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Topics on fractional Brownian motion and regular variation for stochastic processes

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Abstract

The first part of this thesis studies tail probabilities for elliptical distributions and probabilities of extreme events for multivariate stochastic processes. It is assumed that the tails of the probability distributions satisfy a regular variation condition. This means, roughly speaking, that there is a non-negligible probability for very large or extreme outcomes to occur. Such models are useful in applications including insurance, finance and telecommunications networks. It is shown how regular variation of the marginals, or the increments, of a stochastic process implies regular variation of functionals of the process. Moreover, the associated tail behavior in terms of a limit measure is derived.

The second part of the thesis studies problems related to parameter estimation in stochastic models with long memory. Emphasis is on the estimation of the drift parameter in some stochastic differential equations driven by the fractional Brownian motion or more generally Volterra-type processes. Observing the process continuously, the maximum likelihood estimator is derived using a Girsanov transformation. In the case of discrete observations the study is carried out for the particular case of the fractional Ornstein-Uhlenbeck process. For this model Whittle's approach is applied to derive an estimator for all unknown parameters.

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Contents

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Chapter 1

Introduction

This thesis is divided into two parts. The first part consists of three papers included in Chapters 2–4 and studies tail probabilities for elliptical distributions and extreme events for stochastic processes. The second part of the thesis consists of two papers included in Chapters 5 and 6 and studies parameter estimation problems related to the fractional Brownian motion. We will first give an introduction to the topics covered in the thesis and highlight the main results.

1.1 Regular variation and extremal events

In this section we give an introduction to regular variation and the study of extremal events. We start by discussing random variables with heavy tails. Suppose X is a random variable with distribution function F. The outcome of X may be thought of as the measurement of a sea-level, the daily loss from investing in a stock, the total amount of claims faced by an insurance company in one year etc. In such applications it is relevant to compute the probability of a very large (extreme) outcome, for instance the probability that the sea-level exceeds a high barrier, the probability that we make a large loss from an investment in the stock, or the probability that the total amount of claims faced by the insurance company in one year exceeds a high threshold. This means, we would have to compute $1 - F(x)$ where x is large. For this reason it is of course important to know what the distribution function F looks like for large x, e.g. at which rate the function $1 - F(x)$ tends to zero as $x \to \infty$. If the decay is fast then the probability mass is concentrated around the center of the distribution. As an example we may consider the standard normal distribution where $1-F(x) \sim (x\sqrt{2})^{-1} \exp(-x^2/2)$ as $x \to \infty$. In this case we say that the distribution has *light* (right) tail. If, on the other hand, the decay is slow, then there is a significant amount of probability mass far out in the (right) tail of the distribution. The slow decay of the probability distribution as $x \to \infty$ is often referred to as *heavy* (right) tail. As an example we may consider the Pareto distribution where $1 - F(x) \sim x^{-\alpha}$, as $x \to \infty$ and $\alpha > 0$. Similar considerations can of course be made for the left tail as well. Then we consider the rate at which $F(x)$ tends to zero as $x \to -\infty$. There is no precise definition of how fast or slow the decay must be to say that a probability distribution has light or heavy tails. We will work with a more precise concept called regular variation, which specifies the rate at which $1 - F(x)$ tends to zero.

In heavy-tailed distributions large values may occur in a sample with non-negligible probability. This is often observed in insurance data, for instance in the so-called catastrophe insurances including fire, wind-storm and flooding insurances. The large claims may lead to large fluctuations in the cash-flow process faced by the insurance company, increasing the risk in such portfolios. The situation is similar in finance where extremely large losses sometimes occur, which indicate heavy tails of the return distributions. The probability of extreme stock-movements has to be accounted for when analyzing the risk of a portfolio. Another application is queuing models where extreme service times, modeled by heavy-tailed distributions, result in huge waiting times in the system and large fluctuations in the workload process.

In many applications it is appropriate to use a stochastic process $\{X_t : t \geq 0\}$ to model the evolution of the interesting quantities over time. The notion of heavy tails enters naturally in this context either as an assumption on the marginals X_t or as an assumption on the increments $X_{t+h} - X_t$ of the process. However, it is often the case that the marginals or the increments of the process is not the main concern. Instead some functional of the process is important. A natural example is the supremum of the process during a time interval, $\sup_{0 \leq t \leq T} X_t$, i.e. the largest value reached by the process in the interval $[0, T]$. Another example is the mean of the process during a time interval, $T^{-1} \int_0^T X_t \mathrm{d}t$. We are then typically interested in the probability that the functional exceeds some high level, e.g. –What is the probability that the sea-level exceeds a high barrier sometime during $[0, T]$? It may therefore be important to know how the tail behavior of the marginals X_t (or the increments) is related to the tail behavior of functionals of the process. This problem is studied in a multivariate context in Chapters 3 and 4 below.

The concept of regular variation comes from pure mathematics but has found many applications in probability theory. A function f supported on $(0, \infty)$ is said to be *regularly varying at* ∞ *with index* $\rho \in \mathbb{R}$ if for all $x > 0$,

$$\lim_{u \to \infty} \frac{f(ux)}{f(u)} = x^{\rho}. \tag{1.1}$$

If a nonnegative random variable X is distributed according to F we say that X is regularly varying with index $\alpha > 0$ if $1 - F$ is regularly varying at ∞ with index $-\alpha$. In this case we may write the regular variation condition above as

$$\lim_{u \to \infty} \frac{\mathbb{P}(X > ux)}{\mathbb{P}(X > u)} = x^{-\alpha}. \tag{1.2}$$

If X is regularly varying then we can write $\mathbb{P}(X > x) = x^{-\alpha}L(x)$ where L is a slowly varying function (see e.g. Resnick [44]). Hence, the tail of the distribution decays essentially as $x^{-\alpha}$ for large x.

Let us now consider the multivariate case. Suppose that we are dealing with a d-dimensional random vector $\mathbf{X}$ instead of a univariate random variable. This could be interpreted for instance as the measurements of sea-levels at d different locations, the daily losses of d different stocks or the amount of claims in d different insurances in one year. A notable difference between the multivariate case and the univariate case when analyzing extreme values is the possibility to have dependence between the components of the random vector. Large values can for instance tend to occur simultaneously in the different components. To have a good understanding of the dependence between extreme events in the multivariate case may be of great importance in applications. In some cases the dependence between extreme values is even stronger than the dependence of moderate values. As an example we may consider the daily log-returns of BMW and Siemens AG stocks during the period 020189–020196 (see Figure 1.1). Note that large negative shocks tend to occur

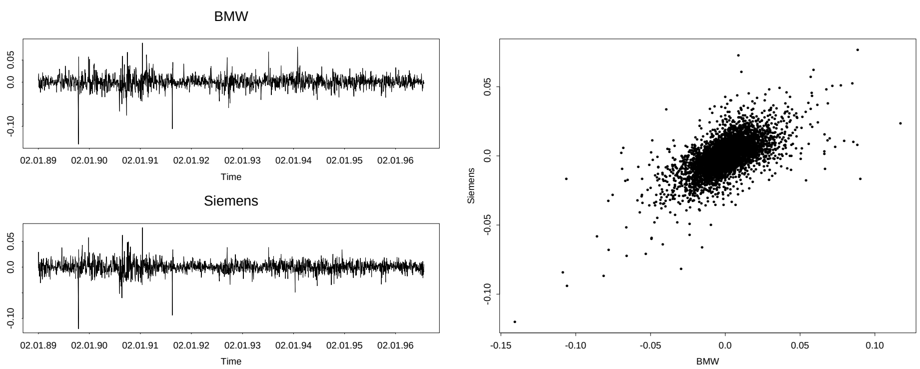

Figure 1.1. The log-returns of BMW and Siemens AG stocks during 020189–020196. Left: The evolution of the log-returns over time. Right: Scatter plot of the log-returns.

simultaneously in both assets. In fact we observe that the dependence seems to be stronger for extreme losses (third quadrant) than for extreme profits (first quadrant) or for moderate values. It is essential to take this fact into account when analyzing the risk of a portfolio with investments in both stocks. For more examples of dependent extreme values in sea-level, storm and wind data see e.g. de Haan and de Ronde [29] and Rootzén and Tajvidi [47]. For examples covering exchange rates in finance see Stărică [52].

In the univariate case we made the assumption that the probability distributions are regularly varying. A similar assumption will be made also in the multivariate setting. This is usually referred to simply as multivariate regular variation. We denote by $\mathbb{S}^{d-1}$ the unit hypersphere in $\mathbb{R}^d$ with respect to a norm $|\cdot|$, and by $\mathcal{B}(\mathbb{S}^{d-1})$ the Borel σ-algebra on $\mathbb{S}^{d-1}$. A d-dimensional random vector $\mathbf{X}$ is said to be *multivariate regularly varying with index* $\alpha > 0$ if there exists a probability measure σ on $\mathbb{S}^{d-1}$ such that for every $x > 0$, as $u \to \infty$,

$$\frac{\mathbb{P}(|\mathbf{X}| > ux, \mathbf{X}/|\mathbf{X}| \in \cdot\,)}{\mathbb{P}(|\mathbf{X}| > u)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot) \text{ on } \mathcal{B}(\mathbb{S}^{d-1}). \tag{1.3}$$

The probability measure σ is referred to as the *spectral measure* of $\mathbf{X}$ and α is referred to as the *tail index* of $\mathbf{X}$.

The concept of multivariate regular variation is formulated in terms of weak convergence $\xrightarrow{w}$ of measures. Since $\mathbb{S}^{d-1}$ is compact weak convergence coincides with vague convergence (denoted $\xrightarrow{v}$) on this space. Therefore we may alternatively formulate multivariate regular variation in terms of vague convergence (see Chapter 2). For a more complete account on weak and vague convergence we refer to Kallenberg [31] or Daley and Vere-Jones [15]. Note that if we plug in the set $\mathbb{S}^{d-1}$ in the definition then

$$\frac{\mathbb{P}(|\mathbf{X}| > ux)}{\mathbb{P}(|\mathbf{X}| > u)} \to x^{-\alpha},$$

and hence $|\mathbf{X}|$ is regularly varying at ∞ with index α according to (1.2). On the other hand, if we put $x = 1$ then, as $u \to \infty$,

$$\mathbb{P}(\mathbf{X}/|\mathbf{X}| \in \cdot \mid |\mathbf{X}| > u) \xrightarrow{w} \sigma(\cdot) \text{ on } \mathcal{B}(\mathbb{S}^{d-1}).$$

The spectral measure gives information on in which directions we are likely to find extreme realizations of the random vector $\mathbf{X}$, whereas α is related to the radial decay of the probability distribution. Note that α does not depend on the direction.

An interesting class of multivariate distributions which is widely applicable is the class of elliptical distributions. This class is an extension of the multivariate normal distributions and may, roughly speaking, be thought of as the class of multivariate distributions whose probability density functions have elliptically shaped level-sets. A thorough introduction to elliptical distributions can be found in Fang, Kotz and Ng [23] and Cambanis, Huang and Simons [9] where many interesting properties are derived. They are also presented in more detail in Chapter 2. The elliptical distributions share many of the tractable properties of the multivariate normal distributions but contrary to the multivariate normal distributions they can also be used in applications where heavy tails are present. This makes the class of elliptical distributions particularly useful, for instance in applications in risk management (see e.g. Embrechts, McNeil and Straumann [22]). As indicated above we sometimes encounter data sets where we have strong dependence of extreme values. Since elliptical distributions have a rather limited flexibility in the dependence structure it is important to understand if their dependence structure is sufficiently rich to accurately model dependent extreme values. In Chapter 2 we study measures of extremal dependence for elliptical distributions.

As previously mentioned it is often appropriate to model the interesting quantities in a dynamical way, as a continuous time stochastic process $\{\mathbf{X}_t : t \in [0,T]\}$. Multivariate regular variation enters naturally as an assumption on the marginals $\mathbf{X}_t$ or as an assumption on the increments $\mathbf{X}_{t+h} - \mathbf{X}_t$ of the process. Similar to the univariate case some functional or vector of functionals of the process may be our primary concern. Natural examples are for instance the componentwise suprema of the process, $(\sup_{0\le t\le T}\mathbf{X}_t^{(1)},\dots,\sup_{0\le t\le T}\mathbf{X}_t^{(d)})$ i.e. the largest value reached by each component of the process in the interval $[0,T]$. Another example is the componentwise mean of the process but other functionals and combinations of functionals may also be of interest. We are then typically interested in the probability that the vector of functionals belongs to some extreme set far away from the origin, e.g. –What is the probability that the sea-level exceeds a high barrier at some (or all) locations sometime during $[0,T]$? To answer this type of questions we need know how the tail behavior of the marginals $\mathbf{X}_t$ is related to the tail behavior of functionals of the process. This is the main problem studied in Chapters 3 and 4.

Let us now give a brief review of the relevant literature. The theory of regularly varying functions is a basic ingredient in the study of weak limits of independent and identically distributed (iid) random variables. The book by Feller [24] is an excellent exposition that makes the connection between regular variation and stable laws. The theory of regularly varying functions is also a fundamental ingredient in the study of extreme values for iid observations, which is explained in de Haan [28]. For an exhaustive treatment of regular variation and its applications in probability theory we refer to Bingham, Goldie and Teugels [7]. For a more recent account on the applications of regularly varying functions and extreme value theory we refer to Embrechts, Klüppelberg and Mikosch [20]. The study of regular variation for stochastic processes is, in the univariate case, often included in more general studies of subexponentiality. Embrechts, Goldie and Veraverbeke [19] proves tail equivalence of a subexponential infinitely divisible random variable and its associated Lévy measure. For results on the tail behavior of functionals of stable processes we refer to Samorodnitsky and Taqqu [50] and references therein. Rosiński and Samorodnitsky [48] derived general results on the tail behavior of subadditive functionals for infinitely divisible stochastic processes. Their paper covers many of the previously known results. Braverman, Mikosch and Samorodnitsky [8] also study the tail behavior of functionals of univariate Lévy processes. Multivariate regular variation was originally used to characterize the domain of attraction of sums of independent identically distributed random vectors that converges in distribution to a multivariate stable distribution (see Rvačeva [49]). The connection between multivariate regular variation and multivariate extreme value theory is explained in Resnick [44]. Kesten [32] used a formulation of multivariate regular variation in terms of linear combinations to conclude that the stationary solution of a multivariate linear stochastic recurrence equation is regularly varying. Several equivalent formulations of multivariate regular variation can be found in the literature. Many of them are documented in Basrak [4]. We refer also to Basrak, Davis and Miksoch

[5] for partial results on the equivalence of the formulation of multivariate regular variation used here (1.3) and the one used by Kesten [32] in terms of linear combinations (see also Chapter 3).

Next, we will give the outline of the first part of the thesis and highlight the main results. Chapter 2 is primarily concerned with the connection between elliptical distributions and multivariate regular variation. Elliptical distributions may be thought of as the class of multivariate distributions whose densities have elliptically shaped level-sets. The main interest in elliptical distributions comes from their usefulness in practice. This class of distributions provides a rich source of multivariate distributions which share many of the tractable properties of the multivariate normal distributions. However, contrary to the multivariate normal distributions, they enable the modeling of multivariate extremes and other forms of non-normal dependences such as tail dependence. It should be noted that, although the elliptical distributions are attractive in many applications, they do not provide a very flexible class of dependence structures. Chapter 2 studies the dependence structure of elliptical distributions and associated dependence measures. Particular emphasis is on the extremal dependence, i.e. dependence of extreme values. The simple dependence structure of elliptical distributions enables explicit computations of interesting dependence measures such as the coefficients of tail dependence and spectral measures associated with regularly varying random vectors.

The main results in Chapter 2 include Theorem 2.22 and a counterexample in Section 2.4. Theorem 2.22 states that for elliptical distributions the existence of tail dependence of the bivariate marginals and the fulfillment of the condition of multivariate regular variation is equivalent to univariate regular variation of the radial random variable in the general representation of elliptical distributions. Moreover, we derive an explicit formula for the coefficient of tail dependence for elliptical distributions. As for the counterexample we show that contrary to Kendall's tau, Spearman's rho is not invariant in the class of elliptical distributions with continuous marginals and a fixed dispersion matrix. We are also able to explicitly compute spectral measures with respect to different norms in Section 2.5.

Chapter 3 is concerned with multivariate stochastic processes with independent increments having marginal distributions which satisfy a multivariate regular variation condition. We find the tail-asymptotics of the distribution of vectors of functionals acting on the process. By tail-asymptotics we mean the limit measure associated with multivariate regular variation. This topic is rather well studied in a univariate context and even though the intuition behind the univariate results to a large extent extends to the multivariate case, proving results in the multivariate case requires other tools such as vague convergence of measures. In the univariate case much of the studies are done for the class of subexponential distributions which includes the class of regularly varying distributions. However, although attempts have been made to formulate subexponentiality in a multivariate framework (see Cline and Resnick [11]), the concept of multivariate subexponential distributions is not well understood. Therefore we have chosen to work with the class of multivariate regularly varying distributions.

The main results in Chapter 3 include Theorem 3.16, Theorem 3.19, Theorem 3.20 and Theorem 3.22. In Theorem 3.16 we prove tail equivalence between an infinitely divisible random vector $\mathbf{X}$ and its associated Lévy measure ν. This can be seen as a multivariate version, in the regularly varying case, of a result in Embrechts, Goldie and Veraverbeke [19] which says that

$$\mathbb{P}(X > x) \sim \nu(\{y \in \mathbb{R} : y > x\}) \quad \text{as } x \to \infty,$$

for X subexponential. In Theorem 3.19 we determine the implications of regular variation of $\mathbf{X}_t$ on the joint tail behavior of vectors of some functionals acting on the process. In particular, we study the tail behavior of the vector of componentwise suprema

$$\mathbf{X}_t^* = \Big(\sup_{0 \le s \le t} X_s^{(1)}, \dots, \sup_{0 \le s \le t} X_s^{(d)} \Big)$$

and the vector of componentwise suprema of the jumps

$$\mathbf{X}_t^\Delta = \Big(\sup_{0 < s \le t} \Delta X_s^{(1)}, \dots, \sup_{0 < s \le t} \Delta X_s^{(d)} \Big).$$

In Theorem 3.20 we give a formulation of regular variation for the graph of an additive process and relates it to regular variation of the marginals of the process. Theorem 3.22 determines the joint tail behavior of the vector of componentwise integrals

$$\mathbf{I}_t = \Big(\int_0^t X_s^{(1)} \mathrm{d}s, \dots, \int_0^t X_s^{(d)} \mathrm{d}s \Big).$$

In Chapter 4 we study general stochastic processes with sample paths in the space $D([0,1], \mathbb{R}^d)$ of right-continuous functions on $\mathbb{R}^d$ with left limits. A natural definition of regular variation for a stochastic process would be to say that it is regularly varying if all its finite dimensional distributions are multivariate regularly varying. However, the finite dimensional distributions alone give limited insights to the extremal behavior of the process and the tail behavior of functionals of the process. Instead we will treat the stochastic process as an element in $D([0,1], \mathbb{R}^d)$ and formulate regular variation on $D([0,1], \mathbb{R}^d)$. The definition is similar to the definition of multivariate regular variation. For explanation of the notation and for technical details we refer to Chapter 4. We say that a stochastic process with sample paths in $D([0,1], \mathbb{R}^d)$ is regularly varying if there exist $\alpha > 0$ and a probability measure σ on $D_1([0,1], \mathbb{R}^d) = \{\mathbf{x} \in D([0,1], \mathbb{R}^d) : \sup_{t \in [0,1]} |\mathbf{x}_t| = 1\}$ such that, as $u \to \infty$,

$$\frac{\mathbb{P}(|\mathbf{X}|_\infty > ux, \mathbf{X}/|\mathbf{X}|_\infty \in \cdot\,)}{\mathbb{P}(|\mathbf{X}|_\infty > u)} \xrightarrow{w} x^{-\alpha} \sigma(\cdot) \quad \text{on } \mathcal{B}(D_1([0,1], \mathbb{R}^d))$$

for every $x > 0$. Here $|\mathbf{x}|_\infty = \sup_{t \in [0,1]} |\mathbf{x}_t|$. The spectral measure σ contains all necessary information for understanding the extremal behavior of the process $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$. One advantage with this formulation is that we can derive a Continuous Mapping Theorem to obtain the tail behavior of many interesting mappings and functionals of the process. In Chapter 3 we studied processes with independent increments. For these processes extreme values are essentially due to one big jump. This intuition may be formalized in terms of the support of the spectral measure. For additive processes the spectral measure concentrates on

$$\{\mathbf{x} \in D_1([0,1],\mathbb{R}^d) : \mathbf{x}(\cdot) = \mathbf{y}1_{[u,1]}(\cdot), u \in [0,1], \mathbf{y} \in \mathbb{S}^{d-1}\},$$

i.e. functions with (at most) one jump. This is actually true also for a larger class of of regularly varying Markov processes which satisfy a condition of asymptotically independent increments. An application of the Continuous Mapping Theorem enables us to study of some filtered stochastic processes of the form

$$\mathbf{Y}_t = \int_0^t f(t,s)\mathrm{d}\mathbf{X}_s, \quad t \in [0,1], \tag{1.4}$$

where $\mathbf{X}$ is a regularly varying Markov process of finite variation. This includes some stable processes and serves as an example on how the formulation of regular variation on $D([0,1],\mathbb{R}^d)$ and the Continuous Mapping Theorem may be applied to obtain the tail behavior of more complicated processes.

The main findings in Chapter 4 include Theorem 4.5, Theorem 4.6, Theorem 4.8, Theorem 4.12 and Theorem 4.18. In Theorem 4.5 we derive a Continuous Mapping Theorem which enables us to obtain the tail behavior of mappings and functionals of a regularly varying stochastic process in $D([0,1],\mathbb{R}^d)$. In Theorem 4.6 we derive necessary and sufficient conditions for a stochastic processes with sample paths in $D([0,1],\mathbb{R}^d)$ to be regularly varying. Theorem 4.8 gives simplified sufficient conditions for regular variation on $D([0,1],\mathbb{R}^d)$ for a class of Markov processes with asymptotically independent increments. In Theorem 4.12 we derive the support of the spectral measure for this class of Markov processes. From this result we may conclude that the extremal behavior for these processes is due to (at most) one big jump. Finally, in Theorem 4.18 we derive the tail behavior of some filtered Markov processes with asymptotically independent increments and paths of finite variation.

1.2 Fractional Brownian motion and parameter estimation

In this section we provide a short introduction to the fractional Brownian motion and some of its applications. For a more thorough introduction we refer to Samorodnitsky and Taqqu [50]. The fractional Brownian motion was first studied by Kolmogorov [34] and it was given its name by Mandelbrot and Van Ness [39]. The study of the fractional Brownian motion appeared within the theory of self-similar processes. Self-similarity is a scaling property of the finite-dimensional distributions. A stochastic process $\{X_t : t \in \mathbb{R}\}$ is self-similar with parameter H

if $\{X_{ct} : t \in \mathbb{R}\}$ and $\{c^H X_t : t \in \mathbb{R}\}$ have the same finite-dimensional distributions for all $c > 0$. This scaling property is surprisingly often observed in various applications. For instance in telecommunications networks, hydrology and finance. It turns out that the fractional Brownian motion is the only Gaussian self-similar process with stationary increments. This is the main motivation for studying it. There are many ways to define the fractional Brownian motion. The simplest, to our knowledge, being the following. The fractional Brownian motion with index $H \in (0,1)$, denoted $\{B_t^H : t \in \mathbb{R}\}$ is the zero mean Gaussian process with covariance function

$$r(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

When $H = 1/2$ then $r(t,s) = \min(t,s)$ and the fractional Brownian motion coincides with the standard Brownian motion. Note that for $H < 1/2$ the increments are negatively correlated whereas they are positively correlated for $H > 1/2$. For $H = 1/2$ they are independent. Let us now have a closer look at the increments of the fractional Brownian motion. Consider for simplicity increments of length 1. Let

$$Y_j = B_{j+1}^H - B_j^H, \quad j \in \mathbb{Z}.$$

This is a stationary process and its covariance function is

$$r(j) = \frac{1}{2}(|j+1|^{2H} + |j-1|^{2H} - 2|j|^{2H}), \quad j \in \mathbb{Z}.$$

The process $\{Y_j : j \in \mathbb{Z}\}$ is often referred to as fractional Gaussian noise. An interesting property is that

$$r(j) \sim H(2H-1)j^{2H-2}, \quad \text{as } j \to \infty.$$

This means that for $H > 1/2$ the sum of correlations diverges, i.e.

$$\sum_{j=0}^{\infty} r(j) = \infty.$$

This property is often referred to as long memory or long-range dependence. It has also served as a motivation for studying the fractional Brownian motion and the fractional Gaussian noise.

We note at this point that according to (1.1) the covariance function $r(j)$ is regularly varying at ∞ with index $\rho = 2H-2$. This is probably the only connection between the first and second part of the thesis.

The most commonly used representation of the fractional Brownian motion is the so-called moving average representation. Let $\{B_t : t \in \mathbb{R}\}$ be a standard

Brownian motion on the real line. Then the fractional Brownian motion with index $H \in (0,1)$ can be represented as

$$B_t^H = C_1^{-1}(H) \int_{-\infty}^{\infty} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right] \mathrm{d}B_s, \quad t \in \mathbb{R},$$

$$C_1(H) = \left\{ \int_0^{\infty} \left((1+s)^{H-1/2} - s^{H-1/2}\right)^2 \mathrm{d}s + \frac{1}{2H} \right\}^{1/2}.$$

The fractional Brownian motion can also be given an integral representation on a compact interval. By self-similarity it is sufficient to give a representation on $[0,1]$. Let $\{B_t : t \in [0,1]\}$ be a standard Brownian motion. Then

$$B_t^H = \int_0^t K_H(t,s) \mathrm{d}B_s, \quad t \in [0,1]$$

is the fractional Brownian motion on $[0,1]$ and

$$K_H(t,s) = \frac{1}{\sqrt{V_H}\Gamma(H+\frac{1}{2})} (t-s)^{H-\frac{1}{2}} {}_1F_2\Big(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{s}\Big) 1_{[0,t)}(s),$$

with ${}_1F_2$ the Gauss hypergeometric function and

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}. \tag{1.5}$$

This representation is due to Decreusefond and Üstünel [16]. See also Norros, Valkeila and Virtamo [42].

We will now give a brief review of the extensive literature on the fractional Brownian motion. This list is far from being complete but aims at giving relevant references for the current thesis. For general self-similar processes we refer to the extensive list of references in Taqqu [53] and for more recent work to Embrechts and Maejima [21]. For many applications of self-similar processes in economics and natural sciences see Mandelbrot [38]. As the increments of the fractional Brownian motion exhibits long memory for $H > 1/2$ new methods for parameter estimation have been developed. Many of the existing estimation methods for discrete time models related to the fractional Brownian motion and the fractional Gaussian noise are summarized in Beran [6]. In particular, Fox and Taqqu [26] have developed the idea of Whittle estimation for such processes. Results that have been extended by Dahlhaus [14]. For a semi-parametric approach to estimation of long memory see Robinson [45]. Models with long memory such as the fractional Gaussian noise may appear as the limit of aggregated dynamic models. This is shown in Granger [27]. Parameter estimation for continuous time models with long memory has also been studied in Comte [12] and Comte and Renault [13]. For applications of the fractional Brownian motion in hydrology we refer to Liu, Molz and Szulga [37]. For applications of the fractional Brownian motion in telecommunications and network traffic we refer Leland, Taqqu, Willinger and Wilson [35]. See also Mikosch, Resnick, Rootzén and Stegeman [41] and the references therein. The use of wavelet analysis in network traffic and processes with long memory has been considered for instance in Abry, Flandrin, Taqqu and Veitch [1] and Abry and Veitch [2]. See also Flandrin [25] and Meyer, Sellan and Taqqu [40] for wavelet analysis and synthesis of the fractional Brownian motion. In economics the fractional Brownian motion was suggested as a model by Mandelbrot and Van Ness [39]. For further applications in finance see Shiryaev [51]. Recently, there has been rather extensive criticism against using the fractional Brownian motion to model asset returns since many of the natural models driven by the fractional Brownian motion admits arbitrage opportunities. See for instance Rogers [46] and Cheridito [10]. However, the fractional Brownian motion may also be used to model the volatility of asset prices, see Djehiche and Eddahbi [17]. The wide use of models driven by the fractional Brownian motion in applied fields has led researchers to develop a stochastic calculus for the fractional Brownian motion. For this reason a lot of efforts have been put into defining stochastic integrals with respect to the fractional Brownian motion. These question have been studied by Lin [36], Zähle [55], Decreusefond and Üstünel [16], Duncan, Hu and Pasik-Duncan [18], Pipiras and Taqqu [43], Alós, Mazét and Nualart [3] and many others.

Next, we will give an outline of the second part of the thesis and highlight the main results. In Chapter 5 we study Gaussian processes admitting representation as a Volterra type stochastic integral with respect to the standard Brownian motion. That is, processes $X = \{X_t : t \in T\}$, where T is an index set, admitting a representation of the form

$$X_t = \int_T V(t,r)\mathrm{d}B_r, \quad t \in T, \tag{1.6}$$

where $\{B_t : t \in T\}$ is a standard Brownian motion. Our primary example is the fractional Brownian motion but other interesting processes also fall within this more general definition. Chapter 5 has two main purposes. The first is to study Mercer-type representations of the Volterra process $\{X_t : t \in T\}$, i.e. representations of the form

$$X_t = \sum_{j,k} \Psi_{j,k}(t)\xi_{j,k}, \tag{1.7}$$

where $\{\Psi_{j,k}\}$ is a basis in a function space and the coefficients $\{\xi_{j,k}\}$ is a sequence of Gaussian random variables. Ideally we want the coefficients to be independent standard normal random variables. This is achieved if $\{\Psi_{j,k}\}$ is an orthonormal basis in the reproducing kernel Hilbert space associated with the process X. This representation have been known for a long time for Gaussian processes but to our knowledge it has received very little attention the literature on the fractional Brownian motion. The first four sections of Chapter 5 illustrates how this representation can be used in the context of fractional Brownian motion.

In the last section we study a Girsanov transformation with respect to the fractional Brownian motion and derive a maximum likelihood estimator of the drift parameter θ in models of the type

$$Y(t,\omega) = \theta A(t,\omega) + X(t,\omega),$$

where $A(\cdot)$ is a sufficiently smooth drift function. Here it is assumed that the process Y is observed continuously. Our results extend previous work by Norros, Valkeila and Virtamo [42], who study the case where $A(t) = t$ and X is the fractional Brownian motion, and work by Kleptsyna and Le Breton [33] who study the fractional Ornstein-Uhlenbeck process.

The main results of Chapter 5 include Corollary 5.17, Proposition 5.29, Proposition 5.34, Theorem 5.31, Theorem 5.32 and Theorem 5.33. In Corollary 5.17 we prove that the representation (1.7) holds almost surely uniformly on compacts if X is an almost surely continuous Gaussian process and $\{\Psi_{j,k}\}$ is an orthonormal base in the associated reproducing kernel Hilbert space. This result was suggested by Janson [30]. In Proposition 5.29 we derive a spectral representation of Skorohod integrals with respect to Volterra type processes. In Proposition 5.34 we derive a formula for computing the coefficients $\{\xi_{j,k}\}$ in the representation (1.7). In Theorem 5.31 we derive a Girsanov transformation for Volterra type processes. Finally, in Theorems 5.32 and 5.33 we derive consistency and asymptotic normality of estimators of the drift parameter in models driven by a Volterra type process.

Although the estimation techniques studied in the last section of Chapter 5 is interesting from a theoretical point of view the estimators seem rather difficult to implement in practice. In particular the derived estimators assume that we have continuous observations of the process, whereas in most applications only discrete observations are at hand. In Chapter 6 we have chosen to study a particular model, namely the fractional Ornstein-Uhlenbeck process, where we consider the joint estimation of all parameters in this model based on discrete observations. The fractional Ornstein-Uhlenbeck process is the stationary solution to the equation

$$X_t - X_s = -\theta \int_s^t X_u \mathrm{d}u + \sigma(B_t^H - B_s^H), \quad s < t,$$

where $\sigma > 0$, $\theta > 0$ and $H \in (0,1)$ and $\{B_t^H : t \in \mathbb{R}\}$ is the fractional Brownian motion with Hurst index H. The increments of the fractional Brownian motion has long memory if $H > 1/2$ and this property is transfered to the fractional Ornstein-Uhlenbeck process. We found it natural and simple to estimate the unknown parameters using an idea introduced in Whittle [54] and further developed for Gaussian sequences with long memory in Fox and Taqqu [26] and Dahlhaus [14].

The main findings in Chapter 6 include the computation of the spectral density and the covariance function of the fractional Ornstein-Uhlenbeck process in Proposition 6.2. Moreover, in Theorem 6.3 we prove consistency and asymptotic normality of the Whittle estimator for the joint estimation of all parameters in the fractional Ornstein-Uhlenbeck process.

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