\acute{e}t})}(j_{p!}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(U_{\acute{e}t})}(\mathcal{F}, j^*\mathcal{G}).$$

It may happen that $j_{p!}\mathcal{F}$ is not a sheaf even though \mathcal{F} is. Hence we make the following definition:

DEFINITION 4.3.2. In the situation of Definition 4.3.1, we define

$$j_!\mathcal{F} = (j_{p!}\mathcal{F})^a.$$

REMARK 4.3.3. The universal property of the sheafification of a presheaf implies that we have bijections

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\acute{e}t})}(j_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(X_{\acute{e}t})}(j_{p!}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(U_{\acute{e}t})}(\mathcal{F}, j^*\mathcal{G}),$$

and since \mathcal{F} and $j^*\mathcal{G}$ are both sheaves, we see that $j_!$ is a left adjoint of j^* .

4.4. The functors f^* and $f_!$ of an object $f: T \rightarrow S$ in \mathcal{C}/S

Let S be a scheme, and choose a class of morphisms \mathbf{E} and a category \mathcal{C}/S as in Remark 3.1.2. Let $f: T \rightarrow S$ be an object in \mathcal{C}/S . Then we get two sites $S_{\mathbf{E}}$ and $T_{\mathbf{E}}$. Every object $U \rightarrow T$ in \mathcal{C}/T may be considered as an object in \mathcal{C}/S by composing with f and every morphism over T may also be considered a morphism over S . We get a functor

$$f_{cat}: \mathcal{C}/T \rightarrow \mathcal{C}/S$$

which takes an object $g: U \rightarrow T$ to the object $f \circ g: U \rightarrow T \rightarrow S$ and which takes a morphism $U' \rightarrow U$ over T to the same morphism $U' \rightarrow U$ considered as a morphism over S . It is clear that f_{cat} is a continuous functor with respect to the topologies on \mathcal{C}/T and \mathcal{C}/S and hence it defines a morphism of sites

$$\hat{f}: S_{\mathbf{E}} \rightarrow T_{\mathbf{E}}.$$

Now consider the functor $f^p: \mathrm{PreSh}(S_{\mathbf{E}}) \rightarrow \mathrm{PreSh}(T_{\mathbf{E}})$ defined as in the case where $S_{\mathbf{E}} = S_{\acute{e}t}$ and $T_{\mathbf{E}} = T_{\acute{e}t}$. That is, if \mathcal{F} is a sheaf on $S_{\mathbf{E}}$ then for every object $U \rightarrow T$ we have $f^p\mathcal{F}(U) = \varinjlim \mathcal{F}(V)$ where the colimit is over commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \varphi \downarrow & & \downarrow \psi \\ T & \xrightarrow{f} & S \end{array}$$

with φ and ψ in \mathcal{C}/T and \mathcal{C}/S respectively. But since $f \circ \varphi: U \rightarrow S$ is an object in \mathcal{C}/S , this colimit is just $\mathcal{F}(U \rightarrow T \rightarrow S) = \hat{f}_*\mathcal{F}(U \rightarrow T)$. Hence $f^p\mathcal{F} = \hat{f}_*\mathcal{F}$ and since $\hat{f}_*\mathcal{F}$ is a sheaf (Lemma 4.2.11), we get that $f^p\mathcal{F}$ is a sheaf. That is

$$f^* = \hat{f}_*.$$

REMARK 4.4.1. Note that if $f: X \rightarrow Y$ is étale and $g: T \rightarrow Y$ is any morphism, we get a diagram

$$\begin{array}{ccc} X \times_Y T & \xrightarrow{p} & T \\ q \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

If \mathcal{F} is a sheaf on $T_{\acute{e}t}$ then we have an isomorphism

$$f^*g_*\mathcal{F} \cong q_*p^*\mathcal{F}.$$

REMARK 4.4.2. Given an object $g: U \rightarrow T$ in $T_{\mathbb{E}}$ and $f: T \rightarrow S$ in \mathcal{C}/S as before, we have that elements in $\hat{f}_*\mathcal{F}(U)$ is the same as commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{F} \\ g \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

in $\text{PreSh}(S_{\mathbb{E}})$. On the other hand, $T(U)$ is the set of morphisms $h: U \rightarrow T$ such that $f \circ h = f \circ g$, and $S(U)$ consists only of the morphism $f \circ g$. Thus, the set of such diagrams is exactly the set of elements in $T(U) \times_{S(U)} \mathcal{F}(U)$. Hence we conclude that

$$f^*\mathcal{F} = \hat{f}_*\mathcal{F} = (T \times_S \mathcal{F})|_{T_{\text{ét}}},$$

where $(T \times_S \mathcal{F})|_{T_{\text{ét}}}$ is the restriction of $T \times_S \mathcal{F}$ to étale schemes over S which factors through T .

By Proposition 4.2.6 we know that \hat{f}_* has a left adjoint \hat{f}^* .

DEFINITION 4.4.3. The left adjoint \hat{f}^* of \hat{f}_* is denoted by $f_!$.

Hence $f_!$ is a left adjoint of f^* . Recall that if \mathcal{F} is representable we have that $\hat{f}^p\mathcal{F}$ is a sheaf when \mathcal{F} is a sheaf. Using Definition 4.2.4 we may describe the functor \hat{f}^p explicitly. Let \mathcal{F} be a sheaf on $T_{\mathbb{E}}$. According to Definition 4.2.4, we have for every object $g: U \rightarrow S$ in \mathcal{C}/S that $\hat{f}^p\mathcal{F}(U) = \varinjlim \mathcal{F}(V)$ where the colimit is over all commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ g \downarrow & & \downarrow \\ S & \xleftarrow{f} & T \end{array}$$

in \mathcal{C}/S . If we fix some $\varphi: U \rightarrow T$ and take the composition $f \circ \varphi: U \rightarrow S$, then the colimit $\varinjlim \mathcal{F}(U)$ over commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ f \circ \varphi \downarrow & & \downarrow \\ S & \xleftarrow{f} & T \end{array}$$

is exactly $\mathcal{F}(U)$. On the other hand, if we take some $U \rightarrow S$ which does not factor through $f: T \rightarrow S$ then there are no diagrams as above and we have $\hat{f}^p\mathcal{F}(U) = \emptyset$. Hence we conclude that for every object $g: U \rightarrow S$ we have

$$\hat{f}^p\mathcal{F}(U) = \coprod_{\varphi} \mathcal{F}(U)$$

where the coproduct is over all morphisms $\varphi: U \rightarrow T$ such that $f \circ \varphi = g$.

If $S_{\mathbb{E}} = S_{\text{ét}}$ and f is an open immersion then there is a unique $\varphi: U \rightarrow T$ such that $f \circ \varphi = g$ and hence $f_!$ is the sheafification of the presheaf

$$U \mapsto \begin{cases} \mathcal{F}(U) & \text{if } U \rightarrow S \text{ factors through } f: T \rightarrow S; \\ \emptyset & \text{otherwise.} \end{cases}$$

This proves that the definition of $f_!$ agrees with the definition of $f_!$ in Section 4.3.

4.5. Operations on sheaves of abelian group/pointed sets

It is also natural to talk about sheaves of abelian groups and sheaves of pointed sets. There are forgetful functors

$$(\mathbf{Ab}) \rightarrow (\mathbf{Set}_*) \rightarrow (\mathbf{Set})$$

where the first one forgets the group structure but remembers the zero, and the second one forgets the zero. The sheaves constructed in Example 3.5.5 and Example 3.5.6 are examples of sheaves of abelian groups since they are even sheaves of rings.

Given a morphism $f: X \rightarrow Y$ of schemes, we may define functors f_* and f^* between categories of sheaves of abelian groups/pointed sets exactly as before. In case f is an \mathbf{E} -morphism, i.e., an object in \mathcal{C}/Y (see Remark 3.1.2), it is easy to see that the functor f^* will be the same as in the case of sheaves of sets. More generally, this holds when the colimit $f^p(U)$ is filtered for every X -scheme U .

Let $f: X \rightarrow Y$ be an object in \mathcal{C}/Y and let \hat{f} be defined as in 4.4. Let \mathcal{F} be a sheaf of abelian groups/pointed sets on $(\mathcal{C}/Y)_{\mathbf{E}}$. Since coproducts are direct sums in (\mathbf{Ab}) and wedge products in (\mathbf{Set}_*) we get that $f_!\mathcal{F}$ is the sheafification of the presheaf which takes $g: U \rightarrow Y$ to

$$\hat{f}^p \mathcal{F}(U) = \bigoplus_{\varphi} \mathcal{F}(U)$$

in the case of abelian groups and the sheafification of

$$\hat{f}^p \mathcal{F}(U) = \bigvee_{\varphi} \mathcal{F}(U)$$

in the case of pointed sets, where the direct sum/wedge product is over all morphisms $\varphi: U \rightarrow X$, such that $f \circ \varphi = g$.

REMARK 4.5.1. The distinguished point of a pointed set will often be denoted by $*$ or 0 .

LEMMA 4.5.2. *Let $f: X \rightarrow Y$ be a morphism of schemes and let \mathcal{F} be a sheaf (of sets, pointed sets, or abelian groups) on $Y_{\text{ét}}$. Let $\bar{x}: \text{Spec } \Omega \rightarrow X$ be a geometric point and put $\bar{y} = f \circ \bar{x}$. Then we have an isomorphism*

$$(f^* \mathcal{F})_{\bar{x}} \cong \mathcal{F}_{\bar{y}}.$$

PROOF. We have that $(f^* \mathcal{F})_{\bar{x}} = \varinjlim f^* \mathcal{F}(U) = \varinjlim \varinjlim \mathcal{F}(V)$ where the first colimit is over étale neighborhoods U of \bar{x} and the second colimit is over commutative diagrams

$$\begin{array}{ccc} & U & \longrightarrow V \\ & \nearrow & \downarrow \\ \text{Spec } \Omega & \longrightarrow X & \longrightarrow Y \\ & & \downarrow \end{array}$$

where $V \rightarrow Y$ is étale. Thus, V is an étale neighborhood of $\text{Spec } \Omega$. On the other hand, $\mathcal{F}_{\bar{y}} = \varinjlim \mathcal{F}(V')$ where the colimit is over étale neighborhoods V' of \bar{y} , i.e., over commutative diagrams

$$\begin{array}{ccc} & X \times_Y V' & \longrightarrow V' \\ & \nearrow & \downarrow \\ \text{Spec } \Omega & \longrightarrow X & \longrightarrow Y \\ & & \downarrow \end{array}$$

where $V' \rightarrow Y$ is étale. Hence we see that $(f^* \mathcal{F})_{\bar{x}} \cong \mathcal{F}_{\bar{y}}$. \square

LEMMA 4.5.3. *Let \mathcal{F} be a sheaf (of sets, pointed sets, or abelian groups) on the small étale site of a scheme X . Let $f: X \rightarrow Y$ be étale and let $g: U \rightarrow Y$ be any morphism. Consider the fiber product*

$$\begin{array}{ccc} X \times_Y U & \xrightarrow{q} & U \\ p \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Then we have an isomorphism

$$g^* f_! \mathcal{F} \cong q_! p^* \mathcal{F}.$$

PROOF. For any sheaf \mathcal{G} on $U_{\text{ét}}$ we have an isomorphism $f^* g_* \mathcal{G} \cong p_* q^* \mathcal{G}$ (Remark 4.4.1). Hence, by the adjoint properties, we have natural bijections

$$\begin{aligned} \text{Hom}_{\text{Sh}_*(U_{\text{ét}})}(g^* f_! \mathcal{F}, \mathcal{G}) &\simeq \text{Hom}_{\text{Sh}_*(Y_{\text{ét}})}(f_! \mathcal{F}, g_* \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Sh}_*(X_{\text{ét}})}(\mathcal{F}, f^* g_* \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Sh}_*(X_{\text{ét}})}(\mathcal{F}, p_* q^* \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Sh}_*((X \times_Y U)_{\text{ét}})}(p^* \mathcal{F}, q^* \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Sh}_*(U_{\text{ét}})}(q_! p^* \mathcal{F}, \mathcal{G}). \end{aligned}$$

This implies that $g^* f_! \mathcal{F} \cong q_! p^* \mathcal{F}$. \square

LEMMA 4.5.4. *Let \mathcal{F} be a sheaf of pointed sets on $T_{\text{ét}}$ and let $f: T \rightarrow S$ be étale. Let $\bar{s}: \text{Spec } \Omega \rightarrow S$ be a geometric point and let $\bar{t}_1, \dots, \bar{t}_n$ be the geometric points of T over \bar{s} . Then we have an isomorphism of pointed sets*

$$(f_! \mathcal{F})_{\bar{s}} \cong \bigvee_{i=1}^n \mathcal{F}_{\bar{t}_i}.$$

PROOF. Suppose first that $S = \text{Spec } \Omega$. Then an étale neighborhood of

$$\text{id}: \text{Spec } \Omega \rightarrow \text{Spec } \Omega$$

is just an étale morphism $U \rightarrow \text{Spec } \Omega$ with a section $\text{Spec } \Omega \rightarrow U$. The geometric points $\bar{t}_1, \dots, \bar{t}_n$ corresponds exactly to the sections $S \rightarrow T$ of f . An étale neighborhood of \bar{t}_i gives an étale neighborhood of $\bar{s} \in \text{Spec } \Omega$ and conversely, any étale $U \rightarrow \text{Spec } \Omega$ with a section $\text{Spec } \Omega \rightarrow U$, yields for a section $\text{Spec } \Omega \rightarrow T$ with image \bar{t}_i , an étale neighborhood $U \times_S T \rightarrow T$ of \bar{t}_i . Furthermore, if there is no section $\text{Spec } \Omega \rightarrow T$ and if $U \rightarrow \text{Spec } \Omega$ is an étale neighborhood, then $U \rightarrow \text{Spec } \Omega$ does not factor through $T \rightarrow \text{Spec } \Omega$. Hence we conclude that

$$(f_! \mathcal{F})_{\bar{s}} \cong \bigvee_{i=1}^n \mathcal{F}_{\bar{t}_i}.$$

Now we use the previous lemma to finish the proof. Let $\bar{s}: \text{Spec } \Omega \rightarrow S$ be a geometric point and consider the diagram

$$\begin{array}{ccc} T \times_S \text{Spec } \Omega & \xrightarrow{q} & \text{Spec } \Omega \\ p \downarrow & & \downarrow \bar{s} \\ T & \xrightarrow{f} & S. \end{array}$$

From Lemma 4.5.2 we have $(f_! \mathcal{F})_{\bar{s}} = (\bar{s}^*(f_! \mathcal{F}))_{\bar{x}}$ where $\bar{x} = \text{id}: \text{Spec } \Omega \rightarrow \text{Spec } \Omega$, and by Lemma 4.5.3 this equals $(q_!(p^* \mathcal{F}))_{\bar{x}}$. From the first part of the proof we have

$(q_!(p^*\mathcal{F}))_{\bar{x}} = \bigvee_i (p^*\mathcal{F})_{\bar{x}_i}$ where the wedge is over all sections $\text{Spec } \Omega \rightarrow T \times_S \text{Spec } \Omega$ of q . But again, by Lemma 4.5.2 we have $\bigvee_i (p^*\mathcal{F})_{\bar{x}_i} = \bigvee_i \mathcal{F}_{p(\bar{x}_i)}$, and hence

$$(f_!\mathcal{F})_{\bar{s}} = \bigvee_{i=1}^n \mathcal{F}_{p(\bar{x}_i)}. \quad \square$$

Algebraic spaces

Algebraic spaces are geometric objects that are more general than schemes. A scheme is obtained by gluing together affine schemes in the Zariski topology, whereas an algebraic space is obtained by gluing together affine schemes in the étale topology (or even finer topologies). Since the étale topology is finer than the Zariski topology, this gives a "bigger" category than the category of schemes. The main references to this chapter are [Art71, Knu71], and [Sta].

Before we give the definition of an algebraic space we introduce some descent theory. This will later be used to extend properties of schemes to properties of sheaves (in particular, algebraic spaces), and to extend properties of morphisms of schemes to properties of morphisms of sheaves (in particular, algebraic spaces).

5.1. Some descent theory

DEFINITION 5.1.1. A morphism $\mathcal{F} \rightarrow \mathcal{G}$ in the category $(\widehat{\text{Sch}}/S)$ of presheaves $(\text{Sch}/S)^{\text{op}} \rightarrow (\text{Set})$

is called *representable* if for all S -schemes X and all morphisms $h_X \rightarrow \mathcal{G}$ in $(\widehat{\text{Sch}}/S)$, the functor $\mathcal{F} \times_{\mathcal{G}} h_X$ is representable.

Remark 3.2.3 implies the following lemma:

LEMMA 5.1.2. *If $X \rightarrow Y$ is a morphism of schemes then $h_X \rightarrow h_Y$ is representable.*

Let \mathbf{P} be a property of morphisms in a category \mathcal{C} with fiber products, let \mathcal{T} be a Grothendieck topology on \mathcal{C} , and let \mathcal{S} be the resulting site. In order to extend properties of morphisms of schemes to properties of morphisms of presheaves, we make the following definitions:

DEFINITION 5.1.3. We say that \mathbf{P} is \mathcal{T} -local on the base if for every covering $\{U_i \rightarrow U\}$ with respect to \mathcal{T} and any morphism $f: X \rightarrow U$, the following are equivalent:

- (1) f has \mathbf{P} ;
- (2) for every i , the morphism $X \times_U U_i \rightarrow U_i$ has \mathbf{P} .

LEMMA 5.1.4. *Let $\varphi: A \rightarrow R$ be a faithfully flat homomorphism of rings and let $\psi: B \rightarrow A$ be a ring homomorphism such that the composition $\varphi \circ \psi$ is flat. Then ψ is flat.*

PROOF. Let $M' \rightarrow M \rightarrow M''$ be any sequence of B -modules. Since the composition $B \rightarrow A \rightarrow R$ is flat, we get that

$$M' \otimes_B R \rightarrow M \otimes_B R \rightarrow M'' \otimes_B R$$

is exact. But this sequence equals

$$M' \otimes_B A \otimes_A R \rightarrow M \otimes_B A \otimes_A R \rightarrow M'' \otimes_B A \otimes_A R$$

and since $A \rightarrow R$ is faithfully flat, Proposition 1.1.8.(3) implies that

$$M' \otimes_B A \rightarrow M \otimes_B A \rightarrow M'' \otimes_B A$$

is exact. That is $B \rightarrow A$ is flat. \square

LEMMA 5.1.5. *The property "flat" is fpqc local on the base.*

PROOF. We only need to prove that (2) implies (1) since a base change of a flat morphism is flat. Let $f: X \rightarrow U$ be any morphism of schemes. Suppose that $\{U_i \rightarrow U\}$ is a flat covering and that $X \times_U U_i \rightarrow U_i$ is flat for each i . Take $x \in X$ and put $u = f(x)$. The question is local and hence we may assume that $X = \text{Spec } B$ and $U = \text{Spec } A$ where $A = \mathcal{O}_{U,u}$ and $B = \mathcal{O}_{X,x}$. Since $U' := \coprod U_i \rightarrow U$ is surjective, there is a $u' \in U'$ mapping to $u \in U$. The induced morphism $\text{Spec } \mathcal{O}_{U',u'} \rightarrow \text{Spec } \mathcal{O}_{U,u}$ is faithfully flat by Lemma 1.1. Put $A' = \mathcal{O}_{U',u'}$.

We have that $B \rightarrow A' \otimes_A B$ is faithfully flat since $A \rightarrow A'$ is faithfully flat. We also have that the composition $A \rightarrow B \rightarrow B \otimes_A A'$ is flat since it equals the composition $A \rightarrow A' \rightarrow B \otimes_A A'$ which is a composition of flat homomorphisms. Hence $A \rightarrow B$ is flat by Lemma 5.1.4. \square

LEMMA 5.1.6. *The property "étale" is fpqc local on the base.*

PROOF. By the previous lemma we only need to show that the property "unramified" is fpqc local on the base. A morphism $X \rightarrow U$ is unramified if and only if it is locally of finite type and the diagonal $\Delta: X \rightarrow X \times_U X$ is an open immersion. But if $U' \rightarrow U$ is an fpqc cover, then $U' \times_U X$ is the fiber product corresponding to the morphisms $\Delta: X \rightarrow X \times_U X$ and the induced morphism $U' \times_U X \times_U X \rightarrow X \times_U X$ which is also an fpqc cover. Hence it suffices to show that "locally of finite type" and "open immersion" are properties that are fpqc local on the base. This follows from [Sta, Tag 02KX] and [Sta, Tag 02L3]. \square

DEFINITION 5.1.7. We say that a stable property \mathbf{P} of morphisms in \mathcal{C} is \mathcal{T} -local on the domain if for any morphism $X \rightarrow Y$ in \mathcal{C} and any \mathcal{T} -covering $\{X_i \rightarrow X\}$, the following are equivalent:

- (1) $X \rightarrow Y$ has \mathbf{P} ;
- (2) $X_i \rightarrow X \rightarrow Y$ has \mathbf{P} for every i .

LEMMA 5.1.8. *The property "flat" is fpqc local on the domain.*

PROOF. We need only prove that if $\{X_i \rightarrow X\}$ is a covering such that $X' := \coprod X_i \rightarrow X$ is faithfully flat, then a morphism $f: X \rightarrow Y$ if $X' \rightarrow X \rightarrow Y$ is flat. Take any $x \in X$ and put $y = f(x)$. The question is local and hence we may assume that $X = \text{Spec } \mathcal{O}_{X,x}$ and $Y = \text{Spec } \mathcal{O}_{Y,y}$. Since $\coprod X_i \rightarrow X$ is surjective, there is an $x' \in X'$ in the fiber over x . Since $\coprod X_i \rightarrow X$ is flat, the local homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X',x'}$ is faithfully flat by Lemma 1.1. Hence $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is flat by Lemma 5.1.4. \square

LEMMA 5.1.9. *The property "étale" is étale local on the domain.*

PROOF. We only need to prove the implication (2) \Rightarrow (1) of Definition 5.1.7. Suppose that $f: X' \rightarrow X$ is an étale cover and that $X \rightarrow Y$ is any morphism such that $X' \rightarrow Y$ is unramified. By [Sta, Tag 0360] we have that $X \rightarrow Y$ is locally of finite type. There is an exact sequence

$$0 \rightarrow f^* \Omega_{X/Y} \rightarrow \Omega_{X'/Y} \rightarrow \Omega_{X'/X} \rightarrow 0$$

[Har77, II.8.11], [AK70, VI.4.9]. We have $\Omega_{X'/Y} = 0$ since $X' \rightarrow Y$ is unramified. Hence $f^* \Omega_{X/Y} = 0$. This implies that the stalk

$$(f^* \Omega_{X/Y})_{x'} = (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is zero for all $x' \in X'$ and $x = f(x')$. But f is surjective and since f is flat, the induced homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X',x'}$ is faithfully flat. Thus we conclude that $(\Omega_{X/Y})_x = 0$ for all $x \in X$ and hence $\Omega_{X/Y} = 0$. That is, $X \rightarrow Y$ is unramified. By Lemma 5.1.8 we get that $X \rightarrow Y$ is étale. \square

DEFINITION 5.1.10. Let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a representable morphism in $(\widehat{\text{Sch}}/S)$ and let \mathbf{P} be a property of morphisms over S which is preserved under base change and is fpqc local on the base. Then we say that α has \mathbf{P} if for every S -scheme U and every morphism $h_U \rightarrow \mathcal{G}$, with V an S -scheme representing $h_U \times_{\mathcal{G}} \mathcal{F}$ we have that the morphism $V \rightarrow U$ (corresponding to the morphism $h_V \cong h_U \times_{\mathcal{G}} \mathcal{F} \rightarrow h_U$) has \mathbf{P} .

5.2. Algebraic spaces

DEFINITION 5.2.1. Let \mathcal{C} be a category with fiber products, and S an object in \mathcal{C} . We call a diagram $R \rightrightarrows U$ in \mathcal{C}/S a *categorical equivalence relation* if for every object V in \mathcal{C}/S , the map

$$\text{Hom}_{\mathcal{C}/S}(V, R) \rightarrow \text{Hom}_{\mathcal{C}/S}(V, U) \times \text{Hom}_{\mathcal{C}/S}(V, U) = \text{Hom}_{\mathcal{C}/S}(V, U \times_S U)$$

is injective and gives an equivalence relation on $\text{Hom}_{\mathcal{C}/S}(V, U)$. In particular, the induced morphism

$$R \rightarrow U \times_S U$$

is a monomorphism.

DEFINITION 5.2.2. Let S be a scheme. A categorical equivalence relation $R \rightrightarrows U$ in (Sch/S) is called an *étale equivalence relation* if the two projection morphisms $p_1, p_2: R \hookrightarrow U \times_S U \rightrightarrows U$ are étale.

DEFINITION 5.2.3. Let S be a scheme. An *algebraic space* over S is a sheaf on $S_{\text{ét}}$ such that there exists an S -scheme U and a representable morphism $h_U \rightarrow \mathcal{A}$, such that for any S -scheme X and morphism $h_X \rightarrow \mathcal{A}$, the morphism $h_U \times_{\mathcal{A}} h_X \rightarrow h_X$ is induced by a surjective and étale morphism of schemes.

DEFINITION 5.2.4. We call a morphism $h_U \rightarrow \mathcal{A}$, as in Definition 5.2.3, a *representable étale covering*. If R is a scheme representing $h_U \times_{\mathcal{A}} h_U$, we say that the diagram $h_R \rightrightarrows h_U \rightarrow \mathcal{A}$ is a *presentation of \mathcal{A}* .

REMARK 5.2.5. An S -scheme X becomes an algebraic space when identifying it with its representable sheaf h_X . Indeed, it is a sheaf in the étale topology, and for the S -scheme U we can just choose $U = X$.

REMARK 5.2.6. If $h_U \rightarrow \mathcal{A}$ is a representable étale covering and $U' \rightarrow U$ is an étale covering of schemes, then $h_{U'} \rightarrow \mathcal{A}$ is a representable étale covering. Indeed, for any morphism $h_X \rightarrow \mathcal{A}$ we have that $h_U \times_{\mathcal{A}} h_X \cong h_Z$ for some scheme Z and $h_{U'} \times_{\mathcal{A}} h_X \cong h_{U'} \times_{h_U} h_Z \cong h_{U' \times_U Z}$. Since $U' \rightarrow U$ is an étale covering, so is $U' \times_U Z \rightarrow Z$, and hence $U' \times_U Z \rightarrow Z \rightarrow X$ is an étale covering.

DEFINITION 5.2.7. A *morphism of algebraic spaces* $\mathcal{A} \rightarrow \mathcal{B}$ is just a morphism $\mathcal{A} \rightarrow \mathcal{B}$ of presheaves (i.e., a natural transformation of functors).

We denote the category of algebraic spaces over a schemes S by (Alg_S) or just (Alg) if no base scheme is specified. Also, we denote the category of algebraic spaces over S over a base space \mathcal{B} by $(\text{Alg}_S/\mathcal{B})$ (or just (Alg/\mathcal{B})). We may also consider the subcategory (Sch/\mathcal{B}) of schemes over an algebraic space \mathcal{B} .

REMARK 5.2.8. By Remark 2.2.6 we have that h_S (given by $\text{id}: S \rightarrow S$) is the final object in (Alg_S) .

PROPOSITION 5.2.9 ([Knu71, 1.3]). *Let \mathcal{A} be an algebraic space over a scheme S and $h_R \rightrightarrows h_U \rightarrow \mathcal{A}$ a presentation of \mathcal{A} . Then*

- (1) *$R \rightrightarrows U$ is an étale equivalence relation in the category (Sch/S) , and*
- (2) *$h_R \rightrightarrows h_U \rightarrow \mathcal{A}$ is a coequalizer diagram in $\text{Sh}(S_{\text{ét}})$, i.e., $h_U \rightarrow \mathcal{A}$ is an effective epimorphism.*

PROOF. To prove (1), note first that the morphisms $R \rightrightarrows U$ are étale by the definition of an algebraic space. Since $h_R = h_U \times_{\mathcal{A}} h_U$, it follows that for any S -scheme V , the morphism

$$\text{Hom}(h_V, h_R) \rightarrow \text{Hom}(h_V, h_U) \times \text{Hom}(h_V, h_U)$$

is injective. Yoneda's lemma now implies that $R \rightrightarrows U$ is a categorial equivalence relation. This proves (1).

To prove (2), Lemma 3.4.9 and Lemma 3.4.8 implies that it is enough to show that $h_U \rightarrow \mathcal{A}$ is locally surjective. Let X be a scheme. An element $\varphi \in \mathcal{A}(X)$ is by Yoneda's lemma a morphism $\varphi: h_X \rightarrow \mathcal{A}$ and by definition, there is an S -scheme V such that $h_V = h_X \times_{\mathcal{A}} h_U$ and $V \rightarrow X$ is surjective and étale. Thus $\{V \rightarrow X\}$ is a covering in $\text{Sh}(S_{\text{ét}})$. Since $h_V = h_X \times_{\mathcal{A}} h_U$ we have that the composition $h_V \rightarrow h_X \rightarrow \mathcal{A}$ is equal to the composition $h_V \rightarrow h_U \rightarrow \mathcal{A}$. That is, the pullback of φ to $\mathcal{A}(V)$ is the image of an element in $h_U(V)$, and hence $h_U \rightarrow \mathcal{A}$ is locally surjective. This proves (2). \square

From now on we will just write X instead of h_X . If $R \rightrightarrows U$ is a categorial equivalence relation, then for every scheme T , the maps $R(T) \rightrightarrows U(T)$ gives an equivalence relation \sim on $U(T)$. Let U/R be the sheaf associated to the presheaf $T \mapsto U(T)/\sim$.

THEOREM 5.2.10. *Let \mathcal{A} be an algebraic space over a scheme S . Then*

- (1) *the diagonal $\Delta: \mathcal{A} \rightarrow \mathcal{A} \times_S \mathcal{A}$ is representable;*
- (2) *for every S -scheme U we have that every S -morphism $f: U \rightarrow \mathcal{A}$ is representable.*

PROOF. Statement (1) is [CLO12, A.1.1]. We show that (1) and (2) are equivalent.

(1) \Rightarrow (2): Suppose that U and V are schemes and we have morphisms $U \rightarrow \mathcal{A}$ and $V \rightarrow \mathcal{A}$. Then $U \times_S V$ is a scheme and we have a morphism $U \times_S V \rightarrow \mathcal{A} \times_S \mathcal{A}$. But then

$$U \times_{\mathcal{A}} V = \mathcal{A} \times_{\mathcal{A} \times_S \mathcal{A}} (U \times_S V)$$

which is a scheme by (1).

(2) \Rightarrow (1): Suppose that Z is a scheme and let $f: Z \rightarrow \mathcal{A} \times_S \mathcal{A}$ be any morphism. Then f factors as $Z \rightarrow Z \times_S Z \rightarrow \mathcal{A} \times_S \mathcal{A}$ and we have

$$\mathcal{A} \times_{\mathcal{A} \times_S \mathcal{A}} (Z \times_S Z) = Z \times_{\mathcal{A}} Z,$$

which is a scheme by (2). Hence

$$\mathcal{A} \times_{\mathcal{A} \times_S \mathcal{A}} Z = (Z \times_{\mathcal{A}} Z) \times_{Z \times_S Z} Z$$

which again is a scheme and hence Δ is representable. \square

Note that Theorem 5.2.10 says that if U and V are S -schemes and we have S -morphisms $U \rightarrow \mathcal{A}$ and $V \rightarrow \mathcal{A}$, then the fiber product $U \times_{\mathcal{A}} V$ is a scheme.

THEOREM 5.2.11 ([CLO12, A.1.2]). *Let U be a scheme over S . If $R \rightrightarrows U$ is an étale equivalence relation, then the quotient sheaf U/R is an algebraic space over S and $R \rightrightarrows U \rightarrow U/R$ is a presentation of U/R .*

COROLLARY 5.2.12. *Let \mathcal{A} be an algebraic space and let $R \rightrightarrows U \rightarrow \mathcal{A}$ be a presentation of \mathcal{A} . Then \mathcal{A} is isomorphic to the quotient sheaf U/R .*

PROOF. Follows from the universal property of a coequalizer. \square

DEFINITION 5.2.13. Let \mathcal{A} be an algebraic space over a scheme S given by an étale equivalence relation $R \rightrightarrows U$. Then we say that

- (1) \mathcal{A} is *locally separated* if the morphism $R \rightarrow U \times_S U$ is an immersion;
- (2) \mathcal{A} is *separated* if $R \rightarrow U \times_S U$ is a closed immersion.

PROPOSITION 5.2.14. *If $\mathcal{A}_1 \rightarrow \mathcal{B}$ and $\mathcal{A}_2 \rightarrow \mathcal{B}$ are morphisms of algebraic spaces, then the presheaf fiber product $\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2$ is an algebraic space.*

As a result, we have that the category (**Alg**) of algebraic spaces has fiber products, products, and coproducts.

PROOF. By Corollary 3.8.2 we only need to show that the sheaf $\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2$ is an algebraic space. Let $U_1 \rightarrow \mathcal{A}_1$, $U_2 \rightarrow \mathcal{A}_2$, and $V \rightarrow \mathcal{B}$ be representable étale coverings. Then we have a morphism $U_1 \times_{\mathcal{B}} U_2 \rightarrow \mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2$. Also, $U_1 \times_{\mathcal{B}} U_2$ is representable by Theorem 5.2.10.

Suppose that we have a morphism $X \rightarrow \mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2$ where X is a scheme. This means that we have morphisms $X \rightarrow \mathcal{A}_1 \rightarrow \mathcal{B}$ and $X \rightarrow \mathcal{A}_2 \rightarrow \mathcal{B}$ and hence we have

$$X \times_{\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2} (U_1 \times_{\mathcal{B}} U_2) \cong (U_1 \times_{\mathcal{A}_1} X) \times_X (U_2 \times_{\mathcal{A}_2} X),$$

where the last fiber product is representable since it is a fiber product of schemes over a scheme. Since $U_1 \times_{\mathcal{A}_1} X \rightarrow X$ and $U_2 \times_{\mathcal{A}_2} X \rightarrow X$ are both induced by surjective étale morphisms of schemes, we conclude that

$$(U_1 \times_{\mathcal{A}_1} X) \times_X (U_2 \times_{\mathcal{A}_2} X) \rightarrow X$$

is also induced by a surjective étale morphism of schemes. \square

PROPOSITION 5.2.15 ([**Knu71**, Proposition 1.4]). *Let \mathcal{A} and \mathcal{B} be algebraic spaces. Every morphism $\mathcal{A} \rightarrow \mathcal{B}$ is induced by a diagram*

$$\begin{array}{ccc} U \times_{\mathcal{A}} U & \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} & U \\ g \downarrow & & \downarrow f \\ V \times_{\mathcal{B}} V & \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} & V \end{array}$$

where $\alpha : U \rightarrow \mathcal{A}$ and $\beta : V \rightarrow \mathcal{B}$ are representable étale coverings such that $\text{pr}_i \circ g = f \circ \text{pr}_i$ for $i = 1, 2$.

PROOF. First, it is clear that such a diagram yields a unique morphism $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ such that $\beta \circ f = \gamma \circ \alpha$ since $U \rightarrow \mathcal{A}$ is a coequalizer of $U \times_{\mathcal{A}} U \rightrightarrows U$ in $\mathbf{Sh}(S_{\text{ét}})$.

Let $\mathcal{A} \rightarrow \mathcal{B}$ be a morphism and let $U' \rightarrow \mathcal{A}$ and $V \rightarrow \mathcal{B}$ be representable étale coverings. The composition $U' \rightarrow \mathcal{A} \rightarrow \mathcal{B}$ is an element $y \in \mathcal{B}(U')$ and since $V \rightarrow \mathcal{B}$ is locally surjective, there is an étale covering $\varphi : U \rightarrow U'$ such that $\varphi^*(y) \in \mathcal{B}(U)$ is in the image of $V(U) \rightarrow \mathcal{B}(U)$. Hence $U \rightarrow U' \rightarrow \mathcal{A}$ is also a representable étale covering. A preimage of $\varphi^*(y)$ is a morphism $U \rightarrow V$ of algebraic spaces and for every choice of preimage we get a morphism $U \times_{\mathcal{A}} U \rightarrow V \times_{\mathcal{B}} V$ by the universal property of $V \times_{\mathcal{B}} V$. \square

5.3. Some descent theory for algebraic spaces

In this section we will extend some properties of schemes (morphisms of schemes) to properties of algebraic spaces (morphisms of algebraic spaces). A good definition will give us the usual properties of schemes (morphisms of schemes) when restricting to the subcategory of schemes inside the category of algebraic spaces.

Let $\mathcal{A} \rightarrow \mathcal{B}$ be a morphism of algebraic spaces over a scheme S and suppose that we have an S -scheme X and a morphism $X \rightarrow \mathcal{B}$. Let $U \rightarrow \mathcal{A}$ and $V \rightarrow \mathcal{B}$ be representable étale coverings. Then $X \times_{\mathcal{B}} V$ is representable, say by Z . Thus we have a commutative diagram

$$\begin{array}{ccccc}
 Z \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{\quad} & X \times_{\mathcal{B}} \mathcal{A} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & V \times_{\mathcal{B}} \mathcal{A} & \xrightarrow{\quad} & \mathcal{A} \\
 \downarrow & & \downarrow & & \downarrow \\
 Z & \xrightarrow{\quad} & X & & \mathcal{B} \\
 & \searrow & & \searrow & \\
 & & V & \xrightarrow{\quad} & \mathcal{B}
 \end{array}$$

If $\mathcal{A} \rightarrow \mathcal{B}$ is representable, then $X \times_{\mathcal{B}} \mathcal{A}$ and $V \times_{\mathcal{B}} \mathcal{A}$ are representable. If \mathbf{P} is a property of morphisms of schemes which is stable under base change and étale-local on the base, then $X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} if $V \times_{\mathcal{B}} \mathcal{A} \rightarrow V$ has \mathbf{P} . Indeed, this follows since $Z \rightarrow X$ is an étale covering. Hence we have proved:

LEMMA 5.3.1. *Let $\mathcal{A} \rightarrow \mathcal{B}$ be a representable morphism of algebraic spaces and let $V \rightarrow \mathcal{B}$ be a representable étale covering. Let \mathbf{P} be a property of morphisms of schemes which is stable under base change and étale local on the base. Then the following are equivalent:*

- (1) $V \times_{\mathcal{B}} \mathcal{A} \rightarrow V$ has \mathbf{P} ;
- (2) $X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} for every scheme X and every morphism $X \rightarrow \mathcal{B}$.

Lemma 5.3.1 implies that the following definition agrees with Definition 5.1.10:

DEFINITION 5.3.2. Let $\mathcal{A} \rightarrow \mathcal{B}$ be a representable morphism of algebraic spaces and let \mathbf{P} be a property of morphisms of scheme which is stable under base change and étale local on the base. Then we say that $\mathcal{A} \rightarrow \mathcal{B}$ has \mathbf{P} if there is a representable étale covering $V \rightarrow \mathcal{B}$ such that the induced morphism $V \times_{\mathcal{B}} \mathcal{A} \rightarrow V$ of schemes has \mathbf{P} .

REMARK 5.3.3. Hence we may speak of morphisms of algebraic spaces being *open immersions*, *closed immersions*, etc., keeping in mind that we are considering representable morphisms only.

For the non-representable case we need to impose stronger conditions on \mathbf{P} to be able to extend it to morphisms of algebraic spaces. A good definition will agree with Definition 5.3.2, when restricting to the class of representable morphisms.

Let X be a scheme and suppose that we have a morphism $X \rightarrow \mathcal{B}$. Then $X \times_{\mathcal{B}} \mathcal{A}$ is an algebraic space. Let $U \rightarrow X \times_{\mathcal{B}} \mathcal{A}$ be a representable étale covering,

$$\begin{array}{ccccc}
 U & \longrightarrow & X \times_{\mathcal{B}} \mathcal{A} & \longrightarrow & X \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mathcal{A} & \longrightarrow & \mathcal{B}
 \end{array}$$

If $X \times_{\mathcal{B}} \mathcal{A}$ is representable, say by a scheme Z , then $U \rightarrow Z$ is an étale covering. Let \mathbf{P} be a property of morphisms of schemes which is étale local on the domain. Then $X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} if and only if the composition $U \rightarrow X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} .

LEMMA 5.3.4. *Let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of algebraic spaces and let $V \rightarrow \mathcal{B}$ and $U' \rightarrow \mathcal{A} \times_{\mathcal{B}} V$ be representable étale coverings. Let \mathbf{P} be a property of morphisms*

of schemes which is stable under base change, étale local on the base, and étale local on the domain. Then the following are equivalent:

- (1) $U' \rightarrow V$ has \mathbf{P} ;
- (2) for every scheme X , every morphism $X \rightarrow \mathcal{B}$, and every representable étale covering $U \rightarrow X \times_{\mathcal{B}} \mathcal{A}$, the composition $U \rightarrow X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} .

PROOF. We obviously need only show that (1) implies (2). Let $X \rightarrow \mathcal{B}$ and $U \rightarrow \mathcal{A} \times_{\mathcal{B}} X$ be as in (2). By definition of a representable étale covering, we have that $X \times_{\mathcal{B}} V$ is representable and hence $U' \times_V (V \times_{\mathcal{B}} X) = U' \times_{\mathcal{B}} X$ is a scheme. Suppose that $U' \rightarrow V$ has \mathbf{P} . Since \mathbf{P} is stable under base change, we get that $U' \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X$ has \mathbf{P} .

The morphism $U' \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X$ factors through

$$(V \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} (\mathcal{A} \times_{\mathcal{B}} X) = \mathcal{A} \times_{\mathcal{B}} V \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X$$

since

$$U' \times_{\mathcal{B}} X = U' \times_{\mathcal{A} \times_{\mathcal{B}} V} (\mathcal{A} \times_{\mathcal{B}} V \times_{\mathcal{B}} X).$$

Hence we have a morphism $U' \times_{\mathcal{B}} X \rightarrow \mathcal{A} \times_{\mathcal{B}} X$ and we may consider the fiber product

$$(U' \times_{\mathcal{B}} X) \times_{\mathcal{A} \times_{\mathcal{B}} X} U = U' \times_{\mathcal{A}} U.$$

By Theorem 5.2.10 we have that $U' \times_{\mathcal{A}} U$ is a scheme and since $U \rightarrow \mathcal{A}$ is a representable étale covering we get that $U' \times_{\mathcal{A}} U \rightarrow U' \times_{\mathcal{B}} X$ is an étale covering. Since \mathbf{P} is étale local on the domain, we get that the composition

$$U' \times_{\mathcal{A}} U \rightarrow U' \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X$$

has \mathbf{P} . But $U' \times_{\mathcal{A}} U$ is the fiber product of $U' \times_{\mathcal{B}} X$ and $U \times_{\mathcal{B}} V$ over $\mathcal{A} \times_{\mathcal{B}} V \times_{\mathcal{B}} X$, which implies that the morphism

$$U' \times_{\mathcal{A}} U \rightarrow U' \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X$$

is the same as the morphism

$$U' \times_{\mathcal{A}} U \rightarrow U \times_{\mathcal{B}} V \rightarrow \mathcal{A} \times_{\mathcal{B}} V \times_{\mathcal{B}} X \rightarrow V \times_{\mathcal{B}} X.$$

We have that $U \times_{\mathcal{B}} V$ is a scheme and $U' \times_{\mathcal{A}} U \rightarrow U \times_{\mathcal{B}} V$ is an étale covering. Again, since \mathbf{P} is étale local on the domain, we get that $U \times_{\mathcal{B}} V \rightarrow V \times_{\mathcal{B}} X$ has \mathbf{P} . But \mathbf{P} is étale local on the base and since $V \times_{\mathcal{B}} X$ is an étale covering and $U \times_{\mathcal{B}} V = U \times_X (V \times_{\mathcal{B}} X)$ we get that $U \rightarrow X$ has \mathbf{P} . This proves the lemma. The argument is summarized in the following diagram:

$$\begin{array}{ccccccc}
 & & U \times_{\mathcal{B}} V & \longrightarrow & U & \longrightarrow & \mathcal{A} \times_{\mathcal{B}} X & \longrightarrow & X & & \\
 & \nearrow & & \searrow & & & \nearrow & & \searrow & & \\
 U' \times_{\mathcal{A}} U & & & & & & \mathcal{A} \times_{\mathcal{B}} V \times_{\mathcal{B}} X & \longrightarrow & V \times_{\mathcal{B}} X & & \\
 & \searrow & & \nearrow & & & \searrow & & \nearrow & & \\
 & & U' \times_{\mathcal{B}} X & \longrightarrow & U' & \longrightarrow & \mathcal{A} \times_{\mathcal{B}} V & \longrightarrow & V & & \\
 & & & & & & \nearrow & & \searrow & & \\
 & & & & & & & & \mathcal{A} & \longrightarrow & \mathcal{B} \\
 & & & & & & & & \nearrow & & \\
 & & & & & & & & & & V & \longrightarrow & \mathcal{B}
 \end{array}$$

□

Hence we make the following definition:

DEFINITION 5.3.5. Let $\alpha: \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of algebraic spaces and let \mathbf{P} be a property of morphisms over S which is stable under base change, étale local on the base, and étale local on the domain. Then we say that α has \mathbf{P} if for some some representable étale covering $V \rightarrow \mathcal{B}$, and some representable étale covering $U \rightarrow X \times_{\mathcal{B}} \mathcal{A}$, the composition $U \rightarrow X \times_{\mathcal{B}} \mathcal{A} \rightarrow X$ has \mathbf{P} .

REMARK 5.3.6. Hence we may speak of morphisms of algebraic spaces being, *surjective, étale, flat, faithfully flat, locally of finite type, unramified*, etc.

LEMMA 5.3.7. *Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in (Alg_S) . The following are equivalent:*

- (1) *f is étale and surjective;*
- (2) *f is étale and an epimorphism in $\text{Sh}(S_{\text{ét}})$;*

PROOF. (1) \Rightarrow (2): We will show that f is locally surjective in $\text{Sh}(S_{\text{ét}})$. Let X be an S -scheme and suppose that we have a morphism $X \rightarrow \mathcal{B}$. Let $U \rightarrow \mathcal{A} \times_{\mathcal{B}} X$ be a representable étale covering. By Lemma 5.3.4, the induced morphism $U \rightarrow X$ is surjective and étale and hence an étale covering of schemes. Thus f is locally surjective and hence an epimorphism in $\text{Sh}(S_{\text{ét}})$.

(2) \Rightarrow (1): If f is an étale epimorphism then f is locally surjective and hence, for every representable étale covering $V \rightarrow \mathcal{B}$, there is an étale covering $V' \rightarrow V$ such that the diagram

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

commutes. Hence $V' \rightarrow V$ factors through $\mathcal{A} \times_{\mathcal{B}} V$. Let $U \rightarrow \mathcal{A} \times_{\mathcal{B}} V$ be a representable étale covering. Then $U \rightarrow \mathcal{A} \times_{\mathcal{B}} V \rightarrow V$ is étale, since f is étale. Hence $V' \amalg U \rightarrow \mathcal{A} \times_{\mathcal{B}} V$ is a representable étale covering and the composition $V' \amalg U \rightarrow \mathcal{A} \times_{\mathcal{B}} V \rightarrow V$ is surjective and étale. \square

5.4. Étale topology on algebraic spaces

All examples of sites of schemes in Section 3.1 may be extended to algebraic spaces. Here are some examples.

EXAMPLE 5.4.1 (Big étale site on (Alg/\mathcal{B}) or (Alg)). The underlying category has morphisms $\mathcal{X} \rightarrow \mathcal{B}$ of algebraic spaces as objects and commutative diagrams

$$\begin{array}{ccc} \mathcal{X}_1 & \longrightarrow & \mathcal{X}_2 \\ & \searrow & \swarrow \\ & \mathcal{B} & \end{array}$$

as morphisms. The coverings are collections $\{U_i \rightarrow U\}$ of étale morphisms over \mathcal{B} such that $\coprod U_i \rightarrow U$ is étale and surjective. This site is denoted $\mathcal{B}_{\text{ét}}$.

Note that if U and the U_i are schemes, then $\{U_i \rightarrow U\}$ is an étale covering of schemes.

EXAMPLE 5.4.2 (Small étale site on \mathcal{A}). The underlying category is $\text{ét}(\mathcal{A})$, i.e., it has étale morphisms $\mathcal{X} \rightarrow \mathcal{A}$ of algebraic spaces as objects and commutative diagrams

$$\begin{array}{ccc} \mathcal{X}_1 & \longrightarrow & \mathcal{X}_2 \\ & \searrow & \swarrow \\ & \mathcal{A} & \end{array}$$

as morphisms. The coverings are collections $\{U_i \rightarrow U\}$ of étale morphisms over \mathcal{A} such that $\coprod U_i \rightarrow U$ is étale and surjective. This site is denoted $\mathcal{A}_{\text{ét}}$.

Note that if U and the U_i are schemes, then $\{U_i \rightarrow U\}$ is an étale covering of schemes.

REMARK 5.4.3. Let X be a scheme. Every sheaf on $X_{\text{ét}}$ extends uniquely to a sheaf on étale algebraic spaces over X . Conversely, every sheaf on étale algebraic spaces over X is completely determined by its restriction to étale schemes over X . Indeed, if \mathcal{F} is a sheaf on étale algebraic spaces and $R \rightrightarrows U \rightarrow \mathcal{A}$ is an algebraic space, then $\mathcal{F}(\mathcal{A})$ is completely determined by $\mathcal{F}(U) \rightrightarrows \mathcal{F}(R)$. If \mathcal{F} is a sheaf on étale schemes over X , then we may define $\mathcal{F}(\mathcal{A})$ to be the equalizer of $\mathcal{F}(U) \rightrightarrows \mathcal{F}(R)$ in (Set) .

This implies that the presheaf $(\text{Alg}_S/\mathcal{A})^{\text{op}} \rightarrow (\text{Set})$ defined by an algebraic space X over \mathcal{A} is a sheaf on $\mathcal{A}_{\text{ét}}$. Indeed, X is a sheaf when restricted to S -schemes over \mathcal{A} since a covering $\{U_i \rightarrow U\}$ in $\mathcal{A}_{\text{ét}}$ where U and the U_i are schemes is also a covering in $S_{\text{ét}}$ when considered as morphisms of S -schemes. By the discussion above, X extends uniquely to a sheaf on $\mathcal{A}_{\text{ét}}$.

Each scheme X has a structure sheaf $\mathcal{O}_{X_{\text{ét}}}$ defined in Example 3.5.6. The structure sheaf $\mathcal{O}_{\mathcal{A}_{\text{ét}}}$ on $\mathcal{A}_{\text{ét}}$ is defined by $\mathcal{O}_{\mathcal{A}_{\text{ét}}}(U) := \Gamma(U, \mathcal{O}_U)$ when U is an étale scheme over \mathcal{A} . If \mathcal{B} is an étale algebraic space over \mathcal{A} and $R \rightrightarrows V \rightarrow \mathcal{B}$ is a presentation of \mathcal{B} , then $\mathcal{O}_{\mathcal{A}_{\text{ét}}}(\mathcal{B})$ is completely determined by $\mathcal{O}_{\mathcal{A}_{\text{ét}}}(V)$, $\mathcal{O}_{\mathcal{A}_{\text{ét}}}(R)$, and the sheaf criterion.

EXAMPLE 5.4.4 (The big fpqc site on \mathcal{A}). The underlying category is (Alg/\mathcal{A}) and coverings are collections $\{U_i \rightarrow U\}$ such that $\coprod U_i \rightarrow U$ is fpqc.

5.5. The sheaf space (espace étalé)

We will show that every sheaf \mathcal{F} on $S_{\text{ét}}$ is represented on $(\text{ét}/S)$ by a unique (up to isomorphism) étale algebraic space $\mathcal{F}^{\text{ét}}$ over S , i.e., every sheaf \mathcal{F} on $S_{\text{ét}}$ is the restriction of a unique (up to isomorphism) étale algebraic space $\mathcal{F}^{\text{ét}}$ to $S_{\text{ét}}$.

First we will show that the fiber product of two étale S -schemes over a sheaf on $S_{\text{ét}}$ is a scheme.

LEMMA 5.5.1. *Let \mathcal{F} be a sheaf on $S_{\text{ét}}$ and let $\Delta: \mathcal{F} \rightarrow \mathcal{F} \times_S \mathcal{F}$ be the diagonal. Then*

- (1) *for every étale S -scheme Z , and any S -morphism $Z \rightarrow \mathcal{F} \times_S \mathcal{F}$, we have that $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} Z$ is represented by an open subscheme of Z ;*
- (2) *for every étale S -schemes U and V , and any elements $\alpha \in \mathcal{F}(U)$ and $\beta \in \mathcal{F}(V)$, the sheaf fiber product $U \times_{\mathcal{F}} V$ in $\text{Sh}(S_{\text{ét}})$ is represented by an open subscheme of $U \times_S V$.*

PROOF. We first show that (1) and (2) are equivalent. It is clear that (1) implies (2) since $U \times_{\mathcal{F}} V = \mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} (U \times_S V)$. To show that (2) implies (1), note that a morphism $Z \rightarrow \mathcal{F} \times_S \mathcal{F}$ factors through the diagonal $Z \rightarrow Z \times_S Z$. By (1) we have that $Z \times_{\mathcal{F}} Z \rightarrow Z \times_S Z$ is an open immersion, and since

$$\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} Z = (Z \times_{\mathcal{F}} Z) \times_{Z \times_S Z} Z$$

and the property "open immersion" is preserved under base change, we get that $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} Z$ is represented by an open subscheme of Z . Hence (1) and (2) are equivalent.

Now we prove (1). To give a morphism $Z \rightarrow \mathcal{F} \times_S \mathcal{F}$ is to give two morphisms $\alpha, \beta: Z \rightarrow \mathcal{F}$ such that the compositions $Z \rightarrow \mathcal{F} \rightarrow S$ coincide. Let $W \subseteq Z$ be the subset of Z consisting of all points z such that $\alpha_{\bar{z}} = \beta_{\bar{z}}$. The claim is that W is open in Z . Indeed, if $\alpha_{\bar{z}} = \beta_{\bar{z}}$ then there is an étale neighborhood $\varphi: U \rightarrow Z$ of \bar{z} such that $\varphi^*(\alpha) = \varphi^*(\beta)$. The image $\varphi(U)$ is open in Z and W is the union of all such $\varphi(U)$ as z ranges over the points of Z . Thus we have an open immersion

$i : W \hookrightarrow Z$ and $i \circ \alpha = i \circ \beta$. This gives a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{i} & Z \\ i \circ \alpha \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\Delta} & \mathcal{F} \times_S \mathcal{F}. \end{array}$$

Now suppose that X is any étale S -scheme and suppose that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Z \\ \varphi \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\Delta} & \mathcal{F} \times_S \mathcal{F}. \end{array}$$

Then $\psi^*(\alpha) = \psi^*(\beta)$ and hence Lemma 4.5.2 implies that $\alpha_{\bar{z}} = \beta_{\bar{z}}$ for all points $z \in \psi(X)$. That is, ψ factors uniquely through i . Furthermore, $\varphi = \alpha \circ \psi = \beta \circ \psi$ and since ψ factors through i , so does φ . Hence we conclude that $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} Z$ is represented by W . This proves the lemma. \square

LEMMA 5.5.2. *Let S be a scheme and let $\text{ét}(S)$ be the category of étale algebraic spaces over S . The functor $\text{ét}(S) \rightarrow \text{Sh}(S_{\text{ét}})$ that takes an étale algebraic space \mathcal{A} to the restriction $\mathcal{A}|_{S_{\text{ét}}}$ is fully faithful.*

PROOF. Let \mathcal{A} and \mathcal{B} be étale algebraic spaces over S and let $U \rightarrow \mathcal{A}$ be a representable étale covering. Since \mathcal{A} is étale over S , we have that U is étale over S , and hence $\mathcal{A}(U) = (\mathcal{A}|_{S_{\text{ét}}})(U)$ and $\mathcal{B}(U) = (\mathcal{B}|_{S_{\text{ét}}})(U)$. Suppose that we have a morphism $\mathcal{A}|_{S_{\text{ét}}} \rightarrow \mathcal{B}|_{S_{\text{ét}}}$. Since $\mathcal{A}(U) = (\mathcal{A}|_{S_{\text{ét}}})(U)$ we get a morphism $U \rightarrow \mathcal{A}|_{S_{\text{ét}}} \rightarrow \mathcal{B}|_{S_{\text{ét}}}$ and since $\mathcal{B}(U) = (\mathcal{B}|_{S_{\text{ét}}})(U)$ we get a morphism $U \rightarrow \mathcal{B}$ such that the compositions $U \times_{\mathcal{A}} U \rightrightarrows U \rightarrow \mathcal{B}$ coincide. Since $U \times_{\mathcal{A}} U \rightrightarrows U \rightarrow \mathcal{A}$ is a coequalizer diagram in $\text{Sh}(S_{\text{ét}})$, we get a unique morphism $\mathcal{A} \rightarrow \mathcal{B}$ through which $U \rightarrow \mathcal{B}$ factors. \square

THEOREM 5.5.3 ([Mil80, V.1.5]). *Let S be a scheme and let \mathcal{F} be a sheaf on the site $S_{\text{ét}}$. Let $\pi : S_{\text{ét}} \rightarrow S$ be the restriction morphism (see Remark 4.1.3). Then there exists a unique locally separated algebraic space $\mathcal{F}^{\text{ét}}$ which is étale over S , and such that $\mathcal{F} = \pi_*(\mathcal{F}^{\text{ét}})$.*

PROOF. Define

$$U = \coprod_{(V, \varphi)} V,$$

where the disjoint union is over all pairs (V, φ) where V is an étale S -scheme and $\varphi \in \mathcal{F}(V)$. Hence we get a canonical element $s \in \mathcal{F}(U)$. By Lemma 5.5.1, $U \times_{\mathcal{F}} U$ is represented by an open subscheme R of $U \times_S U$. The morphisms $U \times_S U \rightrightarrows U$ are étale since U is étale over S , and $R \rightarrow U \times_S U$ is an open immersion. Hence $R \rightrightarrows U$ is an étale equivalence relation which gives us a locally separated algebraic space $\mathcal{F}^{\text{ét}}$.

Since $U \rightarrow \mathcal{F}^{\text{ét}}$ is an epimorphism in $\text{Sh}(S_{\text{ét}})$ it is also an epimorphism in $\text{Sh}(S_{\text{ét}})$. To show that \mathcal{F} and $\mathcal{F}^{\text{ét}}$ are equal as sheaves on $S_{\text{ét}}$, Lemma 3.4.9 implies that it is enough to show that $U \rightarrow \mathcal{F}$ is an epimorphism in $\text{Sh}(S_{\text{ét}})$, i.e., locally surjective. But this is obvious since every morphism $T \rightarrow \mathcal{F}$ from an étale S -scheme T to \mathcal{F} , factors through U .

Uniqueness follows from Lemma 5.5.2. \square

COROLLARY 5.5.4. *Let X be an algebraic space over a scheme S and let \mathcal{F} be a sheaf on $X_{\text{ét}}$. Then \mathcal{F} is represented by an algebraic space $\mathcal{F}^{\text{ét}}$ over S which is étale over X .*

PROOF. As in Theorem 5.5.3 we define

$$U = \coprod_{(V, \varphi)} V$$

where the disjoint union is over all pairs (V, φ) , where V is an étale scheme over X and $\varphi \in \mathcal{F}(V)$. The idea is to show that $U \times_{\mathcal{F}} U$ is represented by an algebraic space which in term is represented by a scheme R . Then we may define $\mathcal{F}^{\text{ét}}$ to be the algebraic space defined by $R \rightrightarrows U$ and check that $\mathcal{F}^{\text{ét}}$ represents \mathcal{F} and is étale over X .

By Theorem 5.2.10 we have that $U \times_X U$ is a scheme and hence Lemma 5.5.1 implies that $U \times_{\mathcal{F}} U$ is represented by an open subscheme $R \subseteq U \times_X U$.

Define $\mathcal{F}^{\text{ét}}$ to be the algebraic space given as the quotient of $R \rightrightarrows U$. We know that \mathcal{F} is completely determined by its restriction to the category of étale schemes over X . It is obvious that the canonical morphism $U \rightarrow \mathcal{F}$ is locally surjective as a morphism of sheaves on schemes over X , since any morphism $\varphi: V \rightarrow \mathcal{F}$ is part of the disjoint union defining U . Thus $R \rightrightarrows U \rightarrow \mathcal{F}$ is a coequalizer diagram in the category of sheaves on the the restriction of the site $X_{\text{ét}}$ to S -schemes that are étale over X . This means that $\mathcal{F}^{\text{ét}}$ and \mathcal{F} agrees as sheaves on étale schemes over X , and by Remark 5.4.3 we conclude that they also agree as sheaves on $X_{\text{ét}}$. \square

DEFINITION 5.5.5. The algebraic space $\mathcal{F}^{\text{ét}}$ of Theorem 5.5.3 and Corollary 5.5.4 is called the *espace étalé* or *sheaf space* of \mathcal{F} .

REMARK 5.5.6. Theorem 5.5.3 and Lemma 5.5.2 implies that there is an equivalence of categories between the category $\text{ét}(S)$ of étale algebraic spaces over S and the category $\text{Sh}(S_{\text{ét}})$ of sheaves on $S_{\text{ét}}$.

Let $\pi: S_{\text{ét}} \rightarrow S_{\text{ét}}$ be the restriction morphism. Let \mathcal{F} be a sheaf on $S_{\text{ét}}$. The pullback $\pi^*\mathcal{F}$ is defined by $\pi^*\mathcal{F}(V) = \varinjlim \mathcal{F}(V')$ where the limit is over commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & V' \\ & \searrow & \swarrow \\ & & S \end{array} \quad \text{ét}$$

Given a morphism $\varphi: V \rightarrow \mathcal{F}$ from an étale S -scheme V to \mathcal{F} , we have $\pi^*\mathcal{F}(V) = \mathcal{F}(V)$ and we get a canonical morphism $\varphi: V \rightarrow \pi^*\mathcal{F}$. Thus, we get a canonical morphism $\psi: U \rightarrow \pi^*\mathcal{F}$ where U is defined as in Theorem 5.5.3. We will show that ψ is an epimorphism and hence that $\pi^*\mathcal{F} = \mathcal{F}^{\text{ét}}$.

LEMMA 5.5.7. *With the setup in Theorem 5.5.3, we have $\mathcal{F}^{\text{ét}} = \pi^*\mathcal{F}$.*

PROOF. It suffices to show that $\psi: U \rightarrow \pi^*\mathcal{F}$ is locally surjective. Let T be an S -scheme and take $\alpha \in \pi^*\mathcal{F}(T)$. Then there exists a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{f} & V \\ & \searrow & \swarrow \\ & & S \end{array} \quad \text{ét}$$

and a $\beta \in \mathcal{F}(V) = \pi^*\mathcal{F}(V)$ such that α is the image of β under the induced map $f^*: \pi^*\mathcal{F}(V) \rightarrow \pi^*\mathcal{F}(T)$. But f^* is the map defined by taking a morphism $V \rightarrow \pi^*\mathcal{F}$ to the composition $T \rightarrow V \rightarrow \pi^*\mathcal{F}$, i.e., $\alpha: T \rightarrow \pi^*\mathcal{F}$ factors through $\beta: V \rightarrow \pi^*\mathcal{F}$. On the other hand, β may be considered as a morphism $V \rightarrow \mathcal{F}$ and hence we have

$V \hookrightarrow U$ and by definition of $\psi: U \rightarrow \pi^*\mathcal{F}$, we also have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ & \searrow \beta & \swarrow \psi \\ & & \pi^*\mathcal{F} \end{array}$$

Hence we conclude that $\alpha: T \rightarrow \pi^*\mathcal{F}$ factors through $U \rightarrow \pi^*\mathcal{F}$. That is, ψ is locally surjective. \square

DEFINITION 5.5.8. Let S be a scheme and let $\pi: S_{\acute{e}t} \rightarrow S_{\acute{e}t}$ be the restriction morphism (see Remark 4.1.3). Then we make the following definitions:

- (1) A sheaf \mathcal{F} on $S_{\acute{e}t}$ is called *locally constructible* if $\mathcal{F} \cong \pi^*\pi_*\mathcal{F}$;
- (2) A sheaf \mathcal{F} on $S_{\acute{e}t}$ is called *constructible* if $\mathcal{F}^{\acute{e}t}$ is of finite type over S ;
- (3) A sheaf \mathcal{F} on $S_{\acute{e}t}$ is called *constructible* if it is locally constructible and $(\pi_*\mathcal{F})^{\acute{e}t}$ is of finite type.

Lemma 5.5.7 implies the following corollary:

COROLLARY 5.5.9. *Let S be a scheme. The following are equivalent:*

- (1) \mathcal{F} is a locally constructible sheaf on $S_{\acute{e}t}$;
- (2) \mathcal{F} is an étale algebraic space over S .

5.6. The functors f_* , f^* , and $f_!$ of a morphism of algebraic spaces

Let $f: X \rightarrow Y$ be a morphism of algebraic spaces. The discussion in the beginning of Section 4.1 also holds for algebraic spaces instead of schemes and hence we get a continuous functor

$$\begin{aligned} f^\times: \acute{e}t(Y) &\rightarrow \acute{e}t(X) \\ (U \rightarrow Y) &\mapsto (U \times_Y X \rightarrow X). \end{aligned}$$

Thus we may define functors $f_*: \text{Sh}(X_E) \rightarrow \text{Sh}(Y_E)$ and $f^*: \text{Sh}(Y_E) \rightarrow \text{Sh}(X_E)$ using the recipe of Section 4.2.

When f is an E-morphism we may show, in the exact same manner as before, that $f^* = \hat{f}_*$ where $\hat{f}: Y_E \rightarrow X_E$ is the morphism given by the continuous functor which takes an étale $U \rightarrow X$ to the composition

$$U \rightarrow X \xrightarrow{f} Y.$$

Again we define $f_!$ to be the left adjoint \hat{f}^* of $\hat{f}_* = f^*$.

Let \mathcal{G} be a sheaf on the small étale site $S_{\acute{e}t}$ of a scheme S and let $f: T \rightarrow S$ be any morphism of schemes (or algebraic spaces). Then f is an object in $S_{\acute{e}t}$ and we may consider the functors in Section 4.4. We showed in Remark 4.4.2 (or Lemma 4.2.9) that

$$f^*(\mathcal{G}^{\acute{e}t}) = T \times_S \mathcal{G}^{\acute{e}t}.$$

Now assume that S and T are algebraic spaces that are not necessarily schemes and that $f: T \rightarrow S$ is any morphism. If \mathcal{G} is a sheaf on $S_{\acute{e}t}$ then $\mathcal{G}^{\acute{e}t}$ is an algebraic space which represents \mathcal{G} on $\acute{e}t(S)$.

LEMMA 5.6.1. *Let $f: T \rightarrow S$ be any morphism of algebraic spaces and let $X \rightarrow S$ be étale. Let π be the restriction morphism from the big étale site to the small étale site, and let h_X be the representable sheaf of X on the big étale site. Then we have*

$$f^*\pi_*h_X \cong \pi_*f^*h_X.$$

PROOF. We have $\pi_* f^* h_X = \pi_* h_{T \times_S X}$. The sheaf $f^* \pi_* h_X$ is the sheafification of the presheaf \mathcal{P} given by

$$\mathcal{P}(U) = \varinjlim \pi_* h_X(V)$$

for every étale $U \rightarrow T$, where the colimit is over all commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

with $V \rightarrow S$ étale. Thus, an element in such a $h_X(V)$ is given by a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & X \\ \downarrow & & \downarrow & \swarrow & \\ T & \longrightarrow & S & & \end{array}$$

and hence we have compatible maps $h_X(V) \rightarrow h_{T \times_S X}(U)$. This gives a morphism $\tau: \mathcal{P} \rightarrow \pi_* h_{T \times_S X}$. Since X is étale over S we get that $\tau_U: \mathcal{P}(U) \rightarrow \pi_* h_{T \times_S X}(U)$ is surjective for each étale $U \rightarrow T$. On the other hand, if we have two morphisms $\alpha, \beta: V \rightrightarrows X$ such that the compositions $U \rightarrow V \rightrightarrows X$ coincides, then we get a commutative diagram

$$\begin{array}{ccccc} U & & & & \\ & \searrow & & \searrow & \\ & & V \times_X V & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \beta \\ & & V & \xrightarrow{\alpha} & X \end{array}$$

where $V \times_X V \rightarrow V$ is étale since β is étale. Hence we conclude that τ_U is also injective and hence τ is an isomorphism. In particular, \mathcal{P} is a sheaf and hence $f^* \pi_* h_X = \mathcal{P} \cong \pi_* f^* h_X$. \square

COROLLARY 5.6.2. *Let S and T be algebraic spaces. Let \mathcal{G} be a sheaf on $S_{\text{ét}}$ and let $f: T \rightarrow S$ be any morphism. Then we have an isomorphism*

$$f^* \mathcal{G} \cong \pi_*(T \times_S \mathcal{G}^{\text{ét}})$$

of sheaves on $T_{\text{ét}}$, where the fiber product is taken in (Alg).

Let $f: T \rightarrow S$ be an étale morphism of algebraic spaces and let \mathcal{F} be a sheaf on $T_{\text{ét}}$. The espace étalé $\mathcal{F}^{\text{ét}}$ is étale over T and if we compose with f we get an étale morphism $\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S$. We know that the functor f^* exists and has a left adjoint $f_!$. The claim is that $\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S$ is equal to $(f_! \mathcal{F})^{\text{ét}} \rightarrow S$.

LEMMA 5.6.3. *The étale algebraic space $(f_! \mathcal{F})^{\text{ét}} \rightarrow S$ is equal to $\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S$.*

PROOF. We show that the functor $\text{ét}(T) \rightarrow \text{ét}(S)$ which takes $\mathcal{F}^{\text{ét}} \rightarrow T$ to $\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S$ is a left adjoint to f^* . For an étale algebraic space $\mathcal{G}^{\text{ét}}$ over S we

have $f^*\mathcal{G}^{\text{ét}} = T \times_S \mathcal{G}^{\text{ét}}$, and hence this is obvious from the following diagram:

$$\begin{array}{ccccc}
 \mathcal{F}^{\text{ét}} & & & & \\
 \searrow & & & & \\
 & T \times_S \mathcal{G}^{\text{ét}} & \longrightarrow & \mathcal{G}^{\text{ét}} & \\
 \searrow & \downarrow & & \downarrow & \\
 & T & \longrightarrow & S &
 \end{array}$$

There is a one-to-one correspondence between morphisms $\mathcal{F}^{\text{ét}} \rightarrow T \times_S \mathcal{G}^{\text{ét}}$ over T and morphisms $\mathcal{F}^{\text{ét}} \rightarrow \mathcal{G}^{\text{ét}}$ over S . This proves the lemma. \square

With the same setup as above, Lemma 4.2.9 implies that $\hat{f}^p\mathcal{F}$ is a sheaf since it is represented by the algebraic space $\mathcal{F}^{\text{ét}}$. Indeed, we have that $\hat{f}^p\mathcal{F}^{\text{ét}} = (\hat{f})^{\text{cat}}\mathcal{F}^{\text{ét}} = (\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S)$. Let $\psi: V \rightarrow S$ be an étale algebraic space over S . A morphism $V \rightarrow \mathcal{F}^{\text{ét}}$ gives a commutative diagram

$$\begin{array}{ccccc}
 V & & & & \\
 \downarrow & \searrow & & & \\
 \mathcal{F}^{\text{ét}} & \longrightarrow & T & \longrightarrow & S
 \end{array}$$

and hence we conclude that

$$\text{Hom}_{\text{ét}(S)}(V, (f_!\mathcal{F})^{\text{ét}}) = \coprod_{\varphi} \mathcal{F}^{\text{ét}}(V) = \coprod_{\varphi} \mathcal{F}(V)$$

where the disjoint unions are over all morphisms $\varphi: V \rightarrow T$ such that $\psi = f \circ \varphi$. But this is just $\hat{f}^p\mathcal{F}(V)$. Since $\text{Hom}_{\text{ét}(S)}(V, (f_!\mathcal{F})^{\text{ét}}) = f_!\mathcal{F}(V)$ and $(f_!\mathcal{F})^{\text{ét}} \rightarrow S$ is just the étale algebraic space $\mathcal{F}^{\text{ét}} \rightarrow T \rightarrow S$ we get the following explicit description of $f_!\mathcal{F}$:

LEMMA 5.6.4. *Let $f: T \rightarrow S$ be an étale morphism of algebraic spaces (or just schemes) and let \mathcal{F} be a sheaf on $T_{\text{ét}}$. Then for any étale $\psi: V \rightarrow S$, we have*

$$f_!\mathcal{F}(V) = \coprod_{\varphi} \mathcal{F}(V),$$

where the disjoint union is over all morphisms $\varphi: V \rightarrow T$ such that $\psi = f \circ \varphi$.

5.7. The functor $f_!$ for sheaves of pointed sets

Again, let $f: T \rightarrow S$ be an étale morphism of algebraic spaces. In case \mathcal{F} is a sheaf of pointed sets, we have the presheaf construction

$$(\psi: U \rightarrow S) \mapsto \bigvee_{\varphi} \mathcal{F}(U)$$

where the wedge is over all morphisms $\varphi: U \rightarrow T$ such that $f \circ \varphi = \psi$. Denote this presheaf by \mathcal{P} . The presheaf \mathcal{P} need not be a sheaf and hence we also need to take the sheafification of \mathcal{P} to get the sheaf $f_!\mathcal{F}$ of pointed sets. The following lemma gives a more explicit description of $f_!\mathcal{F}$.

LEMMA 5.7.1. *Let $f: T \rightarrow S$ be an étale morphism of algebraic spaces and let \mathcal{F} be a sheaf of pointed sets on $T_{\text{ét}}$. Let $\{0\}_T$ and $\{0\}_S$ be the constant sheaves of the set $\{0\}$ on $T_{\text{ét}}$ and $S_{\text{ét}}$ respectively. Let $f_!$ denote the functor of pointed sets*

defined as the left adjoint of f^* and let $f'_!$ be the corresponding functor of sheaves of sets. Then $f'_!\mathcal{F} \in \mathbf{Sh}_*(S_{\acute{e}t})$ is the pushout in $\mathbf{Sh}(S_{\acute{e}t})$ of the diagram

$$\begin{array}{ccc} f'_!\{0\}_T & \longrightarrow & \{0\}_S \\ \downarrow & & \\ f'_!\mathcal{F} & & \end{array}$$

PROOF. For any sheaf of sets \mathcal{G} on $T_{\acute{e}t}$ we have the explicit description $f'_!\mathcal{G}(U) = \coprod_{\varphi} \mathcal{G}(U)$ as in Lemma 5.6.4. The morphism $f'_!\{0\}_T \rightarrow f'_!\mathcal{F}$ is induced by the unique morphism $\{0\}_T \rightarrow \mathcal{F}$ and can be described as follows for each étale $\psi: U \rightarrow S$: we have $f'_!\{0\}_T(U) = \coprod_{\varphi} \{0\}_T(U)$ and for each factorization $U \rightarrow T \rightarrow S$ the unique point in $\{0\}_T(U \rightarrow T)$ will be mapped to the distinguished point in $\mathcal{F}(U \rightarrow T)$. The pushout is a sheaf of pointed sets by taking the image of

$$\{0\}_S(U) \rightarrow \left(f'_!\mathcal{F} \coprod_{f'_!\{0\}_T} \{0\}_S \right) (U)$$

as the distinguished point for each étale $U \rightarrow S$. Let \mathcal{P} be the presheaf on $S_{\acute{e}t}$ defined by

$$\mathcal{P}(U) = \bigvee_{\varphi} \mathcal{F}(U),$$

for each étale $\psi: U \rightarrow S$, where the wedge is over all morphisms $\varphi: U \rightarrow T$ such that $f \circ \varphi = \psi$. The lemma now follows from the fact that the diagram

$$\begin{array}{ccc} f'_!\{0\}_T & \longrightarrow & \{0\}_S \\ \downarrow & & \downarrow \\ f'_!\mathcal{F} & \longrightarrow & \mathcal{P} \end{array}$$

is a pushout diagram in the category of presheaves $(\acute{e}t/S)^{\text{op}} \rightarrow (\mathbf{Set})$. Indeed, for every étale $U \rightarrow S$ we have that for any set M , a commutative diagram

$$\begin{array}{ccc} f'_!\{0\}_T(U) & \longrightarrow & \{0\}_S(U) \\ \downarrow & & \downarrow \\ f'_!\mathcal{F}(U) & \longrightarrow & M \end{array}$$

sends the distinguished point in $\mathcal{F}(U \rightarrow T)$ to the image of $\{0\}_S(U)$ in M for each factorization $U \rightarrow T \rightarrow S$. If there is no factorization $U \rightarrow T \rightarrow S$ then $f'_!\{0\}_T(U) = f'_!\mathcal{F}(U) = \emptyset$ and $\{0\}_S(U) = \mathcal{P}(U) = \{0\}$. Hence both maps to M factors uniquely through $\mathcal{P}(U)$. \square

5.8. Direct image with proper support

Let $f: T \rightarrow S$ be a representable, separated, and locally of finite type morphism of algebraic spaces. Define a functor

$$f_c: \mathbf{Sh}_*((\text{Sch}/T)_{\acute{e}t}) \rightarrow \mathbf{Sh}_*((\text{Sch}/S)_{\acute{e}t})$$

called the *direct image with proper support*.

If $\varphi: (M, p) \rightarrow (N, q)$ is a map of pointed sets, then we define its *kernel* $\ker(\varphi)$ to be the sets of points $x \in M$ such that $\varphi(x) = q$, i.e.,

$$\ker(\varphi) = \{x \in M : \varphi(x) = q\}.$$

DEFINITION 5.8.1. Let \mathcal{F} be a sheaf of pointed sets on $(\text{Sch}/T)_{\text{ét}}$. For any étale $U \rightarrow S$ where U is a scheme, we define

$$f_c\mathcal{F}(U) = \bigcup_Z \ker(\mathcal{F}(U \times_S T) \rightarrow \mathcal{F}((U \times_S T) \setminus Z)),$$

where the union is over all closed subschemes $Z \subseteq U \times_S T$ such that the restriction $Z \rightarrow U$ is proper.

DEFINITION 5.8.2. Let \mathcal{F} be a sheaf of pointed sets on $S_{\text{ét}}$ and let $\varphi: U \rightarrow S$ be an étale S -scheme. For every $s \in \mathcal{F}(U)$ and every geometric point $\bar{x}: \text{Spec } \Omega \rightarrow U$, we define $s_{\bar{x}}$ to be the image of s in $(\varphi^*\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}$. We define the *support* of s to be the subspace

$$\text{supp}(s) = \{x \in U : s_{\bar{x}} \neq *\},$$

where $*$ is the distinguished point in $(\varphi^*\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}$.

REMARK 5.8.3. Note that the subset $\text{supp}(s)$ is closed in U . Indeed, if $s_{\bar{x}} = *$ then there exists an étale neighborhood $\psi: V \rightarrow U$ of \bar{x} such that $\psi^*(s) = *$. The image $\psi(V)$ is open in U and the restriction $V \rightarrow \psi(V)$ is an étale covering. Hence we get that s maps to $*$ in $\mathcal{F}(\psi(V))$ since $\mathcal{F}(\psi(V)) \rightarrow \mathcal{F}(V)$ is injective. Hence we get that $\text{supp}(s)$ is just the complement of the union of all open subschemes $i: W \hookrightarrow U$ such that $i^*(s) = *$.

LEMMA 5.8.4. *Let $\varphi: V \rightarrow U$ be an étale morphism of schemes and let \mathcal{F} be a sheaf of pointed sets on $U_{\text{ét}}$. Take any $s \in \mathcal{F}(U)$ and let $s' = \varphi^*(s) \in \mathcal{F}(V)$. Then we have that*

$$\text{supp}(s') = \varphi^{-1}(\text{supp}(s)).$$

PROOF. Suppose that we have a point $u \in \text{supp}(s)$ and $v \in V$ such that $\varphi(v) = u$. Let $\bar{v}: \text{Spec } \Omega \rightarrow V$ be a geometric point with image v and let $\bar{u} = \varphi \circ \bar{v}$. Since $\mathcal{F}_{\bar{v}} = \bar{v}^*\mathcal{F}(\text{Spec } \Omega)$ it follows that $s'_{\bar{v}} = \bar{v}^*\varphi^*(s) = \bar{u}^*(s) = s_{\bar{u}}$. This proves the lemma. \square

LEMMA 5.8.5. *We have that*

$$f_c\mathcal{F}(U) = \{s \in \mathcal{F}(U \times_S T) : \text{supp}(s) \rightarrow U \text{ is proper}\},$$

where $\text{supp}(s)$ is given the reduced induced closed subscheme structure [Har77, Example 3.2.6].

PROOF. Let $M = \{s \in \mathcal{F}(U \times_S T) : \text{supp}(s) \rightarrow U \text{ is proper}\}$. Clearly $M \subseteq f_c\mathcal{F}(U)$ since

$$s \in \ker(\mathcal{F}(U \times_S T) \rightarrow \mathcal{F}(U \times_S T \setminus \text{supp}(s))).$$

Conversely, if $s \in f_c\mathcal{F}(U)$ then there is a closed subscheme $Z \subseteq U \times_S T$ which is proper over U , such that $s \in \ker(\mathcal{F}(U \times_S T) \rightarrow \mathcal{F}(U \times_S T \setminus Z))$. But then $\text{supp}(s) \subseteq Z$ and since $\text{supp}(s)$ is closed in Z we get that $\text{supp}(s)$ is proper over U . \square

LEMMA 5.8.6. *The presheaf $f_c\mathcal{F}$ is a sheaf on $(\text{Sch}/S)_{\text{ét}}$.*

PROOF. It is clear that $f_c\mathcal{F}$ is separated since $f_*\mathcal{F}$ is a sheaf and we have an injective map $f_c\mathcal{F}(U) \rightarrow f_*\mathcal{F}(U)$ for each $U \rightarrow U \times_Y X$, which is natural in U . Now suppose that we have an element $(s_i)_i \in \prod f_c\mathcal{F}(U_i)$ such that the two images of $(s_i)_i$ in $\prod f_c\mathcal{F}(U_i \times_U U_j)$ coincide. Since $f_*\mathcal{F}$ is a sheaf we get that $(s_i)_i$ comes from a unique element $s \in f_*\mathcal{F}(U)$. We need to prove that $\text{supp}(s)$ is proper over T .

If s_i has proper support for each i then the element $s' \in \mathcal{F}((\prod U_i) \times_S T)$ with image $(s_i)_i$ has proper support over $\prod U_i$. Let $V = \prod U_i$ and let $Z = \text{supp}(s')$. The induced morphism $\varphi: V \times_S T \rightarrow U \times_S T$ is an étale covering since $V \rightarrow U$

is an étale covering. Hence Lemma 5.8.4 implies that $Z = \varphi^{-1}(\varphi(Z))$. Hence we conclude that $Z = V \times_U \varphi(Z)$. The property "proper" is étale local on the base [Sta, Tag 02L1] and hence if $Z \hookrightarrow V \times_S T$ is proper over V then $\varphi(Z)$ is proper over U . But $\varphi(Z)$ is the support of s . \square

LEMMA 5.8.7. *If f is proper then $f_c = f_*$.*

PROOF. Let $U \rightarrow S$ be étale. The restriction of a proper morphism to a closed subset is again proper and hence every section $s \in \mathcal{F}(U \times_S T)$ has proper support over U . \square

5.9. Connected fibration of a smooth morphism

Here we follow the construction in [LMB00, 6.8]. Let T and S be schemes and let $\pi: T \rightarrow S$ be a smooth morphism of finite type. For every S -scheme X we define a relation \sim on $\text{Hom}_{\text{Sch}/S}(X, T)$ as follows: Let $\alpha, \beta: X \rightarrow T$ be two morphisms over S . We say that $\alpha \sim \beta$ if for every algebraically closed field k and every morphism $\xi: \text{Spec } k \rightarrow X$, the two images of $\text{Spec } k$ in $\text{Spec } k \times_S T$ are in the same connected component. The relation \sim is an equivalence relation.

LEMMA 5.9.1. *There exists an open subscheme $R \subseteq T \times_S T$ such that $\alpha \sim \beta: X \rightarrow T$ if and only if the morphism $X \rightarrow T \times_S T$ induced by α and β factors uniquely through $R \hookrightarrow T \times_S T$. Furthermore, the induced morphisms $R \rightrightarrows T$ are smooth.*

PROOF. Let $p_1, p_2: T \times_S T \rightarrow T$ be the two projections and let $\Delta: T \rightarrow T \times_S T$ be the diagonal. We have that p_1 is smooth of finite presentation since p_1 is obtained by a base change from π . Let $\alpha, \beta: X \rightarrow T$ be morphisms over S and let x be a point in X . Then we have a morphism

$$\xi: \text{Spec } \overline{\kappa(x)} \rightarrow X.$$

For every $t \in T$, let R_t be the connected component in $p_1^{-1}(t)$ containing $\Delta(t)$ and define

$$R = \bigcup_{t \in T} R_t \subseteq T \times_S T.$$

But the compositions $\alpha \circ \xi$ and $\beta \circ \xi$ maps $\text{Spec } \overline{\kappa(x)}$ into T and hence we have a morphism $\text{Spec } \overline{\kappa(x)} \rightarrow T \times_S T$. Hence we conclude that $\alpha \sim \beta \Leftrightarrow \alpha(x)$ and $\beta(x)$ are in the same connected component in $\pi^{-1}(\pi \circ \alpha(x)) \subseteq T \times_S T \Leftrightarrow$ the image of x in $T \times_S T$ is mapped to the same component in T via the two projections. But this happens if and only if the image of x is in R . Indeed, the $\overline{\kappa(x)}$ -point $z \in T \times_S T$ with $p_1(z) = \alpha(x)$ and $p_2(z) = \beta(x)$ is in the same component in $p_1^{-1}(\alpha(x))$ as $\Delta(\alpha(x))$.

By [Gro67] we have that R is an open subscheme of $T \times_S T$ and hence we conclude that $X \rightarrow T \times_S T$ factors uniquely through $R \rightarrow T \times_S T$. Furthermore, since $f: T \rightarrow S$ is smooth, so are $T \times_S T \rightrightarrows T$. An open immersion is smooth and hence we conclude that the morphisms $R \rightarrow T \times_S T \rightarrow T$ are smooth. \square

LEMMA 5.9.2 ([Sta, Tag 0464, Tag 01TX]). *Let $f: X \rightarrow Y$ be a morphism of schemes where Y is locally Noetherian. Then the following are equivalent:*

- (1) f is locally of finite type;
- (2) f is locally of finite presentation.

Any open immersion into a locally Noetherian scheme is quasi-compact [Sta, Tag 01OX]. The following result is due to Artin, and implies that the sheaf quotient T/R is an algebraic space.

PROPOSITION 5.9.3 ([LMB00, Corollaire 10.4]). *Let R and T be schemes over S . Suppose that $R \rightrightarrows T$ is a categorial equivalence relation such that the morphisms $R \rightrightarrows T$ are flat and locally of finite presentation and the morphism $R \hookrightarrow T \times_S T$ is quasi-compact. Then T/R is an algebraic space.*

DEFINITION 5.9.4. The algebraic space T/R of Lemma 5.9.1 is denoted by $\pi_0(T/S)$.

LEMMA 5.9.5. *Let $\varphi: U \rightarrow S$ be any morphism. Then we have*

$$U \times_S \pi_0(T/S) \cong \pi_0(U \times_S T/U).$$

PROOF. Let R' be the open subscheme of $(T \times_S U) \times_U (T \times_S U)$ representing the equivalence relation on $T \times_S U$ giving rise to $\pi_0(T \times_S U/U)$. Since $T \rightarrow \pi_0(T/S)$ is a representable smooth covering, so is $T \times_S U \rightarrow \pi_0(T/S) \times_S U$. Hence it is enough to show that

$$\begin{aligned} R' &= (T \times_S U) \times_{\pi_0(T/S) \times_S U} (T \times_S U) \\ &= (T \times_{\pi_0(T/S)} T) \times_S U \\ &= R \times_S U. \end{aligned}$$

Since $R \hookrightarrow T \times_S T$ is an open immersion, we have that

$$R \times_S U \rightarrow (T \times_S U) \times_U (T \times_S U) = (T \times_S T) \times_S U$$

is an open immersion. The image of this open immersion is exactly R' . \square

LEMMA 5.9.6. *The algebraic space $\pi_0(T/S)$ is étale over S .*

PROOF. We have that $\pi_0(T/S) \rightarrow S$ is flat and locally of finite presentation since $T \rightarrow S$ is flat and locally of finite presentation and $T \rightarrow \pi_0(T/S)$ is a smooth cover. By Lemma 5.9.2, it is enough to check that $\pi_0(T/S) \rightarrow S$ is unramified. By Proposition 1.2.4, it is enough to check this after base change to a geometric point. By the previous lemma, we need only show that $\pi_0(T \times_S \text{Spec } \Omega/\text{Spec } \Omega) \rightarrow \text{Spec } \Omega$ is unramified, where Ω is a separably closed field. But $\pi_0(T \times_S \text{Spec } \Omega/\text{Spec } \Omega)$ is the scheme where we have identified all points in each connected component of $T \times_S \text{Spec } \Omega$. That is,

$$\pi_0(T \times_S \text{Spec } \Omega/\text{Spec } \Omega) = \text{Spec} \left(\Omega^{|\pi_0(T \times_S \text{Spec } \Omega)|} \right).$$

Hence $\pi_0(T \times_S \text{Spec } \Omega/\text{Spec } \Omega)$ is unramified over $\text{Spec } \Omega$ and we conclude that $\pi_0(T/S)$ is étale over S . \square

5.10. The functor $f_!$ of a non-étale morphism.

Wise showed in [Wis15] that f^* has a left adjoint also when f is flat, locally of finite presentation, and with reduced geometric fibers. Recall Remark 5.5.6, i.e., that there is an equivalence of categories between the category of étale algebraic spaces over the base and the category of sheaves on the small étale site on the base.

Existence of the algebraic space $\pi_0(U/V)$ when $U \rightarrow V$ is flat, locally of finite presentation, and with reduced geometric fibers follows from [Rom11, Théorème 2.5.2].

THEOREM 5.10.1 ([Wis15, Theorem 4.5]). *Let $f: T \rightarrow S$ be a morphism of algebraic spaces which is flat, locally of finite presentation, and with reduced geometric fibers. Then the functor $f^*: \text{Sh}(S_{\text{ét}}) \rightarrow \text{Sh}(T_{\text{ét}})$ has a left adjoint.*

PROOF IN THE CASE OF SCHEMES. Suppose that T, S are schemes, $\psi: Z \rightarrow T$ an étale T -scheme, and $\varphi: Y \rightarrow S$ an étale S -scheme. Let $R \rightrightarrows Z \rightarrow \pi_0(Z/S)$ be a presentation of $\pi_0(Z/S)$. We have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{(\mathrm{Sch}/S)}(Z, Y) &\simeq \mathrm{Hom}_{(\mathrm{Sch}/T)}(Z, T \times_S Y) \\ &\simeq \mathrm{Hom}_{\mathrm{ét}(T)}(Z, f^*Y), \end{aligned}$$

where the last bijection follows from Lemma 4.2.9. Hence it is enough to show that there is a natural bijection

$$\mathrm{Hom}_{(\mathrm{Sch}/S)}(Z, Y) \simeq \mathrm{Hom}_{\mathrm{ét}(S)}(\pi_0(Z/S), Y).$$

Let $g: Z \rightarrow Y$ be an S -morphism. Since Y is étale over S , it has discrete fibers and hence $Z \times_Y \mathrm{Spec} k$ is both open and closed in $Z \times_S \mathrm{Spec} k$ for each $\mathrm{Spec} k \rightarrow Y$. This implies that the compositions $R \rightrightarrows Z \rightarrow Y$ agree and hence g factors uniquely through $Z/R = \pi_0(Z/S)$. \square

CHAPTER 6

The étale envelope $E_{X/Y}$

Similarly as in Chapter 3, one may define sheaves of *pointed* sets. In [Ryd11] Rydh shows that every unramified morphism $X \rightarrow Y$ of algebraic spaces (algebraic stacks) has a canonical and universal factorization $X \hookrightarrow E_{X/Y} \rightarrow Y$, where $X \hookrightarrow E_{X/Y}$ is a closed immersion and $E_{X/Y} \rightarrow Y$ is étale.

6.1. The sheaf $E_{X/Y}$

DEFINITION 6.1.1. Let $f: X \rightarrow Y$ be an unramified morphism of algebraic spaces. We define a presheaf $E_{X/Y}: (\text{Sch}/Y)^{\text{op}} \rightarrow (\text{Set})$ as follows: For any Y -scheme U , let $E_{X/Y}(U)$ be the set of commutative diagrams

$$\begin{array}{ccc} & U \times_Y X & \\ & \nearrow & \downarrow \\ V & \longrightarrow & U \end{array}$$

where $V \rightarrow U \times_Y X$ is an open immersion and $V \rightarrow U$ is a closed immersion.

If $g: U' \rightarrow U$ is a Y -morphism then for each V as above, then we get a commutative diagram

$$\begin{array}{ccc} & U' \times_Y X & \\ & \nearrow & \downarrow \\ V \times_U U' & \longrightarrow & U' \end{array}$$

and hence $E_{X/Y}$ is indeed a presheaf. For any Y -scheme U we may choose $V = \emptyset$ to get a diagram as in Definition 6.1.1, and hence $E_{X/Y}$ is a presheaf of *pointed* sets.

PROPOSITION 6.1.2. $E_{X/Y}$ is a sheaf on the the big fpqc site on (Sch/Y) .

PROOF. First it is clear that if $\{U_i \rightarrow U\}$ is an fpqc covering then the compositions

$$E_{X/Y}(U) \rightarrow \prod E_{X/Y}(U_i) \rightrightarrows \prod E_{X/Y}(U_i \times_U U_j)$$

coincide. Put $U' := \coprod U_i$. Furthermore, since $g: U' \rightarrow U$ is surjective we get that preimages of distinct sets are distinct, and hence $E_{X/Y}(U) \rightarrow \prod E_{X/Y}(U_i)$ is injective. Finally, suppose that for every i we have an element

$$\begin{array}{ccc} & U_i \times_Y X & \\ & \nearrow & \downarrow \\ V_i & \longrightarrow & U_i \end{array}$$

in $E_{X/Y}(U_i)$ such that

$$\text{pr}_1^{-1}(V_i) = \text{pr}_2^{-1}(V_j) \in E_{X/Y}(U_i \times_U U_j)$$

for every pair i, j . We want to show that there exists an open subset of $U \times_Y X$ with preimage $V = \bigcup V_i \subseteq U' \times_Y X$. Since $f: U' \times_Y X \rightarrow U \times_Y X$ is fpqc, Proposition 1.1.16 says that $U \times_Y X$ has the quotient topology induced by f . Then we only need to show that $f^{-1}(f(V)) = V$. Clearly $V \subseteq f^{-1}(f(V))$. Suppose to derive a contradiction that there is an element $v' \in f^{-1}(f(V)) \setminus V$. Then there is a $v \in V$ such that $f(v) = f(v')$. But then $v \in V_i$ for some i and (v, v') is in $\text{pr}_1^{-1}(V_i)$ but not in $\text{pr}_2^{-1}(V_j)$ for any j . This is a contradiction and hence we conclude that $f(V)$ is the unique open subset of $U \times_Y X$ with preimage V .

Since $f^{-1}(f(V)) = V$ we have that $V = U' \times_U f(V)$ and since the property "closed immersion" is étale local on the base, we get that $f(V) \rightarrow U$ is a closed immersion. \square

The restriction of $E_{X/Y}$ to the big étale site extends uniquely to a sheaf on $Y_{\text{ét}}$ which we also denote by $E_{X/Y}$. Rydh shows in [Ryd11] that $E_{X/Y}$ is locally constructible. That is, $E_{X/Y}$ is an étale algebraic space over Y .

6.2. The case when f is a monomorphism locally of finite type

When the morphism $f: X \rightarrow Y$ is a monomorphism, we may relate $E_{X/Y}$ to the constant sheaf $\{0, 1\}$ via the functor $f_!$.

Note that a base change of a monomorphism is again a monomorphism. Indeed, given a morphism $U \rightarrow Y$ where U is a scheme, suppose that we have two morphisms $T \rightrightarrows U \times_Y X$ such that the compositions $T \rightrightarrows U \times_Y X \rightarrow U$ coincide. Since $X \rightarrow Y$ is a monomorphism we get that the compositions $T \rightrightarrows U \times_Y X \rightarrow X$ also coincide. Hence, by the universal property of the fiber product, we get that the morphisms $T \rightrightarrows U \times_Y X$ coincide.

Note also that a section $s: S \rightarrow T$ of a monomorphism $g: T \rightarrow S$ is always an isomorphism since the equality $g \circ s \circ g = g$ implies that $s \circ g = \text{id}_T$.

PROPOSITION 6.2.1. *Let $f: X \rightarrow Y$ be a monomorphism locally of finite type of algebraic spaces. Then there is a natural isomorphism*

$$E_{X/Y} \cong f_c\{0, 1\}_X$$

where f_c is the direct image with proper support.

PROOF. Let U be an étale scheme over Y . We have that $U \times_Y X$ is a scheme since f is a monomorphism (see [Sta, Tag 0463] and [Sta, Tag 0418]). The support of a section $s \in f_*\{0, 1\}_X(U) = \{0, 1\}^{\pi_0(U \times_s T)}$ is just the union of the connected components where s takes the value 1. Hence we may define a function

$$\begin{aligned} \tau_U: f_c\{0, 1\}_X(U) &\rightarrow E_{X/Y}(U) \\ s &\mapsto \text{supp}(s), \end{aligned}$$

where $\text{supp}(s)$ is given the open subscheme structure with respect to the open and closed inclusion $\text{supp}(s) \hookrightarrow |U \times_Y X|$. Any scheme $V \hookrightarrow U \times_Y X$ which is proper over U and which is a union of connected components in $U \times_Y X$ defines a unique section $s \in f_c\{0, 1\}_X(U)$, namely the one marking the connected components of V with a 1 and the connected components of $U \times_Y X \setminus V$ with a 0.

Suppose that we have a diagram

$$\begin{array}{ccc} & & U \times_Y X \\ & \nearrow j & \downarrow p \\ V & \xrightarrow{i} & U \end{array}$$

representing an element in $E_{X/Y}(U)$. Then $j: V \rightarrow U \times_Y X$ is an open immersion. But since j and p are injective, we have that $j(V) = p^{-1}(i(U))$ which is closed in

$U \times_Y X$ since $i(V)$ is closed in U . Hence V is both open and closed in $U \times_Y X$. That is, V is a union of connected components in $U \times_Y X$. Furthermore, since $U \times_Y X \rightarrow U$ is a monomorphism, so is i and hence i is a closed immersion if and only if i is proper. Hence we get that τ_U is a bijection and it remains to prove that τ_U is natural in U .

A Y -morphism $U' \rightarrow U$ yields a map $E_{X/Y}(U) \rightarrow E_{X/Y}(U')$ taking V to the scheme V' which is the disjoint union of the connected components in $U' \times_Y X$ mapping to the image of V inside $U \times_Y X$. The map $\{0, 1\}^{\pi_0(U \times_Y X)} \rightarrow \{0, 1\}^{\pi_0(U' \times_Y X)}$ takes the element $(a_1, \dots, a_{|\pi_0(U \times_Y X)|})$, marking the connected components of $U \times_Y X$ that are in $V \subseteq U \times_Y X$ with a 1, to the element in $\{0, 1\}^{\pi_0(U' \times_Y X)}$ marking the components of $U' \times_Y X$ that are mapping to components in $V \subseteq U \times_Y X$ with a 1. Hence we conclude that we have a natural isomorphism

$$\tau: f_c\{0, 1\}_X \xrightarrow{\sim} E_{X/Y}. \quad \square$$

COROLLARY 6.2.2. *Let $f: X \rightarrow Y$ be a closed immersion of algebraic spaces. There is a natural isomorphism*

$$E_{X/Y} \cong f_*\{0, 1\}_X,$$

as sheaves on the big (small) étale site on Y , where $\{0, 1\}_X$ denotes the constant sheaf of the set $\{0, 1\}$ on the big (small) étale site on X .

6.3. The case when f is étale

When $f: X \rightarrow Y$ is étale we know that there exists a functor $f_!: \mathbf{Sh}_*(X_{\text{ét}}) \rightarrow \mathbf{Sh}_*(Y_{\text{ét}})$ which is a left adjoint of f^* . Let $E_{X/Y, \text{ét}}$ denote the restriction of $E_{X/Y}$ to the small étale site on Y .

PROPOSITION 6.3.1. *Let $f: X \rightarrow Y$ be an étale morphism of algebraic spaces. There is a natural isomorphism*

$$E_{X/Y, \text{ét}} \cong f_!\{0, 1\}_X,$$

where $\{0, 1\}_X$ denotes the constant sheaf of the pointed set $\{0, 1\}$ on the small étale site on X .

PROOF. Let \mathcal{P} be the presheaf on $Y_{\text{ét}}$ defined by

$$\mathcal{P}(U) = \bigvee_{\varphi} \{0, 1\}^{\pi_0(U)}$$

for all étale $\psi: U \rightarrow Y$, where the wedge product is over all morphisms $\varphi: U \rightarrow X$ such that $f \circ \varphi = \psi$. Then $f_!\{0, 1\}_X$ is the sheafification of \mathcal{P} .

It is clear that if $U \rightarrow Y$ factors through f , then we get a section $U \rightarrow U \times_Y X$ which is open since $U \times_Y X \rightarrow U$ is unramified. An element in $\{0, 1\}^{\pi_0(U)}$ gives a collection of connected components in U and if we take the union we get a subscheme V and a clopen immersion $V \rightarrow U$. Since $U \rightarrow U \times_Y X$ is open, so is the restriction to V and hence we get an element in $E_{X/Y, \text{ét}}(U)$. If we argue as in the end of the proof of Proposition 6.2.2, we conclude that this gives a natural transformation $\tau: \mathcal{P} \rightarrow E_{X/Y, \text{ét}}$. Hence it suffices to show that $\tau_{\bar{y}}: \mathcal{P}_{\bar{y}} \rightarrow (E_{X/Y, \text{ét}})_{\bar{y}}$ is an isomorphism for each geometric point $\bar{y}: \text{Spec } \Omega \rightarrow Y$.

Let $\bar{x}_1, \dots, \bar{x}_n$ be the geometric points in X over \bar{y} . Then by Lemma 4.5.4 and Example 3.6.4, we have that

$$\mathcal{P}_{\bar{y}} \cong \bigvee_{i=1}^n (\{0, 1\}_X)_{\bar{x}_i} \cong \bigvee_{|X_{\bar{y}}|} \{0, 1\} \cong |X_{\bar{y}}| \cup \{*\}.$$

By Lemma 4.5.2 we have that $(E_{X/Y, \acute{e}t})_{\bar{y}} = (\bar{y}^* E_{X/Y, \acute{e}t})(\text{Spec } \Omega)$. Since $E_{X/Y}$ is locally constructible we have that $E_{X/Y}$ is an étale algebraic space over Y and by Lemma 5.6.1 we get that

$$\bar{y}^* E_{X/Y, \acute{e}t} \cong \pi_*(\text{Spec } \Omega \times_Y E_{X/Y}),$$

where $\pi : (\text{Spec } \Omega)_{\acute{e}t} \rightarrow (\text{Spec } \Omega)_{\acute{e}t}$ is the restriction morphism. Since $\text{id} : \text{Spec } \Omega \rightarrow \text{Spec } \Omega$ is étale, this implies that

$$(E_{X/Y, \acute{e}t})_{\bar{y}} = E_{X/Y}(\text{Spec } \Omega) \cong |X_{\bar{y}}| \cup \{\emptyset\}. \quad \square$$

6.4. Final remark

Propositions 6.2.1 and 6.3.1 gives us reason to believe that that one may extend this construction to the general case. Hence we have the following conjecture:

CONJECTURE. *Let X and Y be algebraic spaces and let $f : X \rightarrow Y$ be a morphism locally of finite type. There exists a functor*

$$f_{\#} : \text{Sh}_*(X_{\acute{e}t}) \rightarrow \text{Sh}_*(Y_{\acute{e}t})$$

of sheaves of pointed sets such that:

- (1) *if f is unramified, we have $E_{X/Y} = f_{\#}\{0, 1\}_X$;*
- (2) *if f is étale we have $f_{\#} = f_!$;*
- (3) *if f is a monomorphism we have $f_{\#} = f_c$.*

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