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Dynamical Properties of Quasi-periodic Schrödinger Equations

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ABSTRACT

This thesis deals with the investigation of dynamical properties of quasi-periodic Schrödinger equations. It contains the following two papers:

Paper I. Positive Lyapunov exponent for a class of 1-D quasi-periodic Schrödinger equations — the discrete case. For a non-constant C^1 potential function $V : \mathbb{T} \to \mathbb{R}$ and for large λ, we prove that for an almost full measure set of irrational frequencies $\omega \in \mathbb{T}$ and for a large set of energies $E \in \mathbb{R}$, all lying in the spectrum of the Schrödinger operator

$$(H_\theta u)_n = -(u_{n+1} + u_{n-1}) + \lambda V(\theta + n\omega)u_n,$$

the (maximal) Lyapunov exponent associated with the equation $H_\theta u = Eu$, is positive. Moreover, for these energies, the projective flow corresponding to the fundamental solution of the system

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda V(\theta + n\omega) - E \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix},$$

is shown to be minimal.

Paper II. Positive Lyapunov exponent for a class of 1-D quasi-periodic Schrödinger equations — the continuum case. We prove that in the bottom of the spectrum of the Schrödinger operator

$$(H_\theta u)(t) = -\frac{d^2}{dt^2}u(t) + \lambda V(t, \theta + \omega t)u(t),$$

the Lyapunov exponent is positive for a large set of energies E and frequencies $\omega \in \mathbb{R} \setminus \mathbb{Q}$, provided that λ is large and that the potential function $V : \mathbb{T}^2 \to \mathbb{R}$ satisfies some regularity conditions. We also prove that the projective flow corresponding to the fundamental solution of the system

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda V(t, \theta + \omega t) - E & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix},$$

obtained from the Schrödinger equation $H_\theta u = Eu$, is minimal in these cases.

Key Words: Schrödinger equations, Schrödinger operators, positive Lyapunov exponents, invariant measures, minimality, measure of spectrum.

Mathematics Subject Classification (MSC 2000): Primary 34Cxx, 34L40, 37Cxx, 37Nxx.

SUMMARY

1.1. **Dramatis personae.** The purpose of this thesis is the study of one-dimensional Schrödinger equations; both the time-continuous and its discrete analogue. More precisely, the equations under consideration are of the form

$$(*)^c \qquad -\frac{d^2}{dt^2}u(t) + q(t)u(t) = Eu(t)$$

and

$$(*)^d \qquad -(u_{n+1} + u_{n-1}) + v(n)u_n = Eu_n,$$

respectively. Both these equations are well-studied objects and there is a vast literature (to say the least), using a fruitful blend of techniques from operator theory, ergodic theory, ordinary differential equations and other fields.

The potentials $q(t)$ and $v(n)$ shall be quasi-periodic, i.e., of the form

$$q(t) = f(\theta + t\omega), \quad v(n) = f(\theta + n\omega)$$

where $f : \mathbb{T}^d \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) and $\theta, \omega \in \mathbb{T}^d$, with the elements of the frequency vector $\omega = (\omega_1, \ldots, \omega_d)$ rationally independent. So, the potential is obtained by evaluating the potential function f along the dense 'line' $\theta + t\omega$ (resp. $\theta + n\omega$) in $\mathbb{T}^d$.

What one would like to investigate is the spectral and dynamical properties of equation $(*)$ (this will be made precise below). In comparison with the periodic case — Floquet theory —, i.e., when q and v are periodic functions, there is a much richer flora of phenomena arising in the quasi-periodic case. Some of them depend very delicately on the arithmetical properties of the frequency vector ω. It is to the investigation of these phenomena that this thesis is devoted.

One of the main achievements in this thesis, is the detailed analysis of the diffeomorphisms associated with the two equations (see below) and the proof that they under certain conditions are minimal with exactly two invariant ergodic probability measures, i.e., that they really have a 'chaotic' behavior. This analysis also enables to estimate the measure of the spectrum of the operators associated with $(*)$ from below.

Although there are many similarities between the two equations $(*)^c$ and $(*)^d$, we have separated the analysis into two independent parts, one for the discrete version (part I) and one for the continuous (part II), and we shall keep this separation in this summary as well.

1.2. **Part** *I* **(The time-discrete Schrödinger equation).** In this part we study the family of equations,

$$(H_\theta u)_n := -(u_{n+1} + u_{n-1}) + \lambda V(\theta + n\omega)u_n = Eu_n, \tag{1.1}$$

parametrised by the *phase* θ, where $V : \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is (at least) continuous, the *frequency* $\omega \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$ and the *energy* E is real. The *coupling factor* $\lambda \in \mathbb{R}$ is used as a 'perturbation parameter'.

We shall also derive results about the associated operator H_θ, acting on the Hilbert space $l^2(\mathbb{Z})$.

1.2.1. *The operator* H_θ*.* Since the operator H_θ is bounded and self-adjoint, it follows that the spectrum — $\sigma(H_\theta)$ — is a compact non-void set on the real line. In fact, it is easy to check that

$$\sigma(H_\theta) \subset [\inf \lambda V - 2, \sup \lambda V + 2].$$

Since, moreover, $\omega \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$ (in which case $\overline{\{\theta + n\omega\}_{n\in\mathbb{Z}}} = \mathbb{T}$), one can show that the spectrum is independent of the phase θ. This justifies omitting θ in the notation $\sigma(H_\theta)$.

1.2.2. *Equation (1.1) as a Dynamical System.* Writing equation (1.1) as a first order system, we obtain

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda V(\theta + n\omega) - E \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}. \tag{1.2}$$

The dynamics of (1.2) is given by the skew-product mapping

$$F_E : (\theta, x) \in \mathbb{T} \times \mathbb{R}^2 \mapsto (\theta + \omega, A_E(\theta)x) \in \mathbb{T} \times \mathbb{R}^2, \tag{1.3}$$

(see fig. 1) where

$$A_E(\theta) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda V(\theta) - E \end{pmatrix} \in SL(2, \mathbb{R})$$

and the time evolution of this map is

$$A_E^n(\theta) = \begin{cases} A_E(\theta + (n-1)\omega) \cdots A_E(\theta + \omega)A_E(\theta), & n > 0 \\ Id, & n = 0 \\ A_E^{-1}(\theta - n\omega) \cdots A_E^{-1}(\theta - \omega), & n < 0. \end{cases}$$

To investigate the behavior of (1.3) we introduce the (maximal) *Lyapunov exponent* – $\gamma(E)$ – which is defined by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_E^n(\theta)\| d\theta.$$

This limit exists by subadditivity and is ≥ 0 since $A_E^n(\theta) \in SL(2, \mathbb{R})$. Moreover, it is independent of the matrix norm $\| \cdot \|$.

If $\gamma(E) > 0$ for some fixed E, it follows from the Oseledets theorem (c.f. [14]) that there for Lebesgue a.e. $\theta \in \mathbb{T}$, exists two one-dimensional and

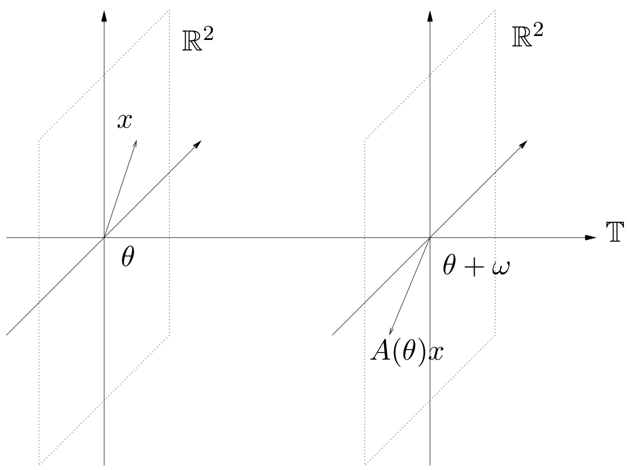


FIGURE 1. The mapping F_E.

invariant subspaces – $W^u(\theta)$,$W^s(\theta)$ – of $\mathbb{R}^2$, varying measurably with θ, such that

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|A_E^n(\theta)x\| = \gamma(E), \quad x\in W^u(\theta)\setminus\{0\},$$
$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|A_E^n(\theta)x\| = -\gamma(E), \quad x\in W^s(\theta)\setminus\{0\},$$

(the subspaces are invariant in the sense that if $x\in W^\beta(\theta)$, then $A(\theta)x\in W^\beta(\theta+\omega)$, $\beta=s,u$). We say that F_E is *uniformly hyperbolic* if $\gamma(E)>0$ and if the two subspaces $W^u(\theta)$,$W^s(\theta)$ exist for *all* $\theta\in\mathbb{T}$ and vary continuously with θ.

There is a correspondence between F_E being uniformly hyperbolic and the spectrum of H_θ, namely

$$F_E \text{ is uniformly hyperbolic} \quad\Longleftrightarrow\quad E\notin\sigma(H)$$

(see [12]). In particular, we always have $\gamma(E)>0$ for energies E outside the spectrum of H_θ.

The skew-product (1.3) induces the mapping

$$G_E:(\theta,l)\in\mathbb{T}\times\mathbb{P}^1(\mathbb{R})\mapsto(\theta+\omega,A_E(\theta)l)\in\mathbb{T}\times\mathbb{P}^1(\mathbb{R}), \tag{1.4}$$

where $\mathbb{P}^1(\mathbb{R})$ is the projective space (the space of all lines through $(0,0)$ in $\mathbb{R}^2$). Since $\mathbb{P}^1(\mathbb{R})\cong\mathbb{T}^2$, we can also view G_E as a homeomorphism on the two-torus $\mathbb{T}^2$.

If $\gamma(E)>0$, then G_E has exactly two ergodic invariant probability measures μ^s,μ^u (see [11]). Moreover, they are supported on the closure of the graphs Γ^u,Γ^s of the two measurable functions

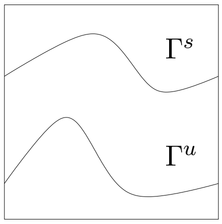
$$\theta\mapsto l^\beta(\theta), \quad \beta=s,u,$$

respectively, where $l^\beta(\theta)$ is the direction of the space $W^\beta(\theta)$.

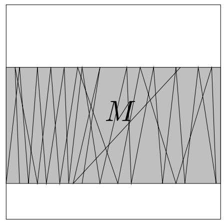
In the case when $E \notin \sigma(H)$, i.e. F_E is uniformly hyperbolic, the two invariant curves are continuous and separated from each other (since the subspaces $W^s(\theta), W^u(\theta)$ vary continuously with θ). The situation when $\gamma(E) > 0$ and $E \in \sigma(H)$ is much more complicated. In this case G_E has a unique non-empty invariant minimal set M and the two measures above satisfy

$$\text{supp } \mu^s = \text{supp } \mu^u = M,$$

(see [11]). So, the combination $E \in \sigma(H)$ and $\gamma(E) > 0$ really gives interesting dynamical behavior of the homeomorphism G_E.



$\mathbb{T} \times \mathbb{P}^1(\mathbb{R})$



$\mathbb{T} \times \mathbb{P}^1(\mathbb{R})$

FIGURE 2. The situation when $\gamma(E) > 0$ and E is outside the spectrum of H (left); $\gamma(E) > 0$, $E \in \sigma(H)$ (right).

1.2.3. *A brief history.* The fact that we can have positive Lyapunov exponent for energies in the spectrum is the content of several important papers. The most classical is the work of M. Herman [10] where he (among many other results) develops an ingenious technique to estimate $\gamma(E)$ when V is a real non-constant trigonometric polynomial. More precisely, he shows that, if

$$V(t) = \sum_{|m| \le M} a_m \exp(2\pi i m t), \quad a_m = \overline{a_{-m}},\ a_M \ne 0,$$

then $\gamma(E) \ge \log|\lambda a_{-M}|$. Note that the estimate is independent of E. Hence we have a positive Lyapunov exponent for all energies, provided that $|\lambda|$ is sufficiently large. The proof is very short, and the key idea is to complexify the problem (so we view $\mathbb{T}$ as the unit circle in the complex plane) and then use an almost magic subharmonic trick.

Later Sorets-Spencer [SS] obtained a lower bound for $\gamma(E)$ (also independent of E) when V is non-constant and real-analytic. This can not be made by a simple approximation of V by a polynomial (note that the lower bound in Herman's theorem behaves very badly to a high-frequency perturbation), but the strategy is still to complexify and use ideas from complex function theory.

In [4], L.H. Eliasson studies the operator H_θ for a potential function V belonging to some Gevrey class and satisfying a generic transversality condition. For such potential functions he shows that, provided that ω is Diophantine, the spectrum of H_θ is pure-point for a.e. $\theta \in \mathbb{T}$. The

Diophantine condition is necessary, since there exist V and irrational ω (for example $V = \cos$) for which H_θ has a purely singular continuous spectrum for a.e. θ [16]. Eliasson also obtains a lower bound for the Lebesgue measure of the spectrum. By the Kotani theory (see [15] for the discrete case), pure-point spectrum implies that $\gamma(E) > 0$ for Lebesgue a.e. E. Hence there are energies in the spectrum for which we have a positive Lyapunov exponent.

Concerning the minimal set M above, M. Herman [10] gives examples when M is not the whole $\mathbb{T}\times\mathbb{P}^1(\mathbb{R})$. To our knowledge, there are no examples in the literature when one actually has $M = \mathbb{T}\times\mathbb{P}^1(\mathbb{R})$, i.e. that G_E is minimal. Similar types of minimal diffeomorphisms are constructed in [11].

Recently there has been much attention to the operator H_θ for a real-analytic potential: [1], [2], [9].

Finally we would like to mention that there are several nice reviews on the subject: see for example [6], [13], [16].

1.2.4. *Results.* Now we state our results in the discrete case. (The notation is illustrated in fig. 3.)

Theorem 1. *Assume that the potential function $V : \mathbb{T} \to \mathbb{R}$ is C^1 and that there is an interval $\mathcal{E} \subset [\inf V, \sup V]$ such that $V^{-1}(\mathcal{E})$ consists of a finite number $\mathcal{N}$ of intervals and that $|V'(x)| \geq s > 0$ on each of them. Then for every $\varepsilon > 0$ there is a $\lambda_0 = \lambda_0(\varepsilon, s, \mathcal{N}, \|V\|_{C^1}, |\mathcal{E}|) > 0$ such that for all $|\lambda| > \lambda_0$ there exists a measurable set $\mathcal{S} \subset (\lambda\mathcal{E}) \times (\mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z}))$, of measure*

$$|\mathcal{S}| \geq |\lambda\mathcal{E}|(1 - \varepsilon),$$

and for all $(E, \omega) \in \mathcal{S}$ the following holds:

a) $\gamma(E) \geq \frac{\log|\lambda|}{4}$.
b) There is a θ such that E is an eigenvalue of H_θ with an exponentially decaying eigenfunction.
c) The diffeomorphism G_E is minimal and has exactly two ergodic invariant probability measures.

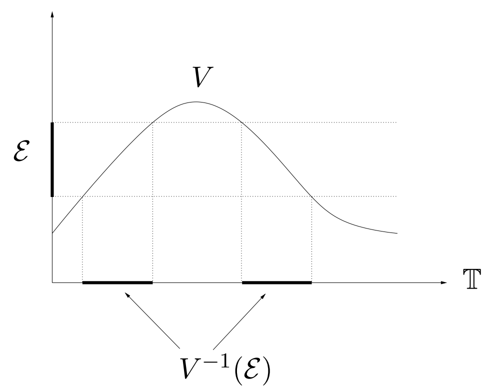


FIGURE 3. Illustration of the assumptions in Theorem 1.

By applying the Fubini theorem to the set S above, we obtain

Corollary 1. *For all* $|\lambda| > \lambda_0$ *there exists a set* $\Omega \subset \mathbb{T}$*, of measure*

$$|\Omega| = 1 - o(1),$$

such that there for each $\omega \in \Omega$ *is associated a set* $\mathcal{E}_\omega \subset \lambda\mathcal{E}$*, satisfying*

$$|\mathcal{E}_\omega| = |\lambda\mathcal{E}| - o(\lambda),$$

and

i) $\gamma(E) \geq \frac{\log|\lambda|}{4}$ *for all* $E \in \mathcal{E}_\omega$*,*

ii) $\mathcal{E}_\omega \subset \bigcup_{\theta\in\mathbb{T}} \sigma_{pp}(H_\theta) \subset \sigma(H)$*.*

Here $\sigma_{pp}(H_\theta)$ denotes the pure-point part of the spectrum. Note that this gives a lower bound for the measure for the spectrum.

1.3. **Part** *II* **(The time-continuous Schrödinger equation).** We shall here consider the quasi-periodic Schrödinger equation

$$(H_\theta u)(t) = -u''(t) + \lambda^2 V(t, \theta + \omega t)u(t) = Eu(t) \tag{1.5}$$

where the *potential function* $V : \mathbb{T}^2 \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) should always be regarded as at least continuous. Moreover we assume that the *coupling factor* λ is real and that the *frequency* $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Note that the *potential* $V(t, \theta + \omega t)$ is obtained by evaluating the potential function along the dense "line" $(t, \theta + \omega t)(t \in \mathbb{R})$ in $\mathbb{T}^2$.

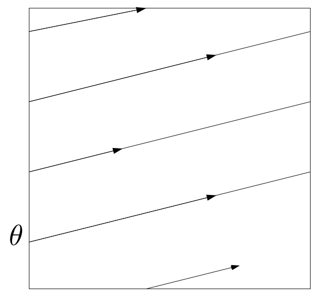


FIGURE 4. Irrational flow on $\mathbb{T}^2$.

1.3.1. *The spectrum of* H_θ*.* The spectrum of the self-adjoint operator H_θ, acting on the Hilbert space $L^2(\mathbb{R})$, is a closed non-void set contained in the interval

$$[\inf \lambda V, \infty).$$

Moreover, it is well known that the spectrum is independent of the phase θ.

1.3.2. *Equation (1.5) as a Dynamical System.* By writing equation (1.5) as a system, we obtain

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda^2 V(t, \theta + \omega t) - E & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}, \tag{1.6}$$

where $' = d/dt$. The fundamental solution of this system – $\Phi_E(t, \theta)$– satisfies

$$\frac{d}{dt}\Phi_E(t,\theta) = \begin{pmatrix} 0 & 1 \\ \lambda^2 V(t, \theta + \omega t) - E & 0 \end{pmatrix} \Phi_E(t,\theta), \quad \Phi_E(0,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\Phi_E(t, \theta) \in SL(2, \mathbb{R})$ for all $t \in \mathbb{R}$. The average exponential growth of the fundamental solution is measured by the Lyapunov exponent

$$\gamma(E) = \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{T}} \log \|\Phi_E(t,\theta)\| d\theta,$$

which exists by subadditivity and which is non-negative since $\Phi_E(t, \theta) \in SL(2, \mathbb{R})$.

One can study the dynamics of (1.6) via the skew-product mapping

$$F_E : (\theta, x) \in \mathbb{T} \times \mathbb{R}^2 \mapsto (\theta + \omega, \Phi_E(1, \theta)x) \in \mathbb{T} \times \mathbb{R}^2. \tag{1.7}$$

To this mapping we associate the homeomorphism

$$G_E : (\theta, l) \in \mathbb{T} \times \mathbb{P}^1(\mathbb{R}) \mapsto (\theta + \omega, \Phi_E(1, \theta)l) \in \mathbb{T} \times \mathbb{P}^1(\mathbb{R}). \tag{1.8}$$

F_E and G_E have the same general properties as their discrete counterparts, described in Part I.

1.3.3. *A brief history.* The continuous quasi-periodic Schrödinger equation is much less studied than its discrete analogue. Firstly, in [5], L.H Eliasson, generalizing the pioneer work of Dinaburg-Sinai [3], shows that if V is real-analytic and ω Diophantine, then the upper part of the spectrum $\sigma(H)$ is purely absolutely continuous and, moreover, the Lyapunov exponent $\gamma(E) = 0$ for a.e. E in this region. For small coupling factors λ, this result extends to the whole spectrum. For the behavior in the bottom of the spectrum in the large coupling regime, there are essentially only results for the potential

$$V(x, y) = \cos(2\pi x) + \sin(2\pi y).$$

For this particular potential, Fröhlich-Spencer-Wittwer [8] show that

$$\sigma(H_\theta) \cap [\inf \sigma(H_\theta), \inf \sigma(H_\theta) + \mathrm{const}(\lambda, \omega)]$$

is pure-point, provided λ is large and ω Diophantine. Moreover, in [17] Sorets-Spencer prove that in the same region (minus some small gaps), $\gamma(E) \geq \lambda + o(1)$ for sufficiently large λ and ω irrational. Similar problems have been studied in [7].

1.3.4. *Results.* Our main result is the following:

Theorem 2. *Assume that the potential function* $V : \mathbb{T}^2 \to \mathbb{R}$ *is* C^3 *and satisfies:*

- $\|V\|_{C^0} = 1$,
- V *has a unique minimum at* $(1/2, 1/2)$ *and* $V(1/2, 1/2) = 0$,
- $V(x, y) = (x-1/2)^2+(y-1/2)^2+\dots$ *in a neighborhood of* $(1/2, 1/2)$.

Then for all $\varepsilon > 0$ *there exists a* $\lambda_0 > 0$ *and a constant* $c > 0$ *such that for all* $\lambda > \lambda_0$ *there is a measurable set* $\mathcal{S} \subset [1.1\lambda, 1.2\lambda] \times ((0, 1/4) \setminus \mathbb{Q})$, *of size*

$$|\mathcal{S}| \geq 0.1\lambda(1/4 - \varepsilon),$$

and for all $(E, \omega) \in \mathcal{S}$ *we have:*

a) $\gamma(E) \geq c\lambda$.
b) There is a $\theta \in \mathbb{T}$ *such that* E *is an eigenvalue of* H_θ *with an exponentially decaying eigenfunction.*
c) The diffeomorphism G_E *is minimal and has exactly two ergodic invariant probability measures.*

Remark 1. The assumptions on V, stated in the theorem, are made for easier presentation of the proof, which already is quite involved. One can show similar results for a generic C^3 potential function V.

From this theorem follows:

Corollary 2. *For all* $\lambda > \lambda_0$ *there exists a set* $\Omega \subset (0, 1/4)$, *of measure*

$$|\Omega| = 1/4 - o(1),$$

such that there for each $\omega \in \Omega$ *is associated a set* $\mathcal{E}_\omega \subset [1.1\lambda, 1.2\lambda]$, *satisfying*

$$|\mathcal{E}_\omega| = 0.1\lambda - o(\lambda),$$

and

i) $\gamma(E) \geq c\lambda$ *for all* $E \in \mathcal{E}_\omega$,

ii) $\mathcal{E}_\omega \subset \bigcup_{\theta \in \mathbb{T}} \sigma_{pp}(H_\theta) \subset \sigma(H)$.

1.4. **Some short comments on the proofs of Theorem 1 and 2.** There are many similarities between the proofs of Theorems 1 and 2. Especially, the general philosophy is the same. Moreover, chapters 5, 7 and 8 in Part I are almost identical to chapters 5, 8 and 9 in Part II. Despite all these similarities, we have chosen to present the two proofs totally independently of each other. One can begin with any of the two parts, but we recommend the reader to start with Part I, since the discrete case is more explicit and therefore the main ideas are more transparent.

1.5. **Acknowledgments.** I wish to thank my advisor Håkan Eliasson for initiating me into the subject and for his support and encouragement during the preparation of this work. I am also very grateful to Michael Benedicks for all his help and support.

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Paper I

POSITIVE LYAPUNOV EXPONENT FOR A CLASS OF 1-D QUASI-PERIODIC SCHRÖDINGER EQUATIONS — THE DISCRETE CASE

KRISTIAN BJERKLÖV

1. Introduction

1.1. The Schrödinger equation. In this paper we study discrete quasi-periodic Schrödinger equations

$$(H_\theta u)_n = -(u_{n+1} + u_{n-1}) + \lambda^2 V(\theta + (n-1)\omega)u_n = Eu_n, \tag{1.1}$$

and the corresponding Schrödinger operator H_θ, acting on the Hilbert space $l^2(\mathbb{Z})$ of square summable sequences. The potential function $V : \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is assumed to be continuous and the coupling factor λ real. Moreover, the frequency $\omega \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$.

It is well known that the spectrum of the bounded self-adjoint operator H_θ, which we shall denote by $\sigma(H_\theta)$ or just $\sigma(H)$, is a closed non-empty subset of the interval

$$[\inf \lambda^2 V - 2, \sup \lambda^2 V + 2],$$

which is independent of θ.

To understand the dynamics of equation (1.1), we write it as the system

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda^2 V(\theta + (n-1)\omega) - E \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}. \tag{1.2}$$

By letting

$$A_E : \ \theta \in \mathbb{T} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & \lambda^2 V(\theta) - E \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

we see that the dynamics of (1.2) is given by the skew-product mapping

$$F_E : \ (\theta, x) \in \mathbb{T} \times \mathbb{R}^2 \mapsto (\theta + \omega, A_E(\theta)x) \in \mathbb{T} \times \mathbb{R}^2.$$

Here the energy E in the Schrödinger equation (1.1) appears just as a free parameter.

The time evolution of the map F_E is given by

$$A_E^n(\theta) = \begin{cases} A_E(\theta + (n-1)\omega) \cdots A_E(\theta + \omega) A_E(\theta), & n > 0, \\ Id, & n = 0, \\ A_E^{-1}(\theta - n\omega) \cdots A_E^{-1}(\theta - \omega), & n < 0. \end{cases}$$

The average exponential growth of the norm of $A_E^n(\theta)$ is measured by the (maximal) *Lyapunov exponent*

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_E^n(\theta)\| \mathrm{d}\theta \geq 0. \tag{1.3}$$

Note that the limit exists by subadditivity and that it is non-negative by the fact that $A_E^n(\theta) \in \mathrm{SL}(2,\mathbb{R})$. Moreover, the limit is independent of the matrix norm $\|\cdot\|$.

Definition 1.1. We say that F_E is *uniformly hyperbolic* if $\gamma(E) > 0$ and if there for each $\theta \in \mathbb{T}$ exists two one-dimensional invariant subspaces of $\mathbb{R}^2$, $W^s(\theta)$ and $W^u(\theta)$, varying continuously with θ, and satisfying

$$\begin{aligned} \lim_{n\to\pm\infty} \frac{1}{n} \log \|A_E^n(\theta)x\| &= \gamma(E), \quad x \in W^u(\theta) \setminus \{0\}, \\ \lim_{n\to\pm\infty} \frac{1}{n} \log \|A_E^n(\theta)x\| &= -\gamma(E), \quad x \in W^s(\theta) \setminus \{0\}. \end{aligned} \tag{1.4}$$

There is a correspondence between F_E being uniformly hyperbolic and the spectrum of H (see [5]), namely

$$F_E \text{ is uniformly hyperbolic} \iff E \in \mathbb{R} \setminus \sigma(H).$$

Remark 1. We have for convenience restricted E to be real, but one could, with the obvious modifications above, consider complex E also (which of course is more natural when studying the operator). Then we would have that F_E is uniformly hyperbolic for all E in the resolvent set.

If we have $\gamma(E) > 0$ for some E in $\sigma(H)$, then, by the Oseledets theorem (see for example [8]), the two subspaces $W^s(\theta)$, $W^u(\theta)$ exist for Lebesgue almost every θ, varying measurably with θ, and (1.4) holds.

The fact that we can have positive Lyapunov exponents for energies in the spectrum, is the contents of several important papers. The most famous is, perhaps, [3], where M. Herman (among other things) shows that under the condition that V is a non-constant trigonometric polynomial, we have $\gamma(E) > 0$ for all E and all irrational ω, provided that λ is large, depending only on V (this result also holds for potential functions defined on $\mathbb{T}^d$, $d \geq 1$). Herman's result (for V defined on $\mathbb{T}$) was later generalized by Sorets-Spencer [12] to cover all non-constant real-analytic potential functions. (This can not be done by just approximating the analytic function by trigonometric polynomials). Recently Bourgain-Goldstein [1] extended Herman's result to non-constant real-analytic potentials V defined on $\mathbb{T}^d$, provided that the vector $\overline{\omega}$ is Diophantine.

In [2], L.H. Eliasson shows that if V is Gevrey and satisfies a generic transversality condition (in particular satisfied by all real-analytic functions), and if ω is Diophantine, then for large λ, the spectrum $\sigma(H_\theta)$ is pure point for a.e. θ. He also shows that the measure of the spectrum is large. By the Kotani theory (see [9] for the discrete case), p.p. spectrum implies that $\gamma(E) > 0$ for a.e. E. Hence, there are energies in the spectrum for which we have a positive Lyapunov exponent. It should be noted that the Diophantine condition on ω in [2] can not be removed, since there exist examples with purely singular continuous spectrum for Liouville frequencies ω (for example $V(\theta) = \cos(\theta)$) (see [10]). In [1], Bourgain and Goldstein show Anderson localization for analytic potential functions defined on $\mathbb{T}^2$ in the large coupling regime.

Some results for weaker regularity assumptions on V can be found in [11]. In [13], L.-S. Young studies related problems.

Our first theorem is the following:

Theorem 1. *Assume that the potential function $V : \mathbb{T} \to \mathbb{R}$ is $\mathcal{C}^1$ and that there is an interval $\mathcal{E}_0 \subset [\inf V, \sup V]$ and a number $s > 0$ such that $V^{-1}(\mathcal{E}_0)$ consists of $0 < \mathcal{N} < \infty$ components and $|V'(\theta)| \geq s$ on each of them.*

Then there exists a $\lambda_0 = \lambda_0(s, \mathcal{N}, \|V\|_{C^1}, |\mathcal{E}_0|) > 0$ so that for all $\lambda > \lambda_0$ there is a measurable set $\mathcal{S} \subset (\lambda^2\mathcal{E}_0) \times (\mathbb{T} \setminus \mathbb{Q})$, of measure

$$|\mathcal{S}| \geq \lambda^2 |\mathcal{E}_0|(1 - 1/\lambda^{1/5}),$$

and the following hold for $(E, \omega) \in \mathcal{S}$:

(∗) $\gamma(E) \geq \frac{\log \lambda}{2}$.

(∗∗) *There is a $\theta \in \mathbb{T}$ such that E is an eigenvalue of H_θ with an exponentially decaying eigenfunction.*

As an immediate consequence, we get

Corollary 1.2. *For all $\lambda > \lambda_0$ there exists a set $\Omega \subset \mathbb{T}$, of measure*

$$|\Omega| = 1 - o(1), \quad \text{as } \lambda \to \infty,$$

such that there to each $\omega \in \Omega$ there corresponds a set $\mathcal{E} = \mathcal{E}(\omega) \subset \lambda^2 \mathcal{E}_0$,

$$|\mathcal{E}| = \lambda^2 |\mathcal{E}_0| - o(\lambda^2), \quad \text{as } \lambda \to \infty,$$

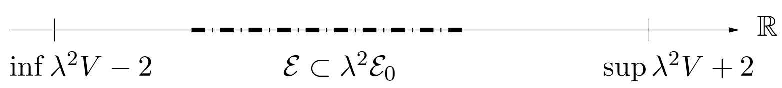
satisfying

(*i*) $\gamma(E) \geq \frac{\log \lambda}{2}$, *for all* $E \in \mathcal{E}$.

(*ii*) $\mathcal{E} \subset \bigcup_{\theta \in \mathbb{T}} \sigma_{pp}(H_\theta) \subset \sigma(H)$.

Remark 2. Here $\sigma_{pp}(H_\theta)$ denotes the pure-point part of the spectrum. In opposite to the case $\sigma(H_\theta)$, the pure-point part is constant for a.e. θ only. Moreover, it is well known that a fixed $E \in \mathbb{R}$ can be an eigenvalue of H_θ only for a measure-zero set of θ.

Remark 3. Note that corollary 1.2 gives a lower estimate of the measure of the spectrum.



Proof of corollary. The existence of the sets Ω and $\mathcal{E}$, as well as their measure, is obtained by an application of the Fubini theorem to the set $\mathcal{S}$. By $(**)$, for each $E \in \mathcal{E}$ there is a $\theta = \theta(E) \in \mathbb{T}$ such that $E \in \sigma_{pp}(H_\theta) \subset \sigma(H_\theta)$. Since the spectrum is independent of θ, this gives (ii). Statement (i) follows immediately. $\square$

1.2. **The Riccati equation.** The approach we shall use in this paper is, instead of studying the Schrödinger equation (1.1) directly, to investigate the related Riccati equation

$$r_n = -1/r_{n-1} + \lambda^2 V(\theta + (n-1)\omega) - E, \tag{1.5}$$

which we obtain from (1.1) by letting $r_n = u_{n+1}/u_n$. Since for any non-trivial solution of (1.1) we have $u_n = 0 \Rightarrow u_{n+1} \neq 0$, r_n is well defined if we let $r_n = \infty$ when $u_n = 0$.

The study of (1.5) shall be made via the diffeomorphism

$$\Phi_E : (\theta, r) \in \mathbb{T} \times \hat{\mathbb{R}} \to (\theta + \omega, -1/r + \lambda^2 V(\theta) - E) \in \mathbb{T} \times \hat{\mathbb{R}}, \tag{1.6}$$

where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a circle on the Riemann sphere. So, instead of studying the vectors $(u_n, u_{n+1}) \in \mathbb{R}^2$, we work with the directions $u_{n+1}/u_n \in \hat{\mathbb{R}}$.

Since $\hat{\mathbb{R}} \cong \mathbb{T} \cong \mathbb{P}^1(\mathbb{R})$ (where $\mathbb{P}^1(\mathbb{R})$ is the projective 1-space (the set of all lines passing through the origin in $\mathbb{R}^2$), we can also view Φ_E as a diffeomorphism on $\mathbb{T}^2$ or on $\mathbb{T} \times \mathbb{P}^1(\mathbb{R})$.

Since we have assumed that $\omega \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$, it follows that if $\gamma(E) > 0$, then the diffeomorphism Φ_E has exactly two ergodic invariant probability measures μ_s and μ_u, and they are supported on the two graphs

$$\Gamma^\alpha = (\theta, r^\alpha(\theta)), \quad \alpha = u, s,$$

where $r^\alpha(\theta)$ is the direction of the space $W^\alpha(\theta)$ ([4], proposition 6.2).

If, moreover, $E \in \sigma(H)$ and $\gamma(E) > 0$ (in which case F_E is not uniformly hyperbolic), then there exists a unique Φ_E-invariant minimal set $M \neq \emptyset$, and

$$\text{supp}(\mu_s) = \text{supp}(\mu_u) = M,$$

(see for example [3], 4.17). An example when $E \in \sigma(H)$, $\gamma(E) > 0$ and $M \neq \mathbb{T}^2$ can be found in [3]. In [7], R. A. Johnson posed the question if one can have $M = \mathbb{T}^2$, i.e. if Φ_E can be minimal but not ergodic. The following statement gives the positive answer:

Theorem 2. *For all $\lambda > \lambda_0$ and for all $(E, \omega) \in \mathcal{S}(\lambda)$ (where $\mathcal{S}$ is the set in Theorem 1), the diffeomorphism Φ_E*

- *is minimal*
- *has exactly two ergodic invariant probability measures.*

Remark 4. To the author's knowledge, this is the first example when a Schrödinger equation exhibits minimal non-ergodic behavior. Similar results can be found in [4].

2. Preliminaries and notations

For simplicity, we shall assume that

$$\|V\|_{C^0} = 1,$$

which then gives $\mathcal{E}_0 \subset [-1, 1]$. $\mathcal{N}$ and s shall be as in Theorem 1.

We define the diffeomorphisms $\Phi = \Phi_{E,\omega,\lambda}$ by

$$\Phi : (\theta, r) \in \mathbb{T} \times \hat{\mathbb{R}} \to (\theta + \omega, -1/r + \lambda^2 V(\theta) - E) \in \mathbb{T} \times \hat{\mathbb{R}},$$

where ω,E and λ act as parameters (c.f. (1.6)).

2.1. **Notations.** In this paper we shall use the following notations:

2.1.1. *Sets.* For sets I and J in $\mathbb{T}$ and $\alpha \in \mathbb{T}$ we define

$$\begin{aligned} I + \alpha &= \{x + \alpha : x \in I\}, \\ |I| &= \text{Lebesgue measure of } I, \\ \overline{I} &= \text{the closure of } I, \\ \text{comp}(I) &= \text{number of components in } I, \end{aligned}$$

and

$$\text{dist}(I, J) = \begin{cases} \infty, & \text{if } I = \emptyset \text{ or } J = \emptyset, \\ \inf\{|x - y| : x \in I \text{ and } y \in J\}, & \text{otherwise,} \end{cases}$$

where $|\cdot|$ is the standard metric on $\mathbb{T}$.

If $I = I(\omega) = [x(\omega), y(\omega)]$ is an interval, where $x(\omega)$ and $y(\omega)$ are defined on an open set in $\mathbb{T}$ and differentiable, we use the notation

$$|\partial_\omega I(\omega)| = \max\{|x'(\omega)|, |y'(\omega)|\}.$$

Finally, we say that a set $X \subset \mathbb{T}$ is ε-dense in an interval $I \subset \mathbb{T}$ if every interval $J \subset I$ of length $> \varepsilon$ intersects X.

2.1.2. *Iterates.* Let E, λ, ω and a point $(\theta_0, r_0) \in \mathbb{T} \times \hat{\mathbb{R}}$ be fixed. Denote

$$(\theta_k, r_k) = \Phi^k(\theta_0, r_0), \quad k \in \mathbb{Z}.$$

2.1.3. *Products.* Many of the estimates shall be made using products of the iterates r_k. First we note that

$$r_0 = 0 \Rightarrow r_1 = \infty \Rightarrow r_2 \neq \infty.$$

If $r_0 \neq \infty$, then the product

$$r_0 \cdots r_n \ (= u_{n+1}/u_0), \quad n \geq 0$$

is well defined if we let $r_k \cdot r_{k+1} = 1$ if $r_k = 0$ and $k < n$, and let the the whole product equal ∞ if $r_n = \infty$.

If $r_0 \neq 0$, then the product

$$r_{-n} \cdots r_0 \ (= u_1/u_{-n}), \quad n \geq 0$$

is defined in the same way.

2.1.4. *Projections.* We define the projections π_1, π_2 onto the first and second coordinate, respectively, by

$$\pi_1 : (\theta, r) \in \mathbb{T} \times \hat{\mathbb{R}} \to \theta \in \mathbb{T},$$
$$\pi_2 : (\theta, r) \in \mathbb{T} \times \hat{\mathbb{R}} \to r \in \hat{\mathbb{R}}.$$

3. Critical sets and no-fast-returns

3.1. **The critical sets.** We now define the main actors of the proof, namely the nested sequence of intervals, which under certain assumptions, stated in the propositions below, will give us the needed information about the diffeomorphism Φ. These are the *critical sets*.

Definition 3.1. For every $\omega \in \mathbb{T}, \lambda > 0, E \in \mathbb{R}$ and for every sequence $M_0 < M_1 < M_2 < \ldots$ of positive integers we define a nested sequence $I_0 \supset I_1 \supset I_2 \supset \ldots$ of open subsets (possibly void) of $\mathbb{T}$ by

$$I_0 := \left\{\theta \in \mathbb{T} : |\lambda^2 V(\theta) - E| < 3\lambda\right\}$$

and

$$I_{j+1} := \pi_1 \left[\Phi^{M_j}(A_j) \cap \Phi^{-M_j}(B_j)\right], \; j \geq 0,$$

where

$$A_j := \{(\theta, r) : \theta \in I_j - M_j\omega, |r| > \lambda\},$$
$$B_j := \{(\theta, r) : \theta \in I_j + M_j\omega, |r| < 1/\lambda\}.$$

Remark 5. When we write $|r| > \lambda$ we include $r = \infty$.

3.2. **A "no-fast-return" condition.** For the future analysis, it will be important to have a certain "no-fast-return" condition on ω and θ:

Definition 3.2. For any sequence $0 < M_0 < M_1 < \ldots$ of positive integers, $E \in \mathbb{R}$ and $\lambda > 0$, we for each $n \geq 0$ define $\mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ to be the set of $\omega \in \mathbb{T}$ satisfying

$$(\mathcal{F}1)_n \qquad \operatorname{dist}\left(I_j, \bigcup_{m=1}^{2^{j+6}\mathcal{N}^2 M_j} (I_j + m\omega)\right) > \frac{3}{\lambda^{M_{j-1}}}, \quad j = 0, 1, \ldots, n,$$

where $M_{-1} = 3/4$, and

$$(\mathcal{F}2)_n \qquad \operatorname{dist}\left(I_j \pm M_j\omega, \bigcup_{l=0}^{j-1} \bigcup_{|m| \leq 2M_l} (I_l + m\omega)\right) > 0, \quad j = 1, 2, \ldots n.$$

Moreover, we define

$$\Theta_n := \mathbb{T} \setminus \bigcup_{l=0}^{n} \bigcup_{|m| \leq M_l - 1} (I_l + m\omega).$$

Finally, we define

$$\mathcal{F}_{-1} = \Theta_{-1} = \mathbb{T}.$$

Remark 6. The first condition is a "Diophantine condition" which just says that if we enter the critical set I_j, then we stay away from it for a "long" time after. The second condition says that the sets A_n, B_n do not enter over any of the previous critical sets to fast.

Remark 7. Note that $(\mathcal{F}1)_n$ implies

$$\operatorname{dist}\left(I_j, \bigcup_{1\le|m|\le 2^{j+6}\mathcal{N}^2 M_j} (I_j + m\omega)\right) > \frac{3}{\lambda^{M_{j-1}}}, \quad j = 0, 1, \dots, n.$$

Remark 8. For a fixed sequence $0 < M_0 < M_1 < M_2 < \dots$ and fixed $E \in \mathbb{R}$, $\lambda > 0$ we clearly have

$$\mathbb{T} = \mathcal{F}_{-1} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots,$$

and

$$\mathbb{T} = \Theta_{-1} \supset \Theta_0 \supset \Theta_1 \supset \Theta_2 \supset \dots.$$

As an immediate consequence of the above definition we have

Lemma 3.3. *For $0 < M_0 < M_1 < M_2 < \dots$, $E \in \mathbb{R}$ and $\lambda > 0$, the following hold for each $\omega \in \mathcal{F}_n$:*

$$I_n \pm M_n\omega \subset \Theta_n.$$

Proof. This is just a combination of $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$. □

4. Basic estimates

This section contains some basic estimates of the mapping Φ.

Lemma 4.1. *For all $\omega \in \mathbb{T}, \lambda > 0$ and $E \in \mathbb{R}$, the following hold: If $\theta_0 \in \mathbb{T} \setminus I_0$ then*

$$|r_0| \ge \frac{1}{\lambda} \quad \Longrightarrow \quad |r_1| \ge 2\lambda,$$

and

$$|r_1| \le \lambda \quad \Longrightarrow \quad |r_0| \le \frac{1}{2\lambda}.$$

Proof. We see that if $\theta_0 \in \mathbb{T} \setminus I_0$ and $|r_0| \ge 1/\lambda$, then

$$|r_1| = |-1/r_0 + \lambda V(\theta_0) - E| \ge |\lambda V(\theta_0) - E| - |1/r_0| \ge 3\lambda - \lambda = 2\lambda$$

(recall the definition of I_0). The second statement is proved in the same way. □

Lemma 4.2. *For all $\omega, \theta_0 \in \mathbb{T}$, $\lambda \ge 4$ and $|E| \le \lambda^2$ we have*

$$|r_0| \ge \frac{1}{\lambda^3} \quad \Longrightarrow \quad |r_{-k} \cdots r_0| \ge \left(\frac{1}{\lambda^3}\right)^{k+1} \quad \forall k \ge 0,$$

and

$$|r_0| \le \lambda^3 \quad \Longrightarrow \quad |r_0 \cdots r_k| \le \lambda^{3(k+1)} \quad \forall k \ge 0.$$

Proof. We prove the first statement. Suppose that $|r_{-i}| < 1/\lambda^3$ for some $i > 0$. By the assumption on E we have $|\lambda^2 V(\theta) - E| \leq 2\lambda^2$ and consequently

$$|r_{-i} \cdot r_{-i+1}| = \left| r_{-i} \left(-\frac{1}{r_{-i}} + \lambda^2 V(\theta_{-i}) - E \right) \right| \geq \frac{1}{2} \geq \frac{1}{\lambda^3} \geq \left(\frac{1}{\lambda^3} \right)^2 .$$

From this the result follows by induction, using the fact that $|r_0| \geq \frac{1}{\lambda^3}$ by assumption. □

Lemma 4.3. *For any sequence* $0 < M_0 < M_1 < M_2 < \ldots$ *of integers, for any* $\lambda > 0, E \in \mathbb{R}$ *and for any* $\omega \in \mathbb{T}$ *we have*

$$\left. \begin{array}{c} (\theta_0, r_0) \in \Phi^{M_j}(A_j) \\ \theta_0 \in I_j \setminus I_{j+1} \end{array} \right\} \Rightarrow |r_{M_j}| \geq 1/\lambda, \quad \forall j \geq 0 \tag{4.1}$$

and

$$\left. \begin{array}{c} (\theta_0, r_0) \in \Phi^{-M_j}(B_j) \\ \theta_0 \in I_j \setminus I_{j+1} \end{array} \right\} \Rightarrow |r_{-M_j}| \leq \lambda, \quad \forall j \geq 0 \tag{4.2}$$

Proof. We prove (4.1). Recall the definition of I_{j+1}. By the assumptions on θ_0 and r_0 we must have

$$(\theta_0, r_0) \notin \Phi^{-M_j}(B_j).$$

Consequently, $(\theta_{M_j}, r_{M_j}) \notin B_j$. Since $\theta_{M_j} \in I_j + M_j\omega$, this (by the definition of B_j) implies that $|r_{M_j}| \geq 1/\lambda$. □

5. Main estimates

In this section we establish the key estimate which shall be needed for the rest of this paper.

Proposition 5.1. *For all* $\lambda \geq 4$, $|E| \leq \lambda^2$ *and every sequence* $10 < M_0 < M_1 < \ldots$, *the following holds for all* $n \geq 0$*:*

Forward iteration: if

$$(A1)_n \qquad \begin{cases} \omega \in \mathcal{F}_{n-1} \\ \theta_0 \in \Theta_{n-1} \\ |r_0| > \lambda \end{cases}$$

then

$$(C1)_n \qquad |r_k \cdots r_N| > \lambda^{(1/2 + (1/2)^{n+1})(N+1-k)} \quad \forall 0 \leq k \leq N,$$

$$(C2)_n \qquad |r_k| \leq \lambda \Rightarrow \theta_k \in \bigcup_{l=0}^{n-1} \bigcup_{m=1}^{M_l} (I_l + m\omega), \quad k = 0, \ldots, N,$$

where $N \geq 0$ *is the smallest integer such that* $\theta_N \in I_n$.

Backward iteration: if

$$(A2)_n \qquad \begin{cases} \omega \in \mathcal{F}_{n-1} \\ \theta_0 \in \Theta_{n-1} \\ |r_0| < 1/\lambda \end{cases}$$

then

$$(C3)_n \qquad |r_{-N} \cdots r_{-k}| < 1/\lambda^{(1/2+(1/2)^{n+1})(N+1-k)} \quad \forall 0 \le k \le N,$$

$$(C4)_n \qquad |r_{-k}| \ge 1/\lambda \Rightarrow \theta_{-k} \in \bigcup_{l=0}^{n-1} \bigcup_{m=0}^{M_l} (I_l - m\omega), \quad k = 0, \ldots, N,$$

where $N \ge 0$ is the smallest integer such that $\theta_N \in I_n + \omega$.

Remark 9. Recall the notation $\Theta_{-1} = \mathbb{T}$ and $\mathcal{F}_{-1} = \mathbb{T}$.

Remark 10. $(C2)_0$ just say that $|r_k| > \lambda$ for $k = 0, \ldots, N$. Similar for $(C4)_0$.

Remark 11. For $\omega \in \mathcal{F}_n \subset \mathcal{F}_{n-1}$ we note that if $(\theta_0, r_0) \in A_n$, then, by lemma 3.3, $(A1)_n$ is satisfied and $N = M_n$. Similarly, if $(\theta_0, r_0) \in B_n$, then $(A2)_n$ is satisfied and $N = M_n - 1$.

As a consequence of this proposition we get

Corollary 5.2. *With the same assumptions as in proposition 5.1, the following hold for $n \ge 1$: if $\omega \in \mathcal{F}_n$, then*

$$(5.1) \qquad \Phi^{M_n - M_{n-1}}(A_n) \subset \{(\theta, r) : \theta \in I_n - M_{n-1}\omega, |r| \ge 2\lambda\} \subset A_{n-1},$$

and

$$(5.2) \qquad \Phi^{-M_n + M_{n-1}}(B_n) \subset \{(\theta, r) : \theta \in I_n + M_{n-1}\omega, |r| \le 1/2\lambda\} \subset B_{n-1}.$$

Proof. To prove this corollary, we proceed as follows: First we note that $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$ implies that

$$(5.3) \qquad (I_{n-1} - (M_{n-1} + 1)\omega) \cap I_0 = \emptyset$$

and

$$(5.4) \qquad (I_{n-1} - (M_{n-1} + 1)\omega) \cap X_{n-1} = \emptyset,$$

where X_{n-1} is the set in $(C2)_n$, proposition 5.1. If we take $(\theta_0, r_0) \in A_n$, i.e.

$$\theta_0 \in I_n - M_n\omega \overset{\text{L.3.3}}{\subset} \Theta_n$$

and $|r_0| > \lambda$, and note that $N = M_n$ is the smallest positive integer such that $\theta_N \in I_n \subset I_{n-1}$, then it follows from $(C2)_n$ and (5.4) that we must have (note that $0 \le M_n - M_{n-1} - 1 < M_n = N$)

$$|r_{M_n - M_{n-1} - 1}| > \lambda.$$

Since (5.3) holds, it hence follows from lemma 4.1 that

$$|r_{M_{n+1} - M_n}| \ge 2\lambda.$$

The proof of (5.2) is similar. □

Proof of proposition 5.1. We only prove the statement in the forward case, since the backward case is identical. The proof goes by induction over n.

For $n = 0$ we see that $(C1)_0$ and $(C2)_0$ follow from lemma 4.1.

Assume now that the statement holds for n, i.e. that the assumption $(A1)_n$ implies $(C1)_n$ and $(C2)_n$. Let us fix $\omega \in \mathcal{F}_n$, $\theta_0 \in \Theta_n$ and $|r_0| > \lambda$, and let $N > 0$ be the smallest integer such that $\theta_N \in I_{n+1}$.

For easier notation, we denote the set in $(C2)_n$ by X_{n-1}, i.e.

$$X_{n-1} := \bigcup_{l=0}^{n-1} \bigcup_{m=0}^{M_l} (I_l + m\omega).$$

From the fact that $\omega \in \mathcal{F}_n$ (conditions $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$) one easily verifies

$$I_n + (M_n + 1)\omega \subset \Theta_n, \tag{5.5}$$

$$(I_n + M_n\omega) \cap I_0 = \emptyset, \tag{5.6}$$

and

$$(I_n - M_n\omega) \cap X_{n-1} = \emptyset. \tag{5.7}$$

Let $0 < N_1 < N_2 < \ldots < N_j < \ldots < N_J = N$ be the times $0 \le i \le N$ such that $\theta_i \in I_n$. Since $\omega \in \mathcal{F}_n$ we have

$$N_{j+1} - N_j > 2^{n+6} M_n, \quad j \ge 1, \tag{5.8}$$

and since $\theta_0 \in \Theta_n$,

$$N_1 \ge M_n. \tag{5.9}$$

We will now inductively show that $(C1-2)_{n+1}[N_j]$ hold, where we by $(C1-2)_{n+1}[N_j]$ denotes condition $(C1-2)_{n+1}$ with N replaced by N_j.

Since $\theta_0 \in \Theta_n \subset \Theta_{n-1}$, $\omega \in \mathcal{F}_n \subset \mathcal{F}_{n-1}$ and $|r_0| > \lambda$, we see that $(A1)_n$ is satisfied. Hence, by the induction assumption, $(C1-2)_n$ hold (where N in $(C1-2)_n$ is N_1, by the definition of N_1). This implies the weaker condition $(C1-2)_{n+1}[N_1]$.

Assume that we inductively have shown that $(C1-2)_{n+1}[N_j]$ hold for some $j < J$. Since $0 \le N_j - M_n < N_j$ (follows from (5.9)), since $\theta_{N_j - M_n} \in I_n - M_n\omega$ (by the definition of N_j) and since (5.7) hold, it follows from $(C2)_{n+1}[N_j]$ that $|r_{N_j - M_n}| > \lambda$, i.e. $(\theta_{N_j-M_n}, r_{N_j-M_n}) \in A_n$. Hence $(\theta_{N_j}, r_{N_j}) \in \Phi^{M_n}(A_n)$, and, since $\theta_{N_j} \notin I_{n+1}$, it follows from lemma 4.3 that we must have

$$|r_{N_j + M_n}| \ge 1/\lambda. \tag{5.10}$$

From (5.6) we get $\theta_{N_j+M_n} \notin I_0$ and by lemma 4.1 we then get

$$|r_{N_j+M_n+1}| \ge 2\lambda > \lambda. \tag{5.11}$$

Note that we could have $|r_k| \leq \lambda$ for some $k = N_j + 1, \ldots, N_j + M_n$, i.e. for $\theta \in (I_n + \omega) \cup \cdots \cup (I_n + M_n\omega)$. This, together with $(C2)_{n+1}[N_j]$, yields

$$|r_k| \leq \lambda \Rightarrow \theta_k \in X_n, \quad k = 0, \ldots N_j + M_n + 1. \tag{5.12}$$

Now, since $\theta_{N_j+M_n+1} \in I_n + (M_n + 1)\omega \overset{(5.5)}{\subset} \Theta_n$ and since (5.11) holds, i.e. $(A1)_n$ is satisfied, if follows from $(C1)_n$ and $(C2)_n$ that

$$|r_k \cdots r_{N_{j+1}}| > \lambda^{(\frac{1}{2}+\frac{1}{2}^{n+1})(N_{j+1}+1-k)}, \quad k = N_j + M_n + 1, \ldots, N_{j+1}, \tag{5.13}$$

and

$$|r_k| \leq \lambda \ \Rightarrow \theta_k \in X_{n-1}, \quad k = N_j + M_n + 1, \ldots, N_{j+1}. \tag{5.14}$$

(Recall that $N_{j+1} > N_j + M_n + 1$ is the smallest positive integer such that $\theta_{N_{j+1}} \in I_n$).

First we note that (5.14) and (5.12) implies $(C2)_{n+1}[N_{j+1}]$.

Secondly, since (5.11) holds, we can apply lemma 4.2 to conclude that

$$|r_k \cdots r_{N_j+M_n+1}| \geq 1/\lambda^{3(N_j+M_n+2-k)}, \quad \forall k \leq N_j + M_n + 1.$$

Combining this estimate with (5.13) and noticing that (5.8) implies

$$\begin{aligned}(1/2 + (1/2)^{n+1})(N_{j+1} + 1 - (N_j + M_n + 2)) - 3(N_j + M_n + 2 - k)\\ \geq (1/2 + (1/2)^{n+2})(N_{j+1} + 1 - k),\end{aligned}$$

for $k = N_j + 1, \ldots, N_j + M_n + 1$, gives

$$|r_k \cdots r_{N_{j+1}}| > \lambda^{(1/2+(1/2)^{n+2})(N_{j+1}+1-k)},$$

for $N_j + 1 \leq k \leq N_{j+1}$. This, together with $(C1)_{n+1}[N_j]$, now yields $(C1)_{n+1}[N_{j+1}]$.

Recalling that $(C1-2)_{n+1}[N_J] = (C1-2)_{n+1}$ finishes the proof. □

6. Geometry of the critical sets

In this section we shall establish the main properties of the critical sets I_n and the intersections between $\Phi^{M_n}(A_n)$ and $\Phi^{-M_n}(B_n)$.

If $\mathcal{E}_0 = [E_0, E_1]$ (where $\mathcal{E}_0$ is the set in Theorem 1), we see that the interval $[\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$ is non-empty for $\lambda \geq 6/|\mathcal{E}_0|$. The following lemma concerns I_0.

Lemma 6.1. *There is a $\lambda_1 = \lambda_1(s, |\mathcal{E}_0|) > 0$ such that for $\lambda > \lambda_1$ and $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$, the critical set I_0 satisfies*

$$\mathrm{comp(I_0)} = \mathcal{N}, \tag{6.1}$$

$$|V'(\theta)| \geq s, \quad \forall \theta \in I_0, \tag{6.2}$$

and for each component I_0^ι, $\iota = 1, \ldots, \mathcal{N}$, we have

$$0 < |I_0^\iota| \leq 1/\lambda^{M_{-1}}, \tag{6.3}$$

and

$$(\lambda^2 V - E)(I_0^\iota) = (-3\lambda, 3\lambda). \tag{6.4}$$

Remark 12. Recall that we have defined $M_{-1} = 3/4$.

Proof. First we assume that $\lambda > 6/|\mathcal{E}_0|$ and take $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$. From the assumptions on $\mathcal{E}_0$ we have

$$\mathrm{comp}(V^{-1}(\mathcal{E}_0)) = \mathcal{N} \quad \text{and} \quad |V'(\theta)| \geq s \text{ on } V^{-1}(\mathcal{E}_0).$$

This implies that for each subinterval $\mathcal{E}_1 \subset \mathcal{E}_0$ we have

$$\mathrm{comp}(\mathrm{V}^{-1}(\mathcal{E}_1)) = \mathcal{N},$$

$$|V'(\theta)| \geq s \text{ on } V^{-1}(\mathcal{E}_1),$$

and

$$V(V^{-1}(\mathcal{E}_1)^\iota) = \mathcal{E}_1 \text{ for each component } V^{-1}(\mathcal{E}_1)^\iota.$$

Now, by definition

$$I_0 = \{\theta \in \mathbb{T} : |\lambda^2 V(\theta) - E| < 3\lambda\} = \{\theta \in \mathbb{T} : |V(\theta) - E/\lambda^2| < 3/\lambda\}.$$

If we let $\mathcal{E}_1 = (E/\lambda^2 - 3/\lambda, E/\lambda^2 + 3/\lambda)$, then $V^{-1}(\mathcal{E}_1) = I_0$ and, by the choice of E, we have $\mathcal{E}_1 \subset \mathcal{E}_0$. From the above argument, this gives (6.1), (6.2) and (6.4). Finally, since $|V'(\theta)| \geq s$ on I_0^ι, we have

$$0 < |I_0^\iota| < 6/(s\lambda) < 1/\lambda^{3/4} = 1/\lambda^{M_{-1}},$$

provided λ is large, depending on s. □

Remark 13. We make the restriction of the energies to the interval $[\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$ (instead of working with the whole $\lambda^2 \mathcal{E}_0$) just so that we can get the above estimates, and an estimate on λ. Otherwise, if we worked with $E = \lambda^2 E_0$ for example, we would need the second derivative of V.

The next inductive lemma gives us information about the critical sets:

Lemma 6.2. *There exists a constant $\lambda_2 = \lambda_2(s, |\mathcal{E}_0|, \|V\|_{C^1}) > 0$ such that for all $\lambda > \lambda_2$, $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$ and all sequences $\{M_k\}_{k=0}^\infty$ satisfying $M_0 > 10$ and $10 M_k \leq M_{k+1}$, the following hold for $n \geq 0$:*

Assume that

$$(\mathcal{A}1)_n \quad \begin{cases} \mathcal{F}_{n-1} \text{ is open in } \mathbb{T} \\ \omega \in \mathcal{F}_{n-1} \Rightarrow \begin{cases} \mathrm{comp}(\mathrm{I_j}(\omega)) = \mathcal{N}, \\ \quad |I_j^\iota(\omega)| > 0, \quad \iota = 1, \ldots, \mathcal{N}, \qquad j = 0, \ldots, n \\ |\partial_\omega I_j^\iota(\omega)| \text{ exists} \end{cases} \end{cases}$$

and that for $\omega \in \mathcal{F}_n$ (which is open by $(\mathcal{A}1)_n$)

$(\mathcal{A}2)_n$

$$\begin{cases} \Phi^{M_n+1}(A_n) & = \{(\theta, r) : \theta \in I_n(\omega) + \omega, \varphi_n^-(\theta, \omega) < r < \varphi_n^+(\theta, \omega)\} \\ \Phi^{-M_n+1}(B_n) & = \{(\theta, r) : \theta \in I_n(\omega) + \omega, \psi_n^-(\theta, \omega) < r < \psi_n^+(\theta, \omega)\} \end{cases}$$

where $\varphi_n^\pm, \psi_n^\pm : \{(\theta,\omega) : \omega \in \mathcal{F}_n, \theta \in I_n(\omega)+\omega\} \to \mathbb{R}$ *are* C^1*-functions satisfying the following: For fixed* $\omega \in \mathcal{F}_n$*, we have on each interval* $I_n^\iota + \omega$ *:*

$(i)_n$ $$\lambda^{3/2} \le \partial_\theta \varphi_n^\pm \le \lambda^{5/2} \quad \text{or} \quad -\lambda^{5/2} \le \partial_\theta \varphi_n^\pm \le -\lambda^{3/2},$$

$(ii)_n$ $$|\partial_\theta \varphi_n^\pm + \partial_\omega \varphi_n^\pm| < \lambda^{1/2},$$

$(iii)_n$ $$|\partial_\theta \psi_n^\pm|, |\partial_\omega \psi_n^\pm| < \lambda^{1/2},$$

$(iv)_n$ $$1/\lambda^{7M_n} < \frac{\varphi_n^+ - \varphi_n^-}{\psi_n^+ - \psi_n^-} \le 2/\lambda^{M_n},$$

and

$(v)_n$ $$\exists \theta_{\alpha,\beta}^\iota \in I_n^\iota + \omega \text{ such that } \varphi_n^\alpha(\theta_{\alpha,\beta}^\iota) = \psi_n^\beta(\theta_{\alpha,\beta}^\iota), \quad \alpha,\beta = \pm,\pm.$$

Then $(\mathcal{A}1)_{n+1}$ *and* $(\mathcal{A}2)_{n+1}$ *hold. Moreover, for each* $\omega \in \mathcal{F}_n$*:*

$(I1)_{n+1}$ $$1/\lambda^{8M_n} < |I_{n+1}^\iota| \le 1/\lambda^{M_n},$$

$(I2)_{n+1}$ $$|\partial_\omega I_{n+1}^\iota| < 1/\lambda^{1/2},$$

and

$(I3)_{n+1}$ $$\overline{I_{n+1}} \subset I_n.$$

Furthermore, for all $\omega \in \mathcal{F}_{n+1}$ *we have*

$(I4)_{n+1}$ *if* $X \subset A_{n+1}$ *and* $\pi_1(X)$ *is* $\varepsilon-$*dense in* $I_{n+1}^\iota - M_{n+1}\omega$ $(\iota = 1,\dots,\mathcal{N})$, *then* $\pi_2(\Phi^{M_{n+1}+M_n}(X))$ *is* $\max\{\epsilon\lambda^{7M_n}, 1/\lambda^{M_n}\}$*-dense in the interval* $(-1/\lambda, 1/\lambda) \subset \hat{\mathbb{R}}$

and

$(I5)_{n+1}$ *if* $Y \subset B_{n+1}$ *and* $\pi_1(Y)$ *is* $\varepsilon-$*dense in* $I_{n+1}^\iota + M_{n+1}\omega$ $(\iota = 1,\dots,\mathcal{N})$, *then* $1/\pi_2(\Phi^{-M_{n+1}-M_n}(Y))$ *is* $\max\{\epsilon\lambda^{7M_n}, 1/\lambda^{M_n}\}$*-dense in the interval* $(-1/\lambda, 1/\lambda) \subset \hat{\mathbb{R}}$.

Remark 14. In $(I5)_{n+1}$, we have used the following notation: if $X \subset \hat{\mathbb{R}}$, then $1/X = \{1/r : r \in X\}$.

Proof. From now on we assume that $\lambda > 0$ is sufficiently large, so that all the estimates below hold true and so that the application of previous lemmas and propositions is possible. λ is of course only allowed to depend on $s, \|V\|_{C^1}$ and $|\mathcal{E}_0|$.

Assume that $(\mathcal{A}1-2)_n$ hold and take $\omega \in \mathcal{F}_n$. Let us fix $\iota = 1,\dots,\mathcal{N}$ and work with only one of the components in $I_n + \omega$. ι shall be fixed through out the whole proof.

We shall denote by A_n^ι and B_n^ι the sets we get when changing I_n to I_n^ι in the definition of A_n and B_n, respectively.

Just for simplicity we shall in what follows make some abuse of notation and write $I_n = I_n^\iota$ and $A_n = A_n^\iota$. Moreover, we always skip the index ι.

We study the case when the first part of $(i)_n$ holds, i.e.,

$$\lambda^{3/2} \leq \partial_\theta \varphi_n^{\pm} \leq \lambda^{5/2} \quad \text{on } I_n + \omega, \tag{6.5}$$

(the other case is similar).

Proof that $(\mathcal{A}1)_{n+1}$ *holds*: Since (6.5) and $(iii)_n$ hold, and since λ is large ($\lambda^{3/2} > \lambda$), the four intersection points $\theta_{\pm,\pm} \in I_n$, given by $(v)_n$, are unique. Let a_{n+1} be the intersection point between φ_n^+ and ψ_n^-, and let b_{n+1} be the intersection point between φ_n^- and ψ_n^+, so

$$\begin{aligned} \varphi_n^-(a_{n+1}) < \varphi_n^+(a_{n+1}) = \psi_n^-(a_{n+1}) < \psi_n^+(a_{n+1}) \\ \varphi_n^+(b_{n+1}) > \varphi_n^-(b_{n+1}) = \psi_n^+(b_{n+1}) > \psi_n^-(b_{n+1}). \end{aligned} \tag{6.6}$$

Then we notice that (see fig.1)

$$I_{n+1} + \omega = \pi_1 \left[\Phi^{M_n+1}(A_n) \cap \Phi^{-M_n+1}(B_n) \right] = (a_{n+1}, b_{n+1}).$$

Hence $(I3)_{n+1}$ holds. Moreover, an easy calculation, using (6.5), $(iii)_n$, and $(iv)_n$ yields

$$1/\lambda^{8M_n} < \frac{2/\lambda^{7M_n}}{\lambda^{5/2} + \lambda^{1/2}} \leq b_{n+1} - a_{n+1} \leq \frac{4/\lambda^{M_n}}{\lambda^{3/2} - \lambda^{1/2}} \leq 1/\lambda^{M_n}, \tag{6.7}$$

which is the estimate in $(I1)_{n+1}$. Also, by letting

$$p(\omega) = a_{n+1} - \omega$$

be the left endpoint of I_{n+1}, we get by the implicit function theorem, applied to the identity

$$\varphi_n^+(p(\omega) + \omega, \omega) = \psi_n^-(p(\omega) + \omega, \omega),$$

(see (6.6)) that $p(\omega)$ is differentiable in ω and that

$$p'(\omega) = \frac{(\partial_\omega \psi_n^- + \partial_\theta \psi_n^-) - (\partial_\omega \varphi_n^+ + \partial_\theta \varphi_n^+)}{\partial_\theta \varphi_n^+ - \partial_\theta \psi_n^-}.$$

From the estimates $(i - iii)_n$ we now get

$$|p'(\omega)| < 1/\lambda^{1/2}.$$

Since the same holds for the right endpoint, this gives $(I2)_{n+1}$.

We have hence verified that $(\mathcal{A}1)_{n+1}$ holds.

Proof that $(\mathcal{A}_2)_{n+1}$ *holds*: By $(\mathcal{A}1)_{n+1}$, $\mathcal{F}_{n+1}$ is open in $\mathbb{T}$. Now, let

$$r_0^{\pm} = r_0^{\pm}(\theta, \omega) = \pm\lambda, \quad \text{for } \omega \in \mathcal{F}_{n+1}, \theta \in I_{n+1}(\omega) - M_{n+1}\omega$$

be the horizontal boundaries of the set $A_{n+1}(\omega)$ and define

$$\varphi_{n+1}^{\pm}(\theta, \omega) = r_{M_{n+1}+1}^{\mp}(\theta - (M_{n+1} + 1)\omega, \omega)$$

for $\omega \in \mathcal{F}_{n+1}, \theta \in I_{n+1}(\omega) + \omega$ (recall the definition of r_k). From now on we skip the indices $\pm$ and $\mp$.

By an easy induction, we obtain the expressions

$$\begin{aligned}\partial_\theta\varphi_{n+1}(\theta,\omega) = \lambda^2 \Bigg(V'(\theta-\omega) + \frac{V'(\theta-2\omega)}{r^2_{M_{n+1}}} + \dots \\ + \frac{V'(\theta-(M_{n+1}+1)\omega)}{r^2_{M_{n+1}}\cdots r_0^2}\Bigg),\end{aligned} \tag{6.8}$$

$$\begin{aligned}\partial_\omega\varphi_{n+1}(\theta,\omega) = -\lambda^2 \Bigg(V'(\theta-\omega) + \frac{2V'(\theta-2\omega)}{r^2_{M_{n+1}}} + \dots + \\ \frac{(M_{n+1}+1)V'(\theta-(M_{n+1}+1)\omega)}{r^2_{M_{n+1}}\cdots r_0^2}\Bigg),\end{aligned} \tag{6.9}$$

and

$$\varphi^+_{n+1} - \varphi^-_{n+1} = \frac{r_0^+ - r_0^-}{(r_0^+\cdots r^+_{M_{n+1}})(r_0^-\cdots r^-_{M_{n+1}})}. \tag{6.10}$$

From proposition 5.1 we get the estimates

$$\left|r_k\cdots r_{M_{n+1}}\right| \geq \lambda^{(M_{n+1}+1-k)/2}, \quad k=0,\dots,M_{n+1},$$

and

$$|r_{M_{n+1}}| \geq \lambda,$$

(for $\omega \in \mathcal{F}_{n+1}$ we have that $N = M_{n+1}$ is the smallest positive time such that a $\theta \in I_{n+1} - M_{n+1}\omega$ is iterated into I_{n+1}). Moreover, since $|r_0| = \lambda$, lemma 4.2 gives the upper bound

$$\left|r_0\cdots r_{M_{n+1}}\right| \leq \lambda^{3(M_{n+1}+1)}.$$

Applying these three estimates to the above formulas yields:

$$|\partial_\theta\varphi_{n+1}(\theta,\omega)| \geq \lambda^2\left(\underbrace{|V'(\theta-\omega)|}_{\geq s} - \|V'\|_{C^0}\sum_{k=1}^{M_{n+1}}\frac{1}{\lambda^k}\right) \geq \lambda^{3/2},$$

$$|\partial_\theta\varphi_{n+1}(\theta,\omega)| \leq \lambda^2\left(\|V'\|_{C^0} + ||V'||_{C^0}\sum_{k=1}^{M_{n+1}}\frac{1}{\lambda^k}\right) \leq \lambda^{5/2},$$

(note that $\partial_\theta\varphi^\pm_{n+1}$ must have the same sign, since the sign is determined by V' on I_0)

$$|\partial_\theta\varphi_{n+1}(\theta,\omega) + \partial_\omega\varphi_{n+1}(\theta,\omega)| = \lambda^2\left|\frac{V'}{r^2_{M_{n+1}}} + \dots + \frac{M_{n+1}V'}{r^2_{M_{n+1}}\cdots r_1^2}\right| < \lambda^{1/2},$$

and

$$\frac{1}{\lambda^{7M_{n+1}}} < \frac{2\lambda}{\lambda^{6(M_{n+1}+1)}} \leq \left|\varphi^+_{n+1} - \varphi^-_{n+1}\right| \leq \frac{2\lambda}{\lambda^{M_{n+1}+1}} = \frac{2}{\lambda^{M_{n+1}}}.$$

This shows that the functions $\varphi^\pm_{n+1}$ satisfies $(i-ii)_{n+1}$ and $(iv)_{n+1}$.

Similarly we let

$$r_0^{\pm} = r_0^{\pm}(\theta, \omega) = \pm 1/\lambda, \quad \text{for } \omega \in \mathcal{F}_{n+1}, \theta \in I_{n+1}(\omega) + M_{n+1}\omega,$$

be the horizontal boundaries of $B_{n+1}(\omega)$ and define

$$\psi_{n+1}^{\pm}(\theta, \omega) = r_{-M_{n+1}+1}^{\pm}(\theta + (M_{n+1} - 1)\omega, \omega)$$

for $\omega \in \mathcal{F}_{n+1}, \theta \in I_{n+1}(\omega) + \omega$. Then one verifies that

$$\partial_\theta \psi_{n+1}(\theta, \omega) = -\lambda^2 \left(V' \cdot r_{-M_{n+1}+1}^2 + \ldots + V' \cdot r_{-M_{n+1}+1}^2 \cdots r_{-1}^2 \right), \tag{6.11}$$

$$\begin{aligned} \partial_\omega \psi_{n+1}(\theta, \omega) = &\lambda^2 (V' \cdot r_{-M_{n+1}+1}^2 r_{-M_{n+1}+2}^2 + \ldots \\ &+ (M_{n+1} - 2) V' \cdot r_{-M_{n+1}+1}^2 \cdots r_{-1}^2), \end{aligned} \tag{6.12}$$

and

$$\psi_{n+1}^{+} - \psi_{n+1}^{-} = (r_0^{+} - r_0^{-})(r_{-1}^{+} \cdots r_{-M_{n+1}+1}^{+})(r_{-1}^{-} \cdots r_{-M_{n+1}+1}^{-}). \tag{6.13}$$

These formulas, together with the estimates given by proposition 5.1, gives

$$|\partial_\theta \psi_{n+1}(\theta, \omega)|, |\partial_\omega \psi_{n+1}(\theta, \omega)| \leq \lambda^{1/2}$$

and

$$1/\lambda^{7M_{n+1}} < \left| \psi_{n+1}^{+} - \psi_{n+1}^{-} \right| \leq \frac{2/\lambda}{\lambda^{M_{n+1}-1}} = \frac{2}{\lambda^{M_{n+1}}},$$

i.e. $(iii)_{n+1}$ and $(iv)_{n+1}$ hold.

To prove $(v)_{n+1}$, we first note that $(I3)_{n+1}$ and corollary 5.2 implies that for $\omega \in \mathcal{F}_{n+1}$, we have

$$\overline{\Phi^{M_{n+1}+1}(A_{n+1})} \subset \Phi^{M_n+1}(A_n) \tag{6.14}$$

and

$$\overline{\Phi^{-M_{n+1}+1}(B_{n+1})} \subset \Phi^{-M_n+1}(B_n). \tag{6.15}$$

Combining (6.14) and (6.15) with (6.6) yields (see fig. 1)

$$\begin{aligned} \varphi_{n+1}^{+}(a_{n+1}+) &< \varphi_n^{+}(a_{n+1}) = \psi_n^{-}(a_{n+1}) < \psi_{n+1}^{-}(a_{n+1}+) \\ \varphi_{n+1}^{-}(b_{n+1}-) &> \varphi_n^{-}(b_{n+1}) = \psi_n^{+}(b_{n+1}) > \psi_{n+1}^{+}(b_{n+1}-), \end{aligned} \tag{6.16}$$

(when evaluating the functions $\varphi_{n+1}^{\pm}, \psi_{n+1}^{\pm}$ at the endpoints we use the left (right) limit, which exists by the derivative estimates). This shows that $(v)_{n+1}$ holds, and finishes the proof of $(\mathcal{A}2)_{n+1}$.

Proof of the density statement $(I4)_{n+1}$*:* Now we proceed with the proof of $(I4)_{n+1}$. Fix $\omega \in \mathcal{F}_{n+1}$.

From (6.16), $(i)_{n+1}$ and $(iii)_n$ we see that the graphs of φ_{n+1}^{α} and ψ_n^{β} must intersect at a unique point in $I_{n+1} + \omega$ ($\alpha = \pm, \beta = \pm$) (see fig. 1 and fig. 2). Let $p, q \in I_{n+1} + \omega$ be such that

$$\varphi_{n+1}^{-}(p) = \psi_n^{-}(p), \quad \varphi_{n+1}^{+}(q) = \psi_n^{+}(q). \tag{6.17}$$

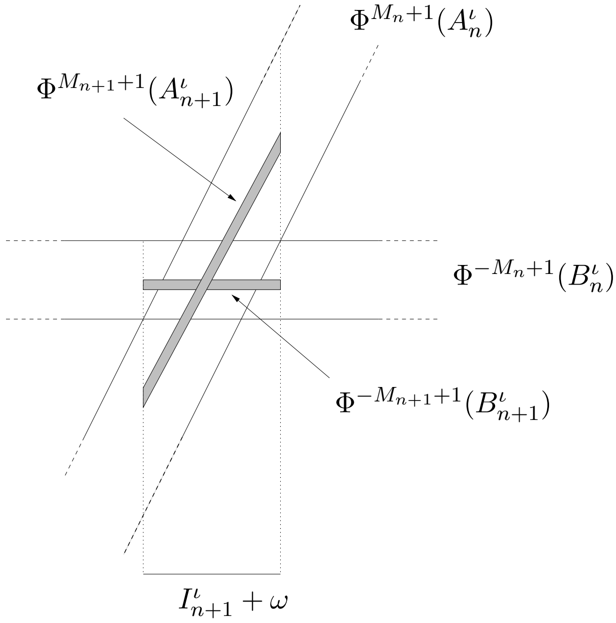


FIGURE 1. One of the $\mathcal{N}$ crossings.

Using the estimates $(i)_{n+1}, (iv)_{n+1}, (iii)_n$ and $(iv)_n$, in addition with the assumption $10M_n \le M_{n+1}$, one easily verifies

$$q - p \ge \frac{1/\lambda^{8M_n} - 2/\lambda^{M_{n+1}}}{\lambda^{5/2} + \lambda^{1/2}} > 1/\lambda^{10M_n}. \tag{6.18}$$

(The important thing is not the lower bound, but the fact that we get a non-empty interval (p, q).)

We now write

$$\Delta + \omega = (p, q) \subset I_{n+1} + \omega$$

and

$$\hat{A}_{n+1} := A_{n+1} \cap ((\Delta - M_{n+1}\omega) \times \hat{\mathbb{R}}).$$

From the choice of p, q we then have that (see fig. 2)

$$\Phi^{M_{n+1}+1}(\hat{A}_{n+1}) = \{(\theta, r) : \theta \in \Delta + \omega, \varphi_{n+1}^-(\theta) < r < \varphi_{n+1}^+(\theta)\}$$

is a subset of $\Phi^{-M_n+1}(B_n)$. Hence

$$\Phi^{M_{n+1}+M_n}(\hat{A}_{n+1}) = \{(\theta, r) : \theta \in \Delta + M_n\omega, \phi^-(\theta) < r < \phi^+(\theta)\}$$

is a subset of B_n, where $\phi^\pm$ are two C^1-functions defined in an open neighborhood of the closure of the interval $\Delta + M_n\omega$. We note that (6.17) implies

$$\phi^+(q + (M_n - 1)\omega) = 1/\lambda$$

and

$$\phi^-(p + (M_n - 1)\omega) = -1/\lambda.$$

(Recall that the graphs of $\psi_n^\pm$ is the image of the (horizontal) boundary of B_n, see fig. 2).

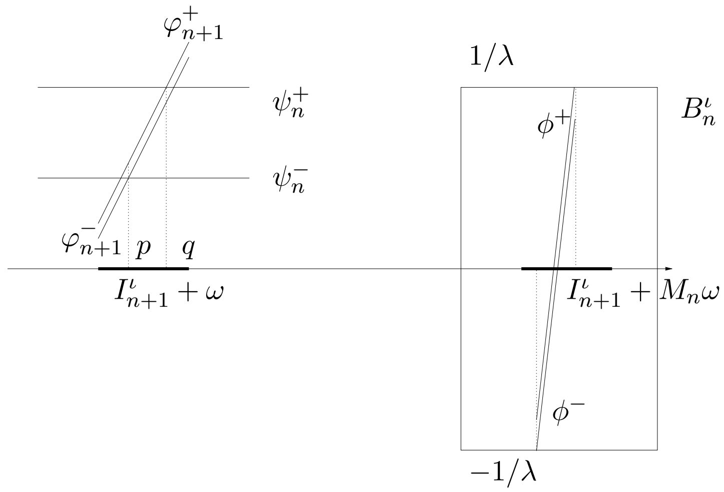


FIGURE 2. The "spreading out" property.

We shall need some estimates of the functions $\phi^\pm$. To get them, let

$$r_1^\pm(\theta) = \varphi_{n+1}^\pm(\theta), \quad \theta \in \Delta + \omega.$$

Then we have

$$\phi^\pm(\theta) = r_{M_n}^\pm(\theta - (M_n - 1)\omega), \quad \theta \in \Delta + M_n\omega.$$

Since this implies that $(\theta, r_{M_n}^\pm) \in B_n$ for $\theta \in \Delta + M_n\omega$, $(C3)_n$ in proposition 5.1 gives

$$|r_1^\pm \cdots r_k^\pm| \leq 1/\lambda^{k/2}, \quad k = 1, \dots, M_n,$$

and by $(C4)_n$ we have

$$|r_1^\pm| < 1/\lambda.$$

Moreover, since $|r_{M_n}^\pm| < 1/\lambda$ (by the definition of B_n) and since $|E| \leq \lambda^2$, we have

$$|r_{M_n-1}^\pm| = \left|\lambda^2 V(\theta_{M_n-1}) - E - r_{M_n}^\pm\right|^{-1} > 1/(2\lambda^2 + 1/\lambda) > 1/\lambda^3.$$

Hence lemma 4.2 yields

$$|r_1^\pm \cdots r_{M_n-1}^\pm| \geq 1/\lambda^{3(M_n-1)}.$$

Using these estimates, and the fact that $\lambda^{3/2} \leq \partial_\theta r_1^\pm \leq \lambda^{5/2}$ and $|r_1^+ - r_1^-| \leq 2/\lambda^{M_{n+1}}$ (recall the definition of $r_1^\pm$), with the two identities

$$\begin{aligned}\partial_\theta r_1^\pm = &-\lambda^2(V' \cdot (r_1^\pm)^2 + \ldots + V' \cdot (r_1^\pm \cdots r_{M_n-1}^\pm)^2) \\ &+ \partial_\theta r_{M_n}(r_1^\pm \cdots r_{M_n-1}^\pm)^2,\end{aligned}$$

and

$$|r_{M_n}^+ - r_{M_n}^-| = \frac{|r_1^+ - r_1^-|}{|r_1^+ \cdots r_{M_n-1}^+||r_1^- \cdots r_{M_n-1}^-|},$$

we get

$$0 < \partial_\theta r_{M_n} \leq (\lambda^{5/2} + \lambda^{1/2})\lambda^{6(M_n-1)} < \frac{\lambda^{6M_n}}{2},$$

and

$$|r^+_{M_n} - r^-_{M_n}| \leq (2/\lambda^{M_{n+1}})\lambda^{6(M_n-1)} < \frac{1}{\lambda^{M_{n+1}-6M_n}} < 1/(2\lambda^{M_n}).$$

Summing up, we have hence showed, writing $(c,d) = \Delta + M_n\omega$,

$$\begin{cases} \phi^-(c) = -1/\lambda, \quad \phi^+(d) = 1/\lambda \\ 0 < \partial_\theta \phi^\pm < \lambda^{6M_n}/2, \quad \text{on } \Delta + M_n\omega \\ \phi^+ - \phi^- < 1/(2\lambda^{M_n}), \quad \text{on } \Delta + M_n\omega. \end{cases}$$

If now $(\theta, r), (\theta', r') \in \Phi^{M_{n+1}+M_n}(\hat{A}_{n+1})$, so $\phi^-(\theta) < r < \phi^+(\theta)$ and $\phi^-(\theta') < r' < \phi^+(\theta')$, it follows from the above estimates of $\phi^\pm$ that

$$|r - r'| \leq |\theta - \theta'|\lambda^{6M_n}/2 + 1/(2\lambda^{M_n}) \leq \max\{|\theta - \theta'|\lambda^{6M_n}, 1/\lambda^{M_n}\}.$$

This finishes the proof of $(I4)_{n+1}$.

Proof of the density statement $(I5)_{n+1}$*:* The proof of $(I5)_{n+1}$ is very similar to that of $(I4)_{n+1}$, but we include it for completeness. Again, fix $\omega \in \mathcal{F}_{n+1}$. From (6.16), $(i)_n$ and $(iii)_{n+1}$ it follows that the graphs of $\varphi^\pm_n$ and $\psi^\pm_{n+1}$ must intersect at unique points in $I_{n+1} + \omega$. Let $p, q \in I_{n+1} + \omega$ be such that

$$\varphi^+_n(p) = \psi^+_{n+1}(p), \quad \varphi^-_n(q) = \psi^-_{n+1}(q). \tag{6.19}$$

Then by the estimates $(i)_n, (iv)_n$ and $(iii)_{n+1}, (iv)_{n+1}$ it follows that

$$p - q > 1/\lambda^{10M_n}$$

(recall assumption (6.5)). Write

$$\Delta + \omega = (p, q) \subset I_{n+1} + \omega$$

and

$$\hat{B}_{n+1} := B_{n+1} \cap ((\Delta + M_{n+1}\omega) \times \hat{\mathbb{R}}).$$

Then

$$\Phi^{-M_{n+1}+1}(\hat{B}_{n+1}) = \{(\theta, r) : \theta \in \Delta + \omega, \psi^-_{n+1}(\theta) < r < \psi^+_{n+1}(\theta)\}$$

is a subset of $\Phi^{M_n+1}(A_n)$ and hence $\Phi^{-M_{n+1}-M_n}(\hat{B}_{n+1})$ is a subset of A_n.

Let $r^\pm_1(\theta) = \psi^\pm_{n+1}(\theta), \theta \in \Delta + \omega$. We then see that the graphs of $1/r^\pm_{-M_n}$ shall be the boundaries of the set

$$\{(\theta, 1/r) : (\theta, r) \in \Phi^{-M_{n+1}-M_n}(\hat{B}_{n+1})\}.$$

We now want to estimate $1/r^\pm_{-M_n}$.

Since $(\theta, r^\pm_{-M_n}) \in A_n$ for $\theta \in \Delta - M_n\omega$ (so $|r^\pm_{-M_n}| > \lambda$), it follows from $(C1)_n$ that

$$|r^\pm_{-k} \cdots r^\pm_0| \geq \lambda^{(k+1)/2}, \quad k = 0, \ldots, M_n$$

and from $(C2)_n$ we get
$$|r_0^\pm| > \lambda.$$
Moreover, by an easy calculation, using $|r_{-M_n}^\pm| > \lambda$ and $|E| \leq \lambda^2$, we get $|r_{-M_n+1}^\pm| < \lambda^3$. By lemma 4.2 we then have
$$|r_{-M_n+1}^\pm \cdots r_0^\pm| \leq \lambda^{3M_n}.$$
From the formula
$$\begin{aligned}\partial_\theta r_1^\pm &= \lambda^2(V' + V'/(r_0^\pm)^2 + \ldots + V'/(r_0^\pm \cdots r_{-M_n}^\pm)^2) \\ &\quad + \partial_\theta r_{-M_n}^\pm/(r_0^\pm \cdots r_{-M_n}^\pm)^2,\end{aligned}$$
we now get, making use of the above estimates and $(iii)_{n+1}$,
$$\left|\partial_\theta \left(\frac{1}{r_{-M_n}^\pm}\right)\right| = \left|-\frac{\partial_\theta r_{-M_n}^\pm}{(r_{-M_n})^2}\right| < \frac{\lambda^{7M_n}}{2}. \tag{6.20}$$
Furthermore, since
$$|1/r_{-M_n}^+ - 1/r_{-M_n}^-| = |r_{-M_n+1}^+ - r_{-M_n+1}^-| = \frac{|r_1^+ - r_1^-|}{|r_1^+ \cdots r_{-M_n+2}^+||r_1^- \cdots r_{-M_n+2}^-|}$$
we need to estimate the products $|r_1^\pm \cdots r_{-M_n+2}^\pm|$. Recalling the definition and construction of ψ_{n+1}, it follows from $(C4)_{n+1}$ that $|r_2^\pm| < 1/\lambda$. Hence we must have $|r_1^\pm| > 1/\lambda^3$. From lemma 4.2 we now get
$$|r_1^\pm \cdots r_{-M_n+2}^\pm| > 1/\lambda^{3M_n}.$$
This, together with $(iv)_{n+1}$ gives
$$|1/r_{-M_n-1}^+ - 1/r_{-M_n-1}^-| \leq \frac{1}{2\lambda^{M_n}}. \tag{6.21}$$
Finally, by (6.19), we have
$$1/r_{-M_n-1}^+(p) = -1/\lambda, \quad 1/r_{-M_n-1}^-(q) = 1/\lambda.$$
From this information, together with (6.20) and (6.21), follows the statement in $(I5)_{n+1}$. □

Lemma 6.3. *For $\lambda > \max\{\lambda_1, \lambda_2\}, E$ and M_0 as in the above proposition, and $\omega \in \mathcal{F}_0$, $(\mathcal{A}1-2)_0$ hold.*

Proof. $(\mathcal{A}1)_0$ is included in lemma 6.1 (recall that $\mathcal{F}_{-1} = \mathbb{T}$). The existence of the C^1-functions $\varphi_0^\pm$ and $\psi_0^\pm$, as well as the estimates $(i-iv)_0$ follows exactly as in the proof of lemma 6.2. It remains to prove $(v)_0$.

We note that lemma 4.1 and $(\mathcal{F}1)_0$ implies
$$(\theta, r) \in \Phi^{M_0}(A_0) \Rightarrow |r| > \lambda$$
$$(\theta, r) \in \Phi^{-M_0+1}(B_0) \Rightarrow |r| < 1/\lambda.$$
Consequently
$$|\psi_0^\pm| \leq 1/\lambda. \tag{6.22}$$

Moreover, since

$$\varphi_0^{\pm}(\theta+\omega) = -1/f_0(\theta)^{\pm} + \lambda^2 V(\theta) - E, \quad \theta \in I_0,$$

where $f_0^{\pm}$ are the boundary of $\Phi^{M_0}(A_0)$, and hence $|f_0^{\pm}| \geq \lambda$, it follows from (6.4) in lemma 6.1 that

$$\varphi_0(I_0^{\iota}+\omega) \supset [-\lambda,\lambda], \quad \iota = 1,\ldots,\mathcal{N}.$$

This, together with (6.22), shows $(v)_0$ (see fig. 3). □

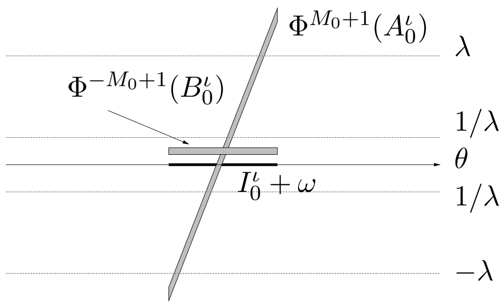


FIGURE 3. One of the $\mathcal{N}$ crossings over I_0.

Summing up, lemma 6.1, 6.2 and 6.3 now gives:

Proposition 6.4. *There exists* $\lambda_3 = \lambda_3(s, |\mathcal{E}_0|, \|V\|_{C^1}) > 0$ *such that for all* $\lambda > \lambda_3$, $E \in [\lambda^2 E_0 + 2\lambda, \lambda^2 E_1 - 2\lambda]$ *and all sequences* $\{M_k\}_{k=0}^{\infty}$ *satisfying* $M_0 > 10$ *and* $M_k \geq 10 M_{k-1}$, *we have for each* $n \geq 0$*:*

$$\omega \in \mathcal{F}_n \quad \Rightarrow \quad \begin{cases} \mathrm{comp(I_j)} = \mathcal{N} \\ 0 < |I_j^{\iota}| < 1/\lambda^{M_{j-1}} \\ |\partial_\omega I_j^{\iota}| \leq 1/\lambda^{1/2} \end{cases} \qquad j = 0,1,\ldots,n+1.$$

Moreover, if $\omega \in \mathcal{F}_n$, $n \geq 1$, *then* $(I3)_k - (I5)_k$ *hold for k=1,…, n.*

7. Good frequencies

Definition 7.1 (The choice of the sequence $\{N_n\}$)**.** For all $\lambda > 0$ we define

$$N_{n+1} = \lambda^{N_n/4}, \quad N_0 = \lambda^{1/4}.$$

From now on N_n shall always as above.

Remark 15. Note that the sequence $\{N_k\}$ grows superexponentially fast.

The main result in this section is:

Proposition 7.2. *There exists a* $\lambda_4 = \lambda_4(s, \mathcal{N}, \|V\|_{C^1}, |\mathcal{E}_0|)$ *such that for all* $\lambda > \lambda_4$ *and* $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$ *(where* $\mathcal{E}_0 = [E_0, E_1]$*), there is a measurable set* $\Omega_E \subset \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$,

$$(\mathcal{S}1) \qquad |\Omega_E| \geq 1 - 2\lambda^{1/4},$$

with the following properties: For each $\omega \in \Omega_E$ there is an infinite sequence $\{M_n\}_{n=0}^{\infty}$ of positive integers ($N_n \leq M_n \leq 2N_n$) such that:

$$\omega \in \bigcap_{n\geq 0} \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E). \tag{$\mathcal{S}2$}$$

The key part for the proof of this proposition is the following lemma:

Lemma 7.3. *There exists a $\lambda_5 = \lambda_5(s, \mathcal{N}, \|V\|_{C^1}, |\mathcal{E}_0|) > 0$ such that for all $\lambda > \lambda_5$ and all $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$, the following holds for $n \geq 0$:*

If $M_j \in [N_j, 2N_j]$ for $j = 0, \ldots, n$, and if $\Omega \subset \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ is an interval, then there is a set $\Omega_+ \subset \Omega$ satisfying

$$|\Omega_+| \geq |\Omega| - 1/N_{n+1}^{3/2}, \tag{7.1}$$

$$comp(\Omega_+) \leq N_{n+1}^4, \tag{7.2}$$

and to each component Ω_+^ι of Ω_+ there is an integer $M_{n+1} \in [N_{n+1}, 2N_{n+1}]$ so that $\Omega_+^\iota \subset \mathcal{F}_{n+1}(M_0, \ldots, M_n, M_{n+1}, \lambda, E)$.

Remark 16. The reason for choosing different M_{n+1} at each component in Ω_+ and not just one, is that condition $(\mathcal{F}2)_{n+1}$ in the definition of $\mathcal{F}_{n+1}$ otherwise will cause us troubles.

Before we start proving lemma 7.3, we need a result concerning the geometry of a no-fast-return-set:

Lemma 7.4. *Assume that for all $\omega \in \mathbb{T}$ there are intervals $I^1, \ldots, I^{\mathcal{N}}$, each of length $\leq \delta$ and satisfying $|\partial_\omega I^\iota| \leq C < 1/2$. Then, for $M > 0$ and $\varepsilon > 0$, the set*

$$\left\{\omega \in \mathbb{T} : dist\left(\cup I^i, \bigcup_{m=1}^{M} (\cup I^i + m\omega)\right) < \varepsilon\right\}$$

has measure $\leq 2\mathcal{N}^2 M \frac{\delta+\varepsilon}{1-2C}$ and consists of at most $\mathcal{N}^2 M^2$ components.

Proof. By standard analysis one easily verifies that if $J_1 = J_1(\omega)$ and $J_2 = J_2(\omega)$ are two intervals in $\mathbb{T}$ such that for all $\omega \in \mathbb{T}$ we have $|\partial_\omega J_i(\omega)| \leq C < 1/2$ and $|J_i(\omega)| \leq \delta_i, i = 1, 2$, then for $m \neq 0$ the set

$$\{\omega \in \mathbb{T} : \text{dist}(J_1, J_2 + m\omega) < \varepsilon\}$$

consists of at most $|m|$ components, each of length

$$\leq \frac{\delta_1 + \delta_2 + 2\varepsilon}{|m| - 2C}.$$

Let now W denote the set in the lemma. Clearly

$$W = \bigcup_{1\leq i,j\leq \mathcal{N}} \bigcup_{m=1}^{M} \{\omega \in \mathbb{T} : \text{dist}(I^i, I^j + m) < \varepsilon\}$$

and hence if follows from the above analysis that the measure of W is bounded by

$$\sum_{1\le i,j\le\mathcal{N}}\sum_{m=1}^{M}|\{\omega\in\mathbb{T}:\operatorname{dist}(I^i,I^j+m)<\varepsilon\}|$$

$$\le\mathcal{N}^2\sum_{m=1}^{M}\frac{2\delta+2\varepsilon}{m-2C}\le 2\mathcal{N}^2M\frac{\delta+\varepsilon}{1-2C}$$

and that the number of components

$$\le\mathcal{N}^2\sum_{m=1}^{M}|m|\le\mathcal{N}^2M^2.$$

□

Now we are ready for the proof of lemma 7.3.

Proof of lemma 7.3. We shall in what follows assume that λ is sufficiently large. Note that we for each $\omega\in\Omega\subset\mathcal{F}_n$ can apply proposition 6.4, since we clearly have $M_0>10$ and $10M_k\le M_{k+1}$ if λ is large (recall that the sequence $\{N_n\}$ grows superexponentially fast). We hence get

$$\begin{cases}\operatorname{comp}(I_j)=\mathcal{N}\\ |I_j^\iota|<1/\lambda^{M_{j-1}}\\ |\partial_\omega I^\iota|<1/4\end{cases}\qquad j=0,\dots,n+1, \tag{7.3}$$

for each $\omega\in\Omega$.

Search for M_{n+1} candidates: We now claim that there always is "room" for the intervals $I_{n+1}\pm M\omega$, for some $N_{n+1}\le M\le 2N_{n+1}$.

Claim: For every $\omega\in\Omega$ there is an integer $M\in[N_{n+1},2N_{n+1}]$ such that

$$\operatorname{dist}\left(I_{n+1}\pm M\omega,\bigcup_{l=0}^{n}\bigcup_{|m|\le 2M_l}(I_l+m\omega)\right)>\frac{1}{\lambda^{M_{n-1}}}.$$

Proof of claim. Fix $\omega\in\Omega$. We write $N=N_{n+1}$ and $\varepsilon_j=1/\lambda^{M_{j-1}}$ ($j=0,\dots,n$). Recall that $M_{-1}=3/4$. Since $\omega\in\Omega\subset\mathcal{F}_n$, $(\mathcal{F}1)_n$ holds, i.e.

$$\operatorname{dist}\left(I_j,\bigcup_{m=1}^{2^{j+6}\mathcal{N}^2M_j}(I_j+m\omega)\right)>3\varepsilon_j,\quad j=0,\dots,n. \tag{7.4}$$

For each $j=0,\dots,n$, if we take $\iota,k\in\{1,\dots,\mathcal{N}\}$ and an integer $L\in\mathbb{Z}$, it follows from (7.4) and (7.3) that there is at most one $p\in J:=[L,L+2^{j+6}\mathcal{N}^2M_j-1]$ such that $\operatorname{dist}(I_j^\iota+p\omega,I_j^k)\le\varepsilon_j$, and at most one $q\in J$ such that $\operatorname{dist}(I_j^\iota - q\omega, I_j^k) \le \varepsilon_j$. Hence there are at most $2(4M_j+1)$ p:s in J such that

$$\operatorname{dist}\left(I_j^\iota \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j^k + m\omega)\right) \le \varepsilon_j.$$

Consequently, there is at most $2(4M_j + 1)\mathcal{N}^2$ different $p \in J$ satisfying

$$\operatorname{dist}\left(I_j \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_j.$$

Now, for each $j = 0, \dots, n$, we divide the interval $[N, N + 2^{n+6}\mathcal{N}^2 M_n - 1]$ into subintervals of length $2^{j+6}\mathcal{N}^2 M_j$. Then we get $[2^{n-j}M_n/M_j]$ intervals, plus maybe one extra, shorter. By the above analysis, and by the fact that $I_{n+1} \subset I_j$ and $\varepsilon_n \le \varepsilon_j$, we get that on each of these subintervals there are at most $2(4M_j + 1)\mathcal{N}^2$ different p:s satisfying

$$\operatorname{dist}\left(I_{n+1} \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_n. \tag{7.5}$$

So, there is at most

$$2(4M_j + 1)\mathcal{N}^2([2^{n-j}M_n/M_j] + 1) \le 20 \cdot 2^{n-j}\mathcal{N}^2 M_n$$

p:s in the interval $[N, N + 2^{n+6}\mathcal{N}^2 M_n - 1]$ such that (7.5) holds. From this we derive that there is at most

$$20(1 + 2 + \ldots + 2^n)\mathcal{N}^2 M_n < 20 \cdot 2^{n+1}\mathcal{N}^2 M_n < 2^{n+6}\mathcal{N}^2 M_n$$

different p:s in $[N, N + 2^{n+6}\mathcal{N}^2 M_n - 1]$ such that

$$\operatorname{dist}\left(I_{n+1} \pm p\omega, \bigcup_{j=0}^{n} \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_n.$$

From this the statement follows, since

$$2^{n+6}\mathcal{N}^2 M_n \le 2^{n+6}\mathcal{N}^2(2N_n) < N_{n+1} = N$$

if λ is large, independent of n (recall the growth rate of the sequence $\{N_n\}$). □

Construction of Ω_+*:* We will now construct the set $\Omega_+ \subset \Omega$ satisfying (7.1) and (7.2). First we let

$$\delta := \frac{1}{12N_{n+1}} \cdot \frac{1}{\lambda^{M_{n-1}}}.$$

Divide the interval Ω into $T - 1$ disjoint intervals $\Omega^1, \dots, \Omega^{T-1}$ of length δ, and one, Ω^T, of length $< \delta$, i.e.

$$\Omega = \bigcup_{i=1}^{T} \Omega^i, \quad |\Omega^i| = \delta, i = 1, \dots, T-1, \quad |\Omega^T| < \delta.$$

Since $\Omega \subset \mathbb{T}$, we clearly have the trivial estimate $T \leq 1/\delta$.

If we now for each interval Ω^i take the midpoint ω_i and apply the above claim, we get an integer $M_{n+1}^i \in [N_{n+1}, 2N_{n+1}]$ so that

$$\operatorname{dist}\left(I_{n+1}(\omega_i) \pm M_{n+1}^i\omega_i, \bigcup_{l=0}^{n} \bigcup_{|m|\leq 2M_l} (I_l(\omega_i) + m\omega_i)\right) > \frac{1}{\lambda^{M_{n-1}}}.$$

Moreover, since $|\Omega^i| \leq \delta$, since $M_0 < \ldots < M_n < M_{n+1}^\iota \in [N_{n+1}, 2N_{n+1}]$ and since $|\partial_\omega I_j^\iota| < 1/4$ for $j = 0, \ldots, n+1$ (proposition 6.4), it is easily seen that

$$\operatorname{dist}\left(I_{n+1}(\omega) \pm M_{n+1}^i\omega, \bigcup_{l=0}^{n} \bigcup_{|m|\leq 2M_l} (I_l(\omega) + m\omega)\right) > 0 \quad \forall\omega \in \Omega^i. \tag{7.6}$$

Hence all $\omega \in \Omega^i$ satisfy $(\mathcal{F}2)_{n+1}$. So, now we only have to remove those ω:s which do not satisfy condition $(\mathcal{F}1)_{n+1}$.

Let Ω_+^i be the set of ω:s in Ω^i which satisfies

$$\operatorname{dist}\left(I_{n+1}(\omega), \bigcup_{m=1}^{2^{(n+1)+6}\mathcal{N}^2 M_{n+1}^i} (I_{n+1}(\omega) + m\omega)\right) > \frac{3}{\lambda^{M_n}}. \tag{7.7}$$

To estimate the size and numbers of intervals in Ω_+^i, we apply lemma 7.4, and think that all "bad" ω:s are concentrated in Ω_+^i. This is of course a very rough estimate, but it is enough for our purpose. We hence get, making use of the above estimates of $I_{n+1}(\omega)$ (equation 7.3) and the fact that $N_{n+1} \leq M_{n+1}^i \leq 2N_{n+1}$ and $M_n \geq N_n$,

$$|\Omega_+^i| \geq |\Omega^i| - 2\mathcal{N}^2(2^{n+7}\mathcal{N}^2 M_{n+1}^i) \cdot \frac{1/\lambda^{M_n} + 3/\lambda^{M_n}}{1/2} \geq |\Omega^i| - \frac{c_1(\mathcal{N})2^n}{N_{n+1}^3}, \tag{7.8}$$

and

$$\operatorname{comp}(\Omega_+^i) \leq \mathcal{N}^2(2^{n+7}\mathcal{N}^2 M_{n+1}^i)^2 \leq c_2(\mathcal{N})4^n N_{n+1}^2. \tag{7.9}$$

Finally we define

$$\Omega_+ = \bigcup_{i=1}^{T} \Omega_+^i.$$

Then, since $T \leq 1/\delta = 12 \cdot N_{n+1} \cdot \lambda^{M_{n-1}} \leq 12 N_{n+1} N_n^8$, it follows from (7.8) and (7.9) that

$$|\Omega_+| \geq \sum_{i=1}^{T} |\Omega^i| - \frac{12 c_1 2^n N_{n+1} N_n^8}{N_{n+1}^3} \geq |\Omega| - \frac{1}{N_{n+1}^{3/2}}$$

and

$$\operatorname{comp}(\Omega_+) \leq \sum_{i=1}^{T} \operatorname{comp}(\Omega_+^i) \leq N_{n+1}^4,$$

provided λ is sufficiently large (independent of n). □

Proof of proposition 7.2. We assume that λ is sufficiently large.

Let us fix $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$. First we let $M_0 := N_0$ and $\Omega_0 = \Omega_0(E)$ be the set of $\omega \in \mathbb{T}$ such that (recall that $M_{-1} = 3/4$)

$$\operatorname{dist}\left(I_0, \bigcup_{m=1}^{2^6\mathcal{N}^2 M_0} (I_0 + m\omega)\right) > \frac{3}{\lambda^{M_{-1}}}. \tag{7.10}$$

(Note that $\Omega_0 = \mathcal{F}_0(M_0, \lambda, E)$). Since $\operatorname{comp}(I_0) = \mathcal{N}$ and $|I_0| < 1/\lambda^{3/4}$ (lemma 6.1) it follows from lemma 7.4 that

$$|\Omega_0| \geq 1 - 2\mathcal{N}^2(2^6\mathcal{N}^2 M_0)\frac{1+3}{\lambda^{3/4}} \geq 1 - \frac{1}{N_0} \tag{7.11}$$

and

$$\operatorname{comp}(\Omega_0) \leq \mathcal{N}^2(2^6\mathcal{N}^2 M_0)^2 \leq N_0^4. \tag{7.12}$$

Assume now that we inductively have gotten sets $\Omega_n \subset \ldots \subset \Omega_0$, where each Ω_j satisfies

- $(O1)_j$ $\operatorname{comp}(\Omega_j) \leq N_0^4 \cdots N_j^4$.
- $(O2)_j$ $|\Omega_j| \geq 1 - \sum_{i=0}^{j} \frac{1}{N_i}$.
- $(O3)_j$ For each $1 \leq \iota \leq \operatorname{comp}(\Omega_j)$ there is a sequence $(M_0, \ldots, M_j)$, $N_i \leq M_i \leq 2N_i$, such that $\Omega_j^\iota \subset \mathcal{F}_j(M_0, \ldots, M_j, \lambda, E)$.

Now we want to construct Ω_{n+1}. To each component in Ω_n we can apply proposition 7.3. Hence, for each Ω_n^ι in Ω_n we get a set $\Omega_{n+1}^\iota \subset \Omega_n^\iota$ which contains at most N_{n+1}^4 components, which satisfies

$$|\Omega_n^\iota| - |\Omega_{n+1}^\iota| \leq \frac{1}{N_{n+1}^{3/2}},$$

and where each component satisfies $(O3)_{n+1}$. Consequently, letting

$$\Omega_{n+1} = \bigcup_{\iota=1}^{\operatorname{comp}(\Omega_n)} \Omega_{n+1}^\iota$$

gives us

$$\operatorname{comp}(\Omega_{n+1}) \leq N_{n+1}^4 \operatorname{comp}(\Omega_n) \leq N_0^4 \cdots N_{n+1}^4$$

and

$$|\Omega_n| - |\Omega_{n+1}| \leq \frac{\operatorname{comp}(\Omega_n)}{N_{n+1}^{3/2}} \leq \frac{N_0^4 \cdots N_n^4}{N_{n+1}^{3/2}} \leq \frac{1}{N_{n+1}}.$$

Hence Ω_{n+1} satisfies $(O1-3)_{n+1}$.

We now let

$$\Omega_E := \bigcap_n \Omega_n \setminus (\mathbb{Q}/\mathbb{Z}).$$

From the above estimates it follows that Ω_E is measurable and satisfies

$$|\Omega_E| \geq 1 - \sum_{i=0}^{\infty} \frac{1}{N_i} \geq 1 - \frac{2}{\lambda^{1/4}}.$$

Moreover, for each $\omega \in \Omega_E$ there is an infinite sequence $(M_0, M_1, \ldots)$, where $N_i \leq M_i \leq 2N_i$, such that $\omega \in \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ for all n. □

8. Proof of theorem 1 and 2

In this section we finish the proof of Theorem 1 and 2. Hence forward we assume that λ is sufficiently large, depending on $s, |\mathcal{E}_0|, \|V\|_{C^1}$ and $\mathcal{N}$.

First we fix $E \in [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$ and let Ω_E be the set given by proposition 7.2. Take $\omega \in \Omega_E$. Then we get a sequence $\{M_k\}_{k=1}^{\infty}$, satisfying

$$N_k \leq M_k \leq 2N_k \quad k \geq 0, \tag{8.1}$$

such that

$$\omega \in (\mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})) \cap \bigcap_n \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E).$$

From now on E, ω and $\{M_k\}$ are fixed as above.

From proposition 6.4 we get

$$\text{comp}(I_n) = \mathcal{N}, \quad 0 < |I_n^\iota| \leq 1/\lambda^{M_{n-1}}, \quad n \geq 0. \tag{8.2}$$

This implies that the set

$$\Theta_\infty := \bigcap_n \Theta_n = \mathbb{T} \setminus \bigcup_{l=0}^{\infty} \bigcup_{|m| \leq M_l - 1} (I_l + m\omega), \tag{8.3}$$

which is closed, satisfies

$$|\Theta_\infty| \geq 1 - \sum_{n=0}^{\infty} (2M_n - 1)|I_n| \geq 1 - 4\mathcal{N} \sum_{n=0}^{\infty} N_n / \lambda^{N_{n-1}} \overset{\lambda \to \infty}{\to} 1.$$

Hence we have $|\Theta_\infty| > 0$ for large λ. (Here we used the estimates (8.1) and (8.2) together with the definition of the N_n:s.)

We can now prove that $(*)$ and $(**)$ in Theorem 1 hold:

Proof of $(*)$. Fix $\theta_0 \in \Theta_\infty$ and take any $|r_0| > \lambda$. Let $0 < T_0 \leq T_1 \leq T_2 \leq \ldots$ be the first times such that

$$\theta_{T_n} \in I_n,$$

i.e. $N = T_n$ is the smallest positive integer such that $\theta_N \in I_n$. These times exists since $\omega \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$ and since $|I_n| > 0$. Moreover, since $\theta_0 \in \Theta_\infty$, we have $T_n \geq M_n$ for all n. From proposition (5.1) it now follows that

$$|r_0 \cdots r_{T_n}| > \lambda^{T_n/2}. \tag{8.4}$$

Recalling that (in view of the Schrödinger equation (1.1))

$$r_0 = u_1/u_0 \quad \text{and} \quad r_0 \cdots r_n = u_{n+1}/u_0,$$

we see that (8.4) implies

$$\limsup_{n\to\infty} \frac{1}{2n} \log(u_n^2 + u_{n+1}^2) \geq \limsup_{n\to\infty} \frac{1}{2T_n} \log(\lambda^{T_n}) = \frac{\log\lambda}{2}. \tag{8.5}$$

By the Oseledets theorem, we have for each $(u_0, u_1) \in \mathbb{R}^2 \setminus \{0\}$:

$$\pm\gamma(E) = \lim_{n\to\infty} \left\| A_E^n(\theta) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\| = \lim_{n\to\infty} \frac{1}{2n}(u_n^2 + u_{n+1}^2), \quad \text{a.e. } \theta.$$

Since Θ_∞ has positive measure, (8.5) implies that

$$\gamma(E) \geq \frac{\log\lambda}{2}.$$

□

Proof of $(**)$. Take

$$(\theta_0, r_0) \in \bigcap_n \left(\Phi^{M_n}(A_n) \cap \Phi^{-M_n}(B_n)\right) \neq \emptyset.$$

The intersection is non-empty by corollary 5.2 and proposition 6.4 ($|I_n| > 0$ and $(I3)_n$ hold for all n). Applying proposition 5.1 to each of the points

$$(\theta_{-M_n}, r_{-M_n}) \in A_n, \quad (\theta_{M_n}, r_{M_n}) \in B_n,$$

gives

$$|u_1/u_{-k}| = |r_{-k} \cdots r_0| > \lambda^{(k+1)/2}, \quad \forall k \geq 0,$$

and

$$|u_{k+1}/u_0| = |r_1 \cdots r_k| < \lambda^{-(k+1)/2}, \quad \forall k \geq 1.$$

□

We now continue with the proof of the statement in Theorem 2.

Invariant measures: Since we have $\gamma(E) > 0$, it follows from [4] that Φ has exactly two invariant and ergodic probability measures (see subsection 1.2).

Proof of the minimality. Let us fix $\iota \in \{1, \ldots, \mathcal{N}\}$ and work with the non-empty intervals

$$I_0^\iota \supset I_1^\iota \supset I_2^\iota \supset \cdots.$$

We recall (8.2).

Forward iterations: Take any $\theta_0 \in \Theta_\infty$ and $|r_0| > \lambda$. We shall now prove that the forward orbit $\{(\theta_k, r_k)\}_{k=0}^\infty$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$.

Firstly, the points $\{\theta_k\}_{k=0}^{\infty}$ are dense in $\mathbb{T}$ since ω is irrational. Since $(C2)_n$ in proposition (5.1) hold for each n, we have

$$|r_k| \leq \lambda \Longrightarrow \theta_k \in \bigcup_{l=0}^{\infty} \bigcup_{m=1}^{M_l} (I_l + m\omega), \quad k \geq 0, \tag{8.6}$$

From $(\mathcal{F}1-2)_n$ we derive

$$(I_n - M_n\omega) \cap \bigcup_{l=0}^{n} \bigcup_{m=1}^{M_l} (I_l + m\omega) = \emptyset.$$

Combining this with (8.6) yields

$$|r_k| \leq \lambda \text{ and } \theta_k \in I_n^{\iota} - M_n\omega \Longrightarrow \theta_k \in \bigcup_{l=n+1}^{\infty} \bigcup_{m=1}^{M_l} (I_l + m\omega), \quad k \geq 0. \tag{8.7}$$

Now, for $n \geq 1$ we let

$$X_n = \{(\theta_k, r_k) : \theta_k \in I_n^{\iota} - M_n\omega, |r_k| > \lambda, k \geq 0\} \subset A_n.$$

Since

$$\left| \bigcup_{l=n+1}^{\infty} \bigcup_{m=1}^{M_l} (I_l + m\omega) \right| \leq 2\mathcal{N} \sum_{l=n+1} \underbrace{N_l/\lambda^{N_{l-1}}}_{1/\lambda^{3N_{l-1}/4}} < 1/\lambda^{N_n/2},$$

it follows from (8.7) that $\pi_1(X_n)$ is $1/\lambda^{N_n/2}$-dense in $I_n^{\iota} - M_n\omega$. Since also

$$\max\{\lambda^{7M_{n-1}}/\lambda^{N_n/2}, 1/\lambda^{M_{n-1}}\} \leq 1/\lambda^{N_{n-1}},$$

(recall (8.1)) it follows from $(I4)_n$ in proposition 6.4 that

$$\pi_2(\Phi^{M_n+M_{n-1}}(X_n)) \text{ is } 1/\lambda^{N_{n-1}}\text{-dense in } (-1/\lambda, 1/\lambda). \tag{8.8}$$

We now note that

$$I_n^{\iota} + M_{n-1}\omega \subset I_{n-1} + M_{n-1}\omega \overset{\text{L. 3.3}}{\subset} \Theta_{n-1} \tag{8.9}$$

and that $(\mathcal{F}1-2)_{n-1}$ gives

$$I_{n-1} + (M_{n-1}+1)\omega \subset \Theta_{n-1}. \tag{8.10}$$

Take any sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n \in I_n^{\iota} + M_{n-1}\omega$. Then, by (8.9) and (8.10) we have

$$p_n, p_n + \omega \in \Theta_{n-1}, \quad n \geq 1. \tag{8.11}$$

By the compactness of $\mathbb{T}$, there exists a subsequence $\{n_j\}$ and a $p \in \mathbb{T}$ such that

$$p_{n_j} \to p \text{ as } n \to \infty.$$

Since the sets $\Theta_{n_j} \subset \Theta_{n_{j-1}}$ are closed, it follows from (8.11) that

$$p, p + \omega \in \Theta_{\infty}. \tag{8.12}$$

Since also $|I_n^\iota| \to 0$ as $n \to \infty$, we must have

$$I_{n_j}^\iota + M_{n_j-1}\omega \to \{p\}$$

in the Hausdorff metric. From the fact that

$$\pi_1\left(\Phi^{M_n+M_{n-1}}(X_n)\right) \subset I_n^\iota + M_{n-1}\omega,$$

it hence follows from (8.8) that

$$\overline{\{(\theta_k, r_k)\}_{k=0}^\infty} \supset \{p\} \times [-1/\lambda, 1/\lambda] := \Gamma. \tag{8.13}$$

We shall now iterate the vertical line-segment Γ. Since $I_0 \cap \Theta_\infty = \emptyset$, lemma 4.1 implies

$$\Phi(\Gamma) = \{p+\omega\} \times (\hat{\mathbb{R}} \setminus (r^-, r^+))$$

where $\lambda < r^- < r^+ < \infty$ or $-\infty < r^- < r^+ < -\lambda$. Now, if we take any $(\theta_0^-, r_0^-), (\theta_0^+, r_0^+) \in \{p+\omega\} \times (r^-, r^+)$, i.e. $\theta_0^\pm = p+\omega \in \Theta_\infty$ (by (8.12)) and $|r_0^\pm| > \lambda$, we can proceed as in the proof of $(*)$, making use of the formula

$$|r_k^+ - r_k^-| = \frac{|r_0^+ - r_0^-|}{(r_0^+ \cdots r_{k-1}^+)(r_0^- \cdots r_{k-1}^-)},$$

to show that the length of the vertical line-segments $\Phi^k(\{p+\omega\} \times (r^-, r^+))$ goes to zero as $k \to \infty$. From this we conclude that

$$\overline{\bigcup_{k=0}^\infty \Phi^k(\Gamma)} = \mathbb{T} \times \hat{\mathbb{R}}.$$

This, together with (8.13) now yields

$$\overline{\{(\theta_k, r_k)\}_{k=0}^\infty} = \mathbb{T} \times \hat{\mathbb{R}}.$$

Backward iterations: Similarly one shows that if $\theta_0 \in \Theta_\infty$ and $|r_0| < 1/\lambda$, then the backward orbit $\{(\theta_{-k}, r_{-k})\}_{k=0}^\infty$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$ (one first proves that $\overline{\{(\theta_{-k}, 1/r_{-k})\}_{k=0}^\infty} \supset \{q\} \times [-1/\lambda, 1/\lambda]$ for some $q \in \mathbb{T}$ (recall $(I5)_n$)).

The minimality: To finish the proof we do as follows: Let

$$\Theta'_\infty := \mathbb{T} \setminus \bigcup_{l=0}^\infty \bigcup_{|m| \le M_l} (I_l + m\omega)$$

(compare with Θ_∞, (8.3)). Then we have

$$\theta \in \Theta'_\infty \quad \Longrightarrow \quad \theta, \theta - \omega \in \Theta_\infty.$$

Note also that $|\Theta'_\infty| > 0$.

Now, take any $\theta_0 \in \Theta'_\infty$ and any $r_0 \in \hat{\mathbb{R}}$. If $|r_0| > \lambda$, it follows from the forward iteration analysis above that $\{(\theta_k, r_k)\}_{k=0}^\infty$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$.

If $|r_0| \le \lambda$, then, by lemma 4.1, $|r_{-1}| < 1/\lambda$. Since $\theta_{-1} \in \Theta_\infty$, we get that $\{(\theta_{-k}, r_{-k})\}_{k=0}^\infty$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$.

From this we conclude that the two graphs, given by the directions in the Oseledet's theorem, are dense in $\mathbb{T} \times \hat{\mathbb{R}}$, and hence that Φ is minimal (recall the discussion in subsection 1.2). $\square$

Selection of the set $\mathcal{S}$: Now it only remains to find a measurable set $\mathcal{S}$ satisfying the statement in Theorem 1. To do this we define the three sets

$$A := \{(E, \omega) \in \mathcal{E}_\lambda \times \mathbb{T} : (*) \text{ hold}\}$$

$$B := \{(E, \omega) \in \mathcal{E}_\lambda \times \mathbb{T} : (**) \text{ hold}\}$$

$$C := \{(E, \omega) \in \mathcal{E}_\lambda \times \mathbb{T} : \Phi_{E,\omega} \text{ is minimal}\}$$

where $\mathcal{E}_\lambda = [\lambda^2 E_0 + 3\lambda, \lambda^2 E_1 - 3\lambda]$. It is standard to verify that each of these sets are measurable. We define

$$\mathcal{S} = A \cap B \cap C.$$

By the above analysis, we have that for each $E \in \mathcal{E}_\lambda$, the set Ω_E, given by proposition 7.2, satisfies

$$\{E\} \times \Omega_E \subset \mathcal{S}.$$

Since $|\Omega_E| > 1 - 2/\lambda^{1/4}$, it hence follows that

$$|\mathcal{S}| \geq |\mathcal{E}_\lambda|(1 - 2/\lambda^{1/4}) \geq \lambda^2 |\mathcal{E}_0|(1 - 1/\lambda^{1/5}).$$

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Paper II

POSITIVE LYAPUNOV EXPONENT FOR A CLASS OF 1-D QUASI-PERIODIC SCHRÖDINGER EQUATIONS — THE CONTINUUM CASE

KRISTIAN BJERKLÖV

1. Introduction

1.1. **The Schrödinger equation.** In this paper we shall study continuous quasi-periodic Schrödinger equations

$$(H_\theta u)(t) = -\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) + \lambda^2 V(t, \theta + \omega t)u(t) = Eu(t), \tag{1.1}$$

for large coupling factors λ and energies E in the bottom of the spectrum of the corresponding Schrödinger operator H_θ, acting on the Hilbert space $L^2(\mathbb{R})$ of square integrable functions. The potential function $V : \mathbb{T}^2 \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is always assumed to be at least continuous and the frequency $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

It is well known that the operator H_θ is self-adjoint and that its spectrum, $\sigma(H_\theta)$, is a closed non-void subset of the interval $[\inf \lambda^2 V, \infty)$, which is independent of θ. Hence we shall often write only $\sigma(H)$.

Since we shall study equation (1.1) from a dynamical systems point of view, we write it as the system

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda^2 V(t, \theta + \omega t) - E & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}, \tag{1.2}$$

where $' = d/dt$. By $F_E(t, \theta)$ we denote the fundamental solution of (1.2), i.e. $F_E(t, \theta)$ satisfies

$$F_E'(t, \theta) = \begin{pmatrix} 0 & 1 \\ \lambda^2 V(t, \theta + \omega t) - E & 0 \end{pmatrix} F_E(t, \theta), \quad F_E(0, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have $F_E(t, \theta) \in SL(2, \mathbb{R})$ for all t. The average exponential growth of the fundamental solution is measured by the Lyapunov exponent

$$\gamma(E) = \lim_{t \to \infty} \frac{1}{t} \int_{\mathbb{T}} \log \|F_E(t, \theta)\| d\theta \geq 0,$$

which exists by subadditivity and which is non-negative since F_E is in $SL(2, \mathbb{R})$. Moreover, the limit is independent of the choice of matrix norm. General properties of the Lyapunov exponent can be found in [8].

For energies E outside the spectrum $\sigma(H)$ one always have $\gamma(E) > 0$ [7], but what happens for E in the spectrum, and in particular in the lower part of the spectrum, is not well known. Firstly, in [2], L. H. Eliasson, generalizing the pioneer work of Dinaburg-Sinai [1], shows that if V is real-analytic and ω Diophantine, then the upper part of the spectrum $\sigma(H)$ is absolutely continuous and, moreover, the Lyapunov exponent $\gamma(E) = 0$ for a.e. E in this region (this result also holds for quasi-periodic potentials V defined on $\mathbb{T}^d$). For small coupling constants λ, this result extends to the whole spectrum. For the behavior in the bottom of the spectrum in the large coupling regime, there are essentially only results for the potential function

$$V(x, y) = \cos(2\pi x) + \cos(2\pi y).$$

For this particular potential, Fröhlich-Spencer-Wittwer [4] show that

$$\sigma(H_\theta) \cap [\inf \sigma(H_\theta), \inf \sigma(H_\theta) + \text{const}(\lambda, \omega)]$$

is pure-point for a.e. θ, provided λ is large and ω Diophantine. Moreover, in [11] Sorets-Spencer prove that in the same region of the spectrum (minus some small gaps), $\gamma(E) \geq \lambda + o(1)$ for λ large and ω irrational.

Similar problems have also been studied in [3]

Our first result is the following:

Theorem 1. *Assume that the potential function $V : \mathbb{T}^2 \to \mathbb{R}$ is C^3 and satisfies*

- $||V||_{C^0} = 1$.
- *V has a unique minimum at $(1/2, 1/2)$ and $V(1/2, 1/2) = 0$.*
- *$V(x, y) = (x-1/2)^2 + (y-1/2)^2 + \dots$ in a neighborhood of $(1/2, 1/2)$.*

Then there exists a $\lambda_0 > 0$ and a constant $c > 0$ such that for all $\lambda > \lambda_0$ there exists a measurable set $\mathcal{S} \in [1.1\lambda, 1.2\lambda] \times ((0, 1/4) \setminus \mathbb{Q})$, of measure

$$|\mathcal{S}| \geq 0.1\lambda \left(\frac{1}{4} - \frac{1}{\lambda^{1/4}} \right),$$

and for all $(E, \omega) \in \mathcal{S}$ we have:

(∗) *$\gamma(E) \geq c\lambda$.*

(∗∗) *There is a $\theta \in \mathbb{T}$ such that E is an eigenvalue of H_θ with an exponentially decaying eigenfunction.*

Remark 1. The assumptions on the potential function V are made only for simplicity. One can obtain similar results for any C^3 potential function with a unique minimum which is quadratic, i.e., for a generic (in the C^3 topology) C^3-function $V : \mathbb{T}^2 \to \mathbb{R}$.

As a consequence of Theorem 1 we have:

Corollary 1.1. *For all $\lambda > \lambda_0$ there is a set $\Omega \in (0, 1/4)$, of measure $1/4 - o(1)$, such that there to each $\omega \in \Omega$ there corresponds a set $\mathcal{E}_\omega \subset [1.1\lambda, 1.2\lambda]$, $|\mathcal{E}_\omega| = 0.1\lambda - o(\lambda)$, satisfying* (*i*) $\gamma(E) > c\lambda,\quad$ *for all* $E \in \mathcal{E}_\omega$

(*ii*) $\mathcal{E}_\omega \subset \bigcup_{\theta\in\mathbb{T}} \sigma_{pp}(H_\theta) \subset \sigma(H)$.

Proof. The existence of the sets Ω and $\mathcal{E}_\omega$, as well as their measures, follows by an application of the Fubini theorem on the set $\mathcal{S}$. By $(**)$, for each $E \in \mathcal{E}_\omega$ we have $E \in \sigma_{pp}(H_\theta)$ for some $\theta \in \mathbb{T}$, and since the spectrum is independent of θ, this gives (*ii*). Statement (*i*) is immediate. □

1.2. **The Riccati equation:** The proof of Theorem 1 will be made through a careful analysis of the flow map (the projective flow)

$$\Phi_E^t : (x_0, \theta_0, r(0)) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \mapsto (x_0 + t, \theta_0 + t\omega, r(t)) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \tag{1.3}$$

($\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) where

$$r(t) = u'(t)/u(t),$$

and $u(t)$ satisfies equation

$$-u''(t) + \lambda^2 V(x_0 + t, \theta_0 + \omega t)u(t) = Eu(t).$$

From the analysis of Φ_E^t we also derive:

Theorem 2. *For $\lambda > \lambda_0$ and for $(E,\omega) \in \mathcal{S}$, the flow map Φ_E^t is minimal.*

1.3. **Skew-products.** One can also investigate the dynamics of system (1.2) via the skew-product mapping

$$G_E : (\theta, x) \in \mathbb{T} \times \mathbb{R}^2 \mapsto (\theta + \omega, F_E(1,\theta)x) \in \mathbb{T} \times \mathbb{R}^2, \tag{1.4}$$

where $F_E(t,\theta)$ is as above. To this skew-product we associate the diffeomorphism

$$\tilde{G}_E : \mathbb{T} \times \mathbb{P}^1(\mathbb{R}) \to \mathbb{T} \times \mathbb{P}^1(\mathbb{R})$$

induced by G_E, where $\mathbb{P}^1(\mathbb{R})$ is the projective 1-space (the set of all lines through $(0,0)$ in $\mathbb{R}^2$). Note that $\tilde{G}_E$ is well-defined since $F_E(t,\theta) \in SL(2,\mathbb{R})$. If $\gamma(E) > 0$, then there exists exactly two $\tilde{G}_E$-invariant ergodic probability measures μ^+, μ^-, and they are supported on the two graphs given by the directions in the Oseledet's theorem [5]. If moreover $\gamma(E) > 0$ and $E \in \sigma(H)$ (E is in the spectrum iff G_E is not uniformly hyperbolic (c.f. [9])), then there exists a unique $\tilde{G}_E$-invariant minimal set $\emptyset \neq M \subset \mathbb{T} \times \mathbb{P}^1(\mathbb{R})$ and

$$\mathrm{supp}(\mu^+) = \mathrm{supp}(\mu^-) = M,$$

(c.f. [6] and [10]). From Theorem 1 and Theorem 2 we derive

Theorem 3. *For $\lambda > \lambda_0$ and $(E,\omega) \in \mathcal{S}$ we have the following: The diffeomorphism $\tilde{G}_E$*

- *is minimal.*
- *has exactly two ergodic invariant probability measures.*

2. Preliminaries and notations

We shall assume that V is C^3 and normalized, so that

$$\|V\|_{C^0} = 1.$$

We also assume that V has a unique minimum, located at $(1/2, 1/2)$, and that $V(1/2, 1/2) = 0$. Moreover, we assume that it is of the form

$$V(x, y) = (x - 1/2)^2 + (y - 1/2)^2 + h.o.t.$$

This implies that there exists a $\delta_0 > 0$ (which we shall take of the form $\delta_0 = 1/k, k \in \mathbb{Z}^+$, i.e. $1/\delta_0$ is an integer) such that

$$V(x, y) > 5\delta_0^2, \quad \forall (x, y) \in [0, 1]^2 \text{ such that } |x - 1/2| \geq 1/20.$$

We shall work with energies E in the interval

$$\mathcal{E} = \mathcal{E}(\lambda) := [1.1\lambda, 1.2\lambda]$$

and frequencies ω in

$$\mathcal{F}_{-1} := (0, 1/4).$$

From now on, $\delta_0, \mathcal{E}$ and $\mathcal{F}_{-1}$ shall always be as above.

2.1. **Notations.** In this paper we shall use the following notations:

2.1.1. *Sets.* For sets I and J in $\mathbb{T}$ and $\alpha \in \mathbb{T}$ we define

$$I + \alpha = \{x + \alpha : x \in I\},$$

$$|I| = \text{Lebesgue measure of } I,$$

$$\overline{I} = \text{the closure of } I,$$

$$\text{comp}(I) = \text{number of components in } I,$$

and

$$\text{dist}(I, J) = \begin{cases} \infty & \text{if } I = \emptyset \text{ or } J = \emptyset \\ \inf\{|x - y| : x \in I \text{ and } y \in J\} & \text{otherwise} \end{cases}$$

where $|\cdot|$ is the standard metric on $\mathbb{T}$.

If $I = I(\omega) = [x(\omega), y(\omega)]$, where $x(\omega)$ and $y(\omega)$ are defined on an open set in $\mathbb{T}$ and differentiable, we use the notation

$$|\partial_\omega I(\omega)| = \max\{|x'(\omega)|, |y'(\omega)|\}.$$

Finally, we say that a set $X \subset \mathbb{T}$ is ε-dense in an interval $I \subset \mathbb{T}$ if every interval $J \subset I$ of length $> \varepsilon$ intersects X.

2.1.2. *Iterates.* Let E, λ, ω and $\theta_0, r(0)$ be fixed. Denote by $u(t)$ the solution of

$$-u''(t) + \lambda^2 V(t, \theta_0 + t\omega)u(t) = Eu(t)$$

with initial conditions $u(0) = \cos(\alpha)$ and $u'(0) = \sin(\alpha)$, where $\tan(\alpha) = r(0)$. Moreover, denote

$$r(t) = u'(t)/u(t).$$

Then $r(t)$ satisfies the Riccatic equation

$$r'(t) = -r(t)^2 + \lambda^2 V(t, \theta_0 + t\omega) - E,$$

whenever $u(t) \neq 0$. Furthermore, the flow map Φ^t is defined as in (1.3), so, in particular,

$$\Phi^t(0, \theta_0, r(0)) = (t, \theta_0 + \omega t, r(t)).$$

Finally, we use the notation

$$\theta_k = \theta_0 + k\omega, \quad k \in \mathbb{Z}.$$

2.1.3. *Projections.* We define the projections π_1, π_2 and π_3 onto the first, second and third coordinate, respectively, by

$$\pi_1 : (x, \theta, r) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \to x \in \mathbb{T},$$
$$\pi_2 : (x, \theta, r) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \to \theta \in \mathbb{T},$$
$$\pi_3 : (x, \theta, r) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \to r \in \hat{\mathbb{R}}.$$

We also define

$$\pi_{23} : (x, \theta, r) \in \mathbb{T}^2 \times \hat{\mathbb{R}} \to (\theta, r) \in \mathbb{T} \times \hat{\mathbb{R}}.$$

3. Critical sets and no-fast-returns

3.1. **The critical sets.** We now define the main actors of the proof, namely the nested sequence of intervals, which under certain assumptions, stated in the propositions below, shall give us the needed estimates on Φ^t. These are the *critical sets*.

Definition 3.1. For every $\lambda > 0, E, \omega \in \mathbb{R}$ and for every sequence $M_0 < M_1 < M_2 < \ldots$ of positive integers we define a nested sequence $I_0 \supset I_1 \supset I_2 \supset \ldots$ of open subsets (possibly void) of $\mathbb{T}$ by

$$I_0 := \{\theta \in \mathbb{T} : (t, \theta + \omega t) \in \mathcal{B} \text{ for some } t \in [0, 1])\},$$

where $\mathcal{B} \subset \mathbb{T}^2$ is the smallest open ball centered at $(1/2, 1/2)$ and containing the set $\{(x, y) : \lambda^2 V(x, y) - E < 0\}$, and

$$I_{j+1} := \pi_2 \left[\Phi^{M_j}(A_j) \cap \Phi^{-M_j}(B_j)\right], \ j \geq 0,$$

where

$$A_j := \{(0, \theta, r) : \theta \in I_j - M_j\omega, \ \delta_0\lambda < r < 2\lambda\},$$
$$B_j := \{(0, \theta, r) : \theta \in I_j + M_j\omega, \ -2\lambda < r < -\delta_0\lambda\}.$$

3.2. **A "no-fast-return" condition.** For the future analysis, it will be important to have a certain "no-fast-return" condition on ω and θ:

Definition 3.2. For any sequence $0 < M_0 < M_1 < \ldots$ of positive integers, $E \in \mathbb{R}$ and $\lambda > 0$, we for each $n \geq 0$ define $\mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ to be the set of $\omega \in \mathcal{F}_{-1}$ satisfying

$$(\mathcal{F}1)_n \qquad \operatorname{dist}\left(I_j, \bigcup_{m=1}^{2^{j+6}M_j/\delta_0} (I_j + m\omega)\right) > \frac{3}{\lambda^{M_{j-1}}}, \quad j = 0, 1, \ldots, n,$$

where $M_{-1} = 3/4$, and

$$(\mathcal{F}2)_n \qquad \operatorname{dist}\left(I_j \pm M_j\omega, \bigcup_{l=0}^{j-1} \bigcup_{|m| \leq 2M_l} (I_l + m\omega)\right) > 0, \quad j = 1, 2, \ldots n.$$

Moreover, we define

$$\Theta_n := \mathbb{T} \setminus \bigcup_{l=0}^{n} \bigcup_{|m| \leq M_l - 1} (I_l + m\omega) \quad \text{and} \quad \Theta_{-1} = \mathbb{T}.$$

Remark 2. The first condition is a "Diophantine condition" which just says that if we enter the critical set I_j, then we stay away from it for a "long" time after.

Remark 3. For a fixed sequence $0 < M_0 < M_1 < M_2 < \ldots$ and fixed $E \in \mathbb{R}$, $\lambda > 0$ we clearly have

$$(0, 1/4) = \mathcal{F}_{-1} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset,$$

and

$$\mathbb{T} = \Theta_{-1} \supset \Theta_0 \supset \Theta_1 \supset \Theta_2 \supset \ldots.$$

4. Basic estimates

Many of the estimates in this paper will be made using the integral

$$\int r(t)\mathrm{d}t = \int u'(t)/u(t)\mathrm{d}t.$$

By regarding this as a principal value integral, one verifies easily that

$$r(0) \neq \infty \quad \Rightarrow \quad \int_{-T}^{0} r(t)\mathrm{d}t = \log\left|\frac{u(0)}{u(-T)}\right| \in \mathbb{R} \cup \{+\infty\}, \quad T \geq 0,$$

$$r(0) \neq \infty \quad \Rightarrow \quad \int_{0}^{T} r(t)\mathrm{d}t = \log\left|\frac{u(T)}{u(0)}\right| \in \mathbb{R} \cup \{-\infty\}, \quad T \geq 0,$$

and that both the integrals are well defined. Note that $u(t) = 0$ implies that $u'(t) \neq 0$.

Lemma 4.1. *There exists a* $\lambda_1 = \lambda_1(\delta_0) > 0$ *such that for all* $\lambda > \lambda_1$, $\omega \in \mathbb{R}$ *and* $E \in \mathcal{E}$, *the following hold:*

Forward iteration: if $\theta_0 \in \mathbb{T} \setminus I_0$, *then*

$$\infty \neq r(0) \geq 0 \quad \Rightarrow \quad \int_T^1 r(t)\,dt > \delta_0\lambda(1-T), \quad 0 \leq T \leq 1,$$

and

$$r(0) \in \hat{\mathbb{R}} \setminus (-2\lambda, -\delta_0\lambda) \quad \Rightarrow \quad r(1) \in (2\delta_0\lambda, 3\lambda/2).$$

Backward iteration: if $\theta_0 \in \mathbb{T} \setminus (I_0 + \omega)$, *then*

$$\infty \neq r(0) \leq 0 \quad \Rightarrow \quad \int_{-1}^T r(t)\,dt < -\delta_0\lambda(T+1), \quad -1 \leq T \leq 0,$$

and

$$r(0) \in \hat{\mathbb{R}} \setminus (\delta_0\lambda, 2\lambda) \quad \Rightarrow \quad r(-1) \in (-3\lambda/2, -2\delta_0).$$

Proof. We begin with the forward case. Assume that λ is sufficiently large, depending only on δ_0. Let

$$w(t) = \lambda^2 V(t, \theta_0 + t\omega) - E,$$

so $r(t)$ satisfies equation $r'(t) = -r(t)^2 + w(t)$. Moreover, we let

$$J_1 = [0, 1/2 - 1/20], \quad J_2 = [1/2 + 1/20, 1].$$

From the choice of δ_0 and the assumptions on E we have

$$w(t) > 5\delta_0^2\lambda^2 - 1.2\lambda > 4\delta_0^2\lambda^2, \quad t \in J_1 \cup J_2 \tag{4.1}$$

and since $||V||_{C^0} = 1$ we have the upper bound

$$w(t) \leq \lambda^2 - E < \lambda^2. \tag{4.2}$$

By the fact that $\theta_0 \in \mathbb{T} \setminus I_0$ we also have

$$w(t) \geq 0, \quad t \in [0,1]. \tag{4.3}$$

First we notice that since (4.1) holds, if $r(t_0) = 2\delta_0\lambda$ for some $t_0 \in J_i$ $(i = 1, 2)$, then $r'(t_0) = -r(t_0)^2 + w(t) > 0$. Hence

$$r(t_0) > 2\delta_0\lambda, t_0 \in J_i \quad \Rightarrow r(t) > 2\delta_0\lambda_0 \text{ for all } t \in J_i \text{ such that } t \geq t_0. \tag{4.4}$$

Next we show that if $r(0) \notin (-2\lambda, -\delta_0\lambda)$, then $r(t_0) > 2\delta_0\lambda$ for some $0 \leq t_0 < 1/20$. There are two cases: If $r(0) \leq -2\lambda$, we compare $r(t)$ with the solution of

$$s'(t) = -s(t)^2 + \lambda^2, \quad s(0) = -2\lambda.$$

By (4.2) we note that $r(t)$ must have a singularity before $s(t)$. Since $s(t)$ is explicit, one verifies that there is a $0 < \tau < 1/20$ such that $s(t)$ has a singularity at $t = \tau$. This shows that there is a $t_0 \in (0, 1/20)$ such that $r(t_0) > 2\delta_0\lambda$. If $r(0) \geq -\delta_0\lambda$, we compare $r(t)$ with $s(t)$ satisfying

$$s'(t) = -s(t)^2 + 4\delta_0^2\lambda^2, \quad s(0) = -\delta_0\lambda,$$

which by (4.1) satisfies $s(t) \leq r(t)$ for $t \in J_1$. Solving the above equation shows that $s(t_0) > 2\delta_0\lambda$ for some $t_0 \in (0, 1/20)$.

Now it follows from (4.4) that

$$r(t) > 2\delta_0\lambda \quad \text{for } 1/20 \le t \le 1/2 - 1/20. \tag{4.5}$$

Since (4.3) hold and since we just have shown that $r(1/2-1/20) > 0$, we have the estimate

$$r(t) \ge 0, \quad t \in [1/2 - 1/20, 1/2 + 1/20]. \tag{4.6}$$

In particular we have $r(1/2+1/20) \ge 0$, so if we proceed as above (as in the case $r(0) \ge -\delta_0\lambda$) we get (again making use of (4.4)) that

$$r(t) > 2\delta_0\lambda \quad \text{for } 1/2 + 1/10 \le t \le 1. \tag{4.7}$$

Noticing that (4.1) implies that if $r(0) \ge 0$, then $r(t) \ge 0$ for $0 \le t \le 1/20$, it now follows from (4.5)-(4.7) that

$$\int_T^1 r(t)\mathrm{d}t > \delta_0\lambda(1-T), \quad 0 \le T \le 1.$$

This proves the first statement.

Finally, since $r(1/2+1/20) \ge 0$ we compare $r(t)$ with the solution of the equation

$$s'(t) = -s(t)^2 + \lambda^2, \quad s(1/2+1/20) = r(1/2+1/20).$$

Then, by (4.2) we have $r(t) \le s(t)$ for $1/2+1/20 \le t \le 1$, and, solving the above equation yields $s(1) < 3/2\lambda$. Together with (4.7) this now gives the second statement.

To prove the backward case, we use the following symmetry: If we want to study equation $r'(t) = -r(t)^2 + w(t)$ for negative t, we let $q(t) = -r(-t)$. Then $q'(t) = -q(t)^2 + w(-t)$ and we can proceed as above.

□

Lemma 4.2. *For all $\omega \in \mathbb{R}, \theta_0 \in \mathbb{T}$, $\lambda > 0$ and $E \ge 0$ we have:*

$$|r(0)| \le 2\lambda \Rightarrow \int_{-T}^0 r(t)dt \ge -2\lambda T, \quad T \ge 0,$$

and

$$|r(0)| \le 2\lambda \Rightarrow \int_0^T r(t)dt \le 2\lambda T, \quad T \ge 0.$$

Proof. Since

$$\int_0^T r(t)dt = \log\frac{|u(T)|}{|u(0)|},$$

the second statement in the lemma is equivalent with

$$|u(T)| \le |u(0)|\exp(2\lambda T), \quad T \ge 0.$$

That this holds follows easily from the fact that

$$-u''(t) + (\lambda^2 V(t, \theta_0 + \omega t) - E)u(t) = 0, \quad |u'(0)/u(0)| \le 2\lambda$$

and $\lambda^2 V(t,\theta_0+\omega t)-E\le\lambda^2$. Indeed, we can assume that $u(0)=1$. By letting $w(t)=\exp(2\lambda t)-u(t)$ we get $w(0)=0$, $w'(0)\ge 0$ and $w''(0)>0$. Hence $w(t)>0$ for small $t>0$. Assume that there is a $t_0>0$ such that $w(t_0)=0$ and $w(t)>0$ for $0<t<t_0$. Then there must be a $0<t_1<t_0$ such that $w''(t_1)\le 0$. But, since $w(t_1)>0$, we have $w''(t_1)>0$. Consequently, $w(t)>0$ for all $t>0$. The lower bound is proved similarly. □

Lemma 4.3. *For any sequence $0<M_0<M_1<M_2<\dots$ of integers, for any $\lambda>0, E\in\mathbb{R}$ and for any $\omega\in\mathbb{R}$ we have*

$$\left.\begin{array}{r}(0,\theta_0,r(0))\in\Phi^{M_j}(A_j)\\ \theta_0\in I_j\setminus I_{j+1}\end{array}\right\}\Rightarrow r(M_j)\in\hat{\mathbb{R}}\setminus(-2\lambda,-\delta_0\lambda),\quad\forall j\ge 0,\tag{4.8}$$

and

$$\left.\begin{array}{r}(0,\theta_0,r(0))\in\Phi^{-M_j}(B_j)\\ \theta_0\in I_j\setminus I_{j+1}\end{array}\right\}\Rightarrow r(-M_j)\in\hat{\mathbb{R}}\setminus(\delta_0\lambda,2\lambda),\quad\forall j\ge 0.\tag{4.9}$$

Proof. We prove the first statement. From the definition of I_{j+1}, the assumptions on θ_0 and $r(0)$ implies that

$$(0,\theta_0,r(0))\notin\Phi^{-M_j}(B_j).$$

Hence we have

$$(0,\theta_{M_j},r(M_j))\notin B_j.$$

Since $\theta_{M_j}\in I_j+M_j\omega$, it now follows from the definition of B_j that $r(M_j)\notin(-2\lambda,-\delta_0\lambda)$. □

5. Main estimates

In this section we establish the key estimates, which shall be used several times in what follows.

Proposition 5.1. *There exists a $\lambda_2=\lambda_2(\delta_0)>0$ such that for all $\lambda\ge\lambda_2$, $E\in\mathcal{E}$ and for every sequence $10<M_0<M_1<\dots$, the following holds for $n\ge 0$:*

Forward iteration: if

$$(A1)_n\qquad\begin{cases}\omega\in\mathcal{F}_{n-1}\\ \theta_0\in\Theta_{n-1}\\ r(0)\in(\delta_0\lambda,2\lambda)\end{cases}$$

then

$$(C1)_n\qquad\int_T^N r(t)dt\ge(1/2+1/2^{n+1})\delta_0(N-T)\lambda,\quad 0\le T\le N,$$

and

$$(C2)_n\qquad r(k)\notin(\delta_0\lambda,2\lambda)\Rightarrow\theta_k\in\bigcup_{l=0}^{n-1}\bigcup_{m=1}^{M_l}(I_l+m\omega),\quad k=0,1,\dots,N,$$

where $N\ge 0$ is the smallest integer such that $\theta_N\in I_n$.

Backward iteration: if

$$(A2)_n \qquad \begin{cases} \omega \in \mathcal{F}_{n-1} \\ \theta_0 \in \Theta_{n-1} \\ r(0) \in (-2\lambda, -\delta_0\lambda) \end{cases}$$

then

$$(C3)_n \qquad \int_{-N}^{-T} r(t)dt \leq -(1/2 + 1/2^{n+1})\delta_0(N-T)\lambda, \quad 0 \leq T \leq N,$$

and

$$(C4)_n \quad r(-k) \notin (-2\lambda, -\delta_0\lambda) \Rightarrow \theta_{-k} \in \bigcup_{l=0}^{n-1} \bigcup_{m=0}^{M_l} (I_l - m\omega), \quad k = 0, 1, \ldots, N,$$

where N is the smallest positive integer such that $\theta_N \in I_n + \omega$.

From this follows:

Corollary 5.2. *With the same assumptions as in proposition 5.1, the following hold for $n \geq 1$: if $\omega \in \mathcal{F}_n$, then*

$$(5.1) \qquad \Phi^{M_n - M_{n-1}}(A_n) \subset \{0\} \times (I_{n-1} - M_{n-1}\omega) \times (2\delta_0\lambda, 3\lambda/2) \subset A_{n-1},$$

and

$$(5.2) \quad \Phi^{-M_n + M_{n-1}}(B_n) \subset \{0\} \times (I_{n-1} + M_{n-1}\omega) \times (-3\lambda/2, 2\delta_0\lambda) \subset B_{n-1}.$$

Proof. We prove (5.1). First we note that $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$ implies that

$$(5.3) \qquad (I_{n-1} - (M_{n-1}+1)\omega) \cap I_0 = \emptyset$$

and

$$(5.4) \qquad (I_{n-1} - (M_{n-1}+1)\omega) \cap X_{n-1} = \emptyset,$$

where X_{n-1} is the set in $(C2)_n$, proposition 5.1. If we now take $(0, \theta_0, r(0)) \in A_n$, i.e.

$$\theta_0 \in I_n - M_n\omega \subset \Theta_n$$

and $r(0) \in (\delta_0\lambda, 2\lambda)$, and note that $N = M_n$ is the smallest positive integer such that $\theta_N \in I_n \subset I_{n-1}$ (since $\omega \in \mathcal{F}_n$), it follows from $(C2)_n$ and (5.4) that we have

$$r(M_n - (M_{n-1}+1)) \in (\delta_0\lambda, 2\lambda).$$

Since (5.3) holds, it follows from lemma 4.1 that

$$r(M_n - M_{n-1}) \in (2\delta_0\lambda, 3\lambda/2).$$

This finishes the proof. □

Proof of proposition 5.1. We only prove the statement in the forward case, since the backward case is identical. The proof goes by induction over n.

For $n = 0$ we see that $(C1)_0$ and $(C2)_0$ follow by repeated use of lemma 4.1.

In the rest of the proof we will use the notation

$$L^+ := (\delta_0\lambda, 2\lambda), \quad L^- := (-2\lambda, \delta_0\lambda).$$

Assume now that the statement holds for n, i.e. that the assumption $(A1)_n$ implies $(C1)_n$ and $(C2)_n$. Let us fix $\omega \in \mathcal{F}_n$, $\theta_0 \in \Theta_n$ and $r(0) \in L^+$, and let $N > 0$ be the smallest integer such that $\theta_N \in I_{n+1}$.

For easier notation, we denote the set in $(C2)_n$ by X_{n-1}, i.e.

$$X_{n-1} := \bigcup_{l=0}^{n-1} \bigcup_{m=0}^{M_l} (I_l + m\omega).$$

From the fact that $\omega \in \mathcal{F}_n$ (conditions $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$) one easily verifies

$$I_n + (M_n + 1)\omega \subset \Theta_n, \tag{5.5}$$

$$(I_n + M_n\omega) \cap I_0 = \emptyset, \tag{5.6}$$

and

$$(I_n - M_n\omega) \cap X_{n-1} = \emptyset. \tag{5.7}$$

Let $0 < N_1 < N_2 < \ldots < N_j < \ldots < N_J = N$ be the times $0 \leq i \leq N$ such that $\theta_i \in I_n$. Since $\omega \in \mathcal{F}_n$ we have

$$N_{j+1} - N_j \geq 2^{n+6} M_n/\delta_0, \quad j \geq 1, \tag{5.8}$$

and since moreover $\theta_0 \in \Theta_n$,

$$N_1 \geq M_n. \tag{5.9}$$

We will now inductively show that $(C1-2)_{n+1}[N_j]$ hold, where we by $(C1-2)_{n+1}[N_j]$ denotes condition $(C1-2)_{n+1}$ with N replaced by N_j.

Since $\theta_0 \in \Theta_n \subset \Theta_{n-1}$, $\omega \in \mathcal{F}_n \subset \mathcal{F}_{n-1}$ and $r(0) \in L^+$, we see that $(A1)_n$ is satisfied. Hence, by the induction assumption, $(C1-2)_n$ hold (where N in $(C1-2)_n$ is N_1, by the definition of N_1). This implies the weaker condition $(C1-2)_{n+1}[N_1]$.

Assume that we inductively have shown that $(C1-2)_{n+1}[N_j]$ hold for some $j < J$. Since $0 \leq N_j - M_n < N_j$ (follows from (5.8) and (5.9)), since $\theta_{N_j - M_n} \in I_n - M_n\omega$ (by the definition of N_j) and since (5.7) hold, it follows from $(C2)_{n+1}[N_j]$ that $r(N_j - M_n) \in L^+$, i.e. $(0, \theta_{N_j - M_n}, r(N_j - M_n) \in A_n$. Hence $(0, \theta_{N_j}, r(N_j)) \in \Phi^{M_n}(A_n)$, and, since $\theta_{N_j} \notin I_{n+1}$, it follows from lemma 4.3 that we must have

$$r(N_j + M_j) \notin L^-. \tag{5.10}$$

From (5.6) and lemma 4.1 we then get

$$r(N_j + M_j + 1) \in L^+. \tag{5.11}$$

Note that we could have $r(k) \notin L^+)$ for some $k = N_j + 1, \ldots, N_j + M_n$, i.e. for $\theta \in (I_n + \omega) \cup \cdots \cup (I_n + M_n\omega)$. This, together with $(C2)_{n+1}[N_j]$, yields

$$r(k) \notin L \Rightarrow \theta_k \in X_n, \quad k = 0, \ldots N_j + M_n + 1. \tag{5.12}$$

Now, since $\theta_{N_j+M_n+1} \in I_n + (M_n+1)\omega \overset{(5.5)}{\subset} \Theta_n$ and since (5.11) hold, i.e. $(A1)_n$ is satisfied, if follows from $(C1)_n$ and $(C2)_n$ that

$$\int_T^{N_{j+1}} r(t)dt \geq (1/2 + 1/2^{n+1})\delta_0(N_{j+1}-T)\lambda \quad \text{for } N_j + M_n + 1 \leq T \leq N_{j+1}, \tag{5.13}$$

and

$$r(k) \notin L^+ \Rightarrow \theta_k \in X_{n-1}, \quad k = N_j + M_n + 1, \dots N_{j+1}. \tag{5.14}$$

(Recall that $N_{j+1} > N_j + M_n + 1$ is the smallest positive integer such that $\theta_{N_{j+1}} \in I_n$).

First we note that (5.14) and (5.12) implies $(C2)_{n+1}[N_{j+1}]$. Secondly, since (5.11) holds, we can apply lemma 4.2 to conclude that

$$\int_T^{N_j+M_n+1} r(t)\mathrm{d}t \geq -2\lambda(N_j + M_n + 1 - T), \ \forall T \leq N_j + M_n + 1.$$

Combining this estimate with (5.13) and noticing that (5.8) implies

$$(1/2 + (1/2)^{n+1})\delta_0(N_{j+1} - (N_j + M_n + 1))\lambda - \lambda(N_j + M_n + 1 - T) \geq (1/2 + (1/2)^{n+2})\delta_0(N_{j+1} - T)\lambda,$$

for $T \in [N_j, N_j + M_n + 1]$, we obtain

$$\int_T^{N_{j+1}} r(t)\mathrm{d}t \geq (1/2 + (1/2)^{n+2})\delta_0(N_{j+1} - T)\lambda, \quad N_j \leq T \leq N_{j+1}.$$

This together with $(C1)_{n+1}[N_j]$ now yields $(C1)_{n+1}[N_{j+1}]$.

Recalling that $(C1-2)_{n+1}[N_J] = (C1-2)_{n+1}$ finishes the proof. □

6. The Harmonic oscillator

Before we can start analyzing the geometry of the critical sets, we need some results about the solutions of the harmonic oscillator.

6.1. **The Harmonic Oscillator.**

We shall now study the harmonic oscillator

$$-u''(t) + (t^2 - x)u(t) = 0. \tag{6.1}$$

By letting

$$r(t) = u'(t)/u(t),$$

we get the Riccati equation

$$r'(t) = -r(t)^2 + t^2 - x. \tag{6.2}$$

It is well known that equation (6.1) has a solution $u(t) = \exp(-t^2/2)$ for $x = 1$, and $u(t) = t\exp(-t^2/2)$ for $x = 3$. Since we in this section shall study the solutions of equation (6.2) up to $t = 0$, we see that we get problems when $x = 3$. To get easier analysis, we shall in fact only work with $x \leq 5/4$. The reason for choosing $5/4$ as the upper bound is just for convenience.

The aim now is to get estimates on $r_x(0)$.

Our first lemma gives some basic estimates of the solutions to (6.2).

Lemma 6.1. *For $T > 0$ sufficiently large, $x \leq 5/4$ and $r(-T) \geq 0$, we have*

$$r(t) \geq 0, \quad -T \leq t \leq -T + 2$$

and

$$-t - 1 \leq r(t) \leq -t + (2 - x), \quad -T + 2 \leq t \leq 0.$$

Proof. That we have $r(t) \geq 0$ for $t \in [-T, -T + 2]$ is obvious. We begin with proof of the upper bound.

Let $\delta = 2 - x > 0$ and set

$$d(t) = -t + \delta - r(t).$$

If we show that $d(t) \geq 0$ in the interval $[-T+2, 0]$ we are done. To accomplish this, we differentiate the above equation to get

$$d'(t) = d(t)^2 + 2(t-\delta)d(t) - 2t\delta + \delta^2 + x - 1, \quad d(-T) = -T + \delta - r(-T). \tag{6.3}$$

Since

$$\delta^2 + x - 1 \geq 3/4 \quad \text{for all } x,$$

we notice that if $d(t_0) = 0$ for some $t_0 \leq 0$, then $d'(t_0) > 0$. Hence, the statement is proved if we only show that $d(-T + 2) \geq 0$. To do this, we assume that $d(-T) < 0$. Hence there is an interval $[-T, -T_0]$, some $T_0 > 0$, on which $d(t)$ is ≤ 0. This implies that

$$2(t - \delta)d(t) - 2t\delta + \delta^2 + x - 1 > 3/4, \quad t \in [-T, -T_0].$$

Hence the solution to

$$p'(t) = p(t)^2 + 3/4, \quad p(-T) = d(-T),$$

must satisfy $p(t) \leq d(t)$ on $[-T, -T_0]$. Since

$$p(t) = \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} t + const.\right),$$

and since the maximal length of an interval were $\tan t$ is negative is $\pi/2$, we must have $p(t_0) = 0$ for some $t_0 \in [-T, -T + \pi/\sqrt{3}] \subset [-T, -T + 2]$.

We now proceed with the proof of the lower bound. Clearly we only have to consider the case $x = 5/4$.

Let

$$d(t) = r(t) + t + 1.$$

Differentiating yields

$$\begin{aligned} d'(t) &= - d(t)^2 + 2d(t)(t + 1) - 2t - x \\ &= - d(t)^2 + 2d(t)(t + 1) - 2t - 5/4, \quad d(-T) \geq -T + 1. \end{aligned}$$

It is not hard to verify that we must have $d(t_0) = 1/2$ for some t_0 in the interval $[-T, -T+2]$ if T is sufficiently large. Furthermore, we see that if $d(t_0) = 1/2$ for some $t_0 < 1/2$, then $d'(t_0) > 0$. Consequently,

$$d(t) \geq 1/2 \quad \text{for } -T+2 \leq t \leq 1/2.$$

By comparing d with the function

$$g'(t) = -1/4 + 2g(t)(t+1) - 2t - 5/4, \quad g(-1/2) = 1/2,$$

which by an explicit calculation can be shown to be positive on $[-1/2, 0]$, we find that $d(t) \geq g(t) > 0$ on $[-1/2, 0]$. This finishes the proof of the lower estimate.

□

The next lemma yields estimates of the function $r(0, x)$.

Lemma 6.2. *Assume that $T > 0$ is large and let $r(t, x)$ be the solution of (6.2) with initial condition $r(-T, x) \equiv 0$. Then*

$$\partial_x r(0, x) < 0, \quad x \leq 5/4.$$

Moreover, for x in the interval $0 \leq x \leq 5/4$ we have

$$-10 < \partial_x r(0, x) < -1/10.$$

Finally,

$$|r(0, 1)| < \exp(-2T).$$

Proof. Denote

$$p(t) = \partial_x r(t, x).$$

Differentiating (6.2) w.r.t. x yields

$$p'(t) = -2r(t)p(t) - 1, \quad p(-T) = 0.$$

Solving this equation gives us

$$p(0) = \left(-\int_{-T}^{0} \exp\left(2\int_{-T}^{t} r(\xi)\mathrm{d}\xi\right)\mathrm{d}t\right)\exp\left(-2\int_{-T}^{0} r(\xi)\mathrm{d}\xi\right) < 0. \tag{6.4}$$

For x in the interval $[0, 5/4]$, we make use of the upper and lower bounds of $r(t)$, given by lemma 6.1. Since (by direct computation)

$$\begin{aligned}\int_{-T}^{0} \exp\left(-2\int_{t}^{0} r(\xi)\mathrm{d}\xi\right)\mathrm{d}t &\leq 2 + \int_{-T+2}^{0} \exp\left(2\int_{t}^{0} (\xi+1)\mathrm{d}\xi\right)\mathrm{d}t \\ &< 2 + \int_{-\infty}^{0} \exp\left(2\int_{t}^{0} (\xi+1)\mathrm{d}\xi\right)\mathrm{d}t < 10\end{aligned}$$

and

$$\begin{aligned}\int_{-T}^{0} \exp\left(-2\int_{t}^{0} r(\xi)\mathrm{d}\xi\right)\mathrm{d}t &\geq \int_{-T+2}^{0} \exp\left(2\int_{t}^{0} (\xi-2)\mathrm{d}\xi\right)\mathrm{d}t \\ &> \int_{-1}^{0} \exp\left(2\int_{t}^{0} (\xi-2)\mathrm{d}\xi\right)\mathrm{d}t > 1/10,\end{aligned}$$

it follows from (6.4) that

$$-10 < p(0) < -1/10.$$

When $x = 1$, we note that $q(t) = -t$ is a solution to

$$q'(t) = -q(t)^2 + t^2 - 1.$$

By letting $d(t) = r(t) - q(t)$, it follows from the fact that q and r satisfies equation (6.2) that

$$d'(t) = -d(t)(r(t) + q(t)).$$

Solving this, and making use of the estimates in 6.1, yields

$$|d(0)| = \left| d(-T+2) \exp\left(- \int_{-T+2}^{0} (r(t) + q(t)) \mathrm{d}t \right) \right| < e^{-2T}.$$

□

The previous lemma allows us to get

Lemma 6.3. *For $T > 0$ sufficiently large, $x \le 5/4$ and $r(-T) \ge 0$ we have*

$$r(0) < -\frac{x-1}{10} + \exp(-T), \quad \textit{if } 1 \le x \le 5/4,$$

$$r(0) > \frac{1-x}{10} - \exp(-T), \quad \textit{if } 0 \le x \le 1,$$

and

$$r(0) > \frac{1}{10} - \exp(-T), \quad \textit{if } x \le 0.$$

Proof. Let $q(t, y)$ be the solution to equation

$$\partial_t q(t, y) = -q(t, y)^2 + t^2 - y, \quad q(-T, y) \equiv 0.$$

From lemma 6.2 we derive

$$q(0, y) < -\frac{y-1}{10} + \exp(-2T), \quad \text{if } 1 \le y \le 5/4,$$
$$q(0, y) > \frac{1-y}{10} - \exp(-2T), \quad \text{if } 0 \le y \le 1,$$
$$q(0, y) > \frac{1}{10} - \exp(-2T), \quad \text{if } y \le 0.$$

As in the proof of lemma 6.2, one verifies that

$$|r(0) - q(0, x)| < \exp(-2T).$$

From this the statement of the lemma follows. □

7. The geometry of the critical sets

7.1. Results about I_0.

Lemma 7.1. *There exists a* $\lambda_3 = \lambda_3(V) > 0$ *such that for all* $\lambda > \lambda_3$, $E \in \mathcal{E}$ *and* $\omega \in \mathcal{F}_{-1}$ *we have*

$$I_0 = \left((1-\omega)/2 - \sqrt{1+\omega^2}\rho, (1-\omega)/2 + \sqrt{1+\omega^2}\rho\right),$$

where $\rho = \rho(\lambda, E)$ *satisfies* $1/\sqrt{\lambda} < \rho < 2/\sqrt{\lambda}$.

Proof. Recall the definition of I_0. Since

$$\lambda^2 V(x,y) - E < 0 \iff V(x,y) < E/\lambda^2,$$

since $1.1/\lambda \le E/\lambda^2 \le 1.2/\lambda$ and since, by assumption, V has a unique minimum at $(1/2, 1/2)$ of the form $V(x,y) = (x-1/2)^2 + (y-1/2)^2 + h.o.t.$, we must for large λ have that the radius ρ of the ball $\mathcal{B}$ satisfies $1/\sqrt{\lambda} < \rho < 2/\sqrt{\lambda}$. The rest of the proof is just easy geometry (see fig. 1). □

7.2. Change of coordinates and the Poincaré map.

For convenience we shall not study the intersection of the sets $\Phi^{M_n}(A_n)$ and $\Phi^{-M_n}(B_n)$ directily. Instead we shall define a Poincaré map $\mathcal{P}$ and study the intersection $\mathcal{P} \circ \Phi^{M_0}(A_0) \cap \mathcal{P} \circ \Phi^{-M_0}(B_0)$ over the line $(1/2, 1/2) + s(-\omega, 1)$.

For fixed $\omega \in \mathcal{F}_{-1}$, we associate to each $\theta \in \mathbb{T}$ two numbers $s(\theta, \omega)$ and $\tau(\theta, \omega)$, satisfying

$$(0, \theta) + \tau(1, \omega) = (1/2, 1/2) + s(-\omega, 1),$$

(see fig. 1). Solving this yields

$$s(\theta, \omega) = \frac{2\theta - (1-\omega)}{2(1+\omega^2)}, \quad \tau(\theta, \omega) = \frac{1 + \omega - 2\theta\omega}{2(1+\omega^2)}.$$

From now on s and τ are as above.

Definition 7.2 (The critical sets in the s-coordinates)**.** Let

$$S_n := \{s(\theta) : \theta \in I_n\}, \quad n \ge 0$$

and

$$S_0^1 = \left(-\frac{1}{2\sqrt{\lambda}}, -\frac{1}{\lambda^{2/3}}\right), \quad S_0^2 = \left(\frac{1}{\lambda^{2/3}}, \frac{1}{2\sqrt{\lambda}}\right).$$

Definition 7.3. We define $\mathcal{P}$ to be the Poincaré map

$$\mathcal{P}(0, \theta, r(0)) = \Phi^{\tau}(0, \theta, r(0)).$$

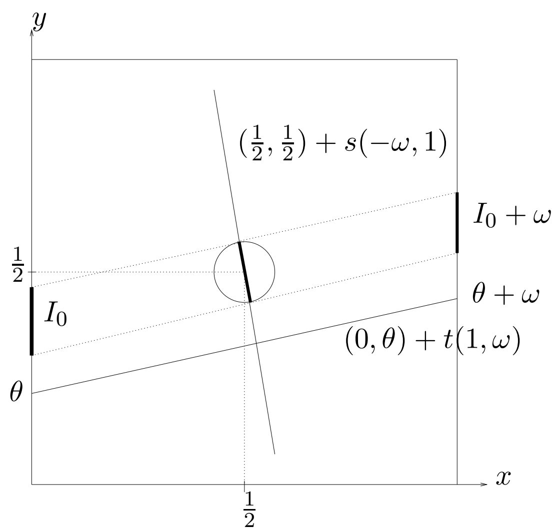

FIGURE 1. The first critical set

7.3. **Perturbations of the Harmonic Oscillator.** The aim now is to study the equation

$$r'(t, s, \omega) = -r(t, s, \omega)^2 + \lambda^2 V(1/2 - s\omega + t, 1/2 + s + \omega t) - E \tag{7.1}$$

for s and t close to zero, where $\partial_t =' $. We have chosen this particular form since exactly equation (7.1) will appear later. This subsection contains the delicate estimates of the solutions when we integrate over the "critical region".

We note that for $\lambda > \lambda_3$, $E \in \mathcal{E}$ and $\omega \in \mathcal{F}_{-1}$, it follows from lemma 7.1 and the definition of S_0 that

$$S_0 = \left(-\rho/\sqrt{1+\omega^2}, \rho/\sqrt{1+\omega^2}\right), \quad \frac{1}{\sqrt{\lambda}} < \rho < \frac{2}{\sqrt{\lambda}}.$$

Hence $S_0^i \subset S_0$, $i = 1, 2$.

Lemma 7.4. *There exists a $\lambda_4 > 0$ such that for all $\lambda > \lambda_4$, $E \in \mathcal{E}$ and $\omega \in \mathcal{F}_{-1}$ we have the following, where $\varepsilon = \lambda^{1/9}/\sqrt{\lambda}$: If the solution to equation (7.1) satisfies*

$$0 \leq r(-\varepsilon, s, \omega) \leq 2\lambda$$

and

$$|\partial_s r(-\varepsilon, s, \omega)|, |\partial_\omega r(-\varepsilon, s, \omega)| < \lambda^2$$

for $s \in S_0$, then for all s in S_0 we have

a) $$r(0, s, \omega) < 0 \quad \textit{if } |s| \leq \frac{1}{\lambda^{2/3}},$$

b) $$r(0, s, \omega) > 0 \quad \textit{if } |s| \geq \frac{1}{2\sqrt{\lambda}},$$

c)
$$\int_{-\varepsilon}^{0} r(t,s,\omega)dt > 0,$$

d)
$$-\lambda^{3/2} < \partial_s r(0,s,\omega) < -\lambda^{3/4} \quad \text{for } s \in S_0^1,$$

e)
$$\lambda^{3/4} < \partial_s r(0,s,\omega) < \lambda^{3/2} \quad \text{for } s \in S_0^2,$$

and

f)
$$|\partial_\omega r(0,s,\omega)| < \lambda^{2/3} \quad \text{for } s \in S_0^1 \cup S_0^2.$$

Remark 4. If we instead have the initial condition $-2\lambda \le r(\varepsilon,s,\omega) \le 0$ and the same derivative estimates as in the lemma, then one verifies easily from the proof that $-r(-t,s,\omega)$ satisfies $a) - f)$. This we shall use for the backward iteration.

Proof. We assume that $\lambda > 0$ is large and we write $T = \lambda^{1/9}$. First we note that since (by the assumption on V)

$$V(x,y) = (x-1/2)^2 + (y-1/2)^2 + \mathcal{O}\left[\left((x-1/2)^2 + (y-1/2)^2\right)^{3/2}\right],$$

we have

$$V(1/2 - s\omega + t, 1/2 + s + \omega t) = (1+\omega)^2(t^2+s^2) + \varphi(t,s,\omega), \tag{7.2}$$

where

$$|\varphi| < C\varepsilon^3, \quad |\partial_s\varphi|, |\partial_\omega\varphi| < C\varepsilon^2 \quad \text{for } |t|, |s| \le \varepsilon. \tag{7.3}$$

To analyse equation (7.1), we shall re-scale the time. Let

$$\beta = (1+\omega^2)^{1/4}\sqrt{\lambda}$$

and let

$$p(t,s,\omega) = r(t/\beta, s, \omega)/\beta.$$

Then (7.1) becomes (using 7.2)

$$p'(t,s,\omega) = -p(t,s,\omega)^2 + t^2 + \lambda\sqrt{1+\omega^2}s^2 - \frac{E}{\lambda\sqrt{1+\omega^2}} + \psi(t,s,\omega), \tag{7.4}$$

where

$$\psi(t,s,\omega) = \lambda\varphi(t/\beta,s,\omega)/\sqrt{1+\omega^2},$$

which, by (7.3), satisfies

$$|\psi| < C/\lambda^{1/6} \quad |\partial_s\psi(t,s,\omega)|, |\partial_\omega\psi(t,s,\omega)| < C\lambda^{2/9}, \quad |t| \le T, |s| \le \varepsilon. \tag{7.5}$$

From the assumptions on $r(-\varepsilon,s,\omega)$ we obtain

$$|\partial_s p(-T,s,\omega)|, |\partial_\omega p(-T,s,\omega)| < \lambda^2 \quad \text{for } s \in S_0.$$

Now we shall prove the following: For $s \in S_0$ we have (recall that $S_0^i \subset S_0, i = 1,2$):

(*i*) $p(0,s,\omega) < 0$ for $|s| \le \frac{1}{\lambda^{2/3}}$,
(*ii*) $p(0,s,\omega) > 0$ for $|s| \ge \frac{1}{2\sqrt{\lambda}}$,

(*iii*) $-1 \le p(0,s,\omega) \le 2 \quad$ for $|s| \le \frac{1}{\lambda^{1/2}}$,
(*iv*) $\int_{-T}^{0} p(t,s,\omega)dt > 0$,
(*v*) $-\frac{\lambda}{2} < \partial_s(0,s,\omega) < -\lambda^{1/4} \quad$ for $s \in S_0^1$,
(*vi*) $\lambda^{1/4} < \partial_s(0,s,\omega) < \frac{\lambda}{2} \quad$ for $s \in S_0^2$,
(*vii*) $|\partial_\omega p(0,s,\omega)| < \frac{\lambda^{1/6}}{2} \quad$ for $s \in S_0^1 \cup S_0^2$.

By the assumptions on E and ω, we have for $|s| \le 1/\lambda^{2/3}$ and $|t| \le T$:

$$\lambda\sqrt{1+\omega^2}s^2 - \frac{E}{\lambda\sqrt{1+\omega^2}} + \psi(t,s,\omega) < -1.03.$$

By comparing $p(t,s,\omega)$ with the solution of

$$q'(t) = -q(t)^2 + t^2 - 1.03, \quad q(-T) = p(-T,s,\omega) \ge 0,$$

it follows from lemma 6.3 that $p(0,s,\omega) \le q(0) < 0$. Similarly, for $s \in S_0$ such that $s \ge 1/(2\sqrt{\lambda})$, and $|t| \le T$,

$$\lambda\sqrt{1+\omega^2}s^2 - \frac{E}{\lambda\sqrt{1+\omega^2}} + \psi(t,s,\omega) > -0.97,$$

so we compare with

$$q'(t) = -q(t)^2 + t^2 - 0.97, \quad q(-T) = p(-T,s,\omega) \ge 0$$

and hence lemma 6.3 implies $p(0,s,\omega) \ge q(0) > 0$. This finishes the proof of (*i*) and (*ii*).

Next, since

$$\lambda\sqrt{1+\omega^2}s^2 - \frac{E}{\lambda\sqrt{1+\omega^2}} + \psi(t,s,\omega) > -5/4$$

for all $|t| \le T$ and $s \in S_0$, we bound $p(t,s,\omega)$ from below with the solution to the equation

$$q'(t) = -q(t)^2 + t^2 - 5/4, \quad q(-T) = p(-T,s,\omega) \ge 0.$$

By lemma 6.1 we hence have

$$p(t,s,\omega) \ge q(t) \ge \begin{cases} 0, & -T \le t \le -T+2 \\ -t-1, & -T+2 \le t \le 0 \end{cases} \tag{7.6}$$

for all $s \in S_0$. This gives (*iv*) and half of (*iii*).

Furthermore, for $|s| \le 1/\sqrt{\lambda}$ and $|t| \le T$ we have

$$\lambda\sqrt{1+\omega^2}s^2 - \frac{E}{\lambda\sqrt{1+\omega^2}} + \psi(t,s,\omega) < 0,$$

and by comparing with the solution of

$$q'(t) = -q(t)^2 + t^2, \quad q(-T) = p(-T,s,\omega) \ge 0,$$

we get from lemma 6.1 that

$$p(t,s,\omega) \le q(t) \le -t+2, \quad \text{for } -T+2 \le t \le 0. \tag{7.7}$$

This gives the second half of (iii).

By differentiating (7.4) w.r.t. s resp. ω and solving the equations obtained gives us:

$$\partial_s p(0,s,\omega) = \partial_s p(-T,s,\omega)\exp\left(-2\int_{-T}^{0} p(\xi,s,\omega)d\xi\right)$$
$$+\int_{-T}^{0}\left(2s\lambda\sqrt{1+\omega^2}+\partial_s\psi(t,s,\omega)\right)\exp\left(-2\int_{t}^{0} p(\xi,s,\omega)d\xi\right)dt$$

and

$$\partial_\omega p(0,s,\omega) = \partial_\omega p(-T,s,\omega)\exp\left(-2\int_{-T}^{0} p(\xi,s,\omega)d\xi\right)+$$
$$\int_{-T}^{0}\left(\frac{\lambda\omega s^2}{\sqrt{1+\omega^2}}+\frac{E\omega}{\lambda(1+\omega^2)^{3/2}}+\partial_\omega\psi(t,s,\omega)\right)\exp\left(-2\int_{t}^{0} p(\xi,s,\omega)d\xi\right)dt.$$

By proceeding as in lemma 6.2, making use of the estimates (7.6), (7.7), the estimates on ψ and the estimates on $p(-T,s,\omega)$, we obtain (v), (vi) and (vii) (in the case of the estimate of $\partial_\omega p(0,s,\omega)$ we can first integrate over $[-T,-\sqrt{T}]$, using the bound $|\partial_\omega\psi| < C\lambda^{2/9}$, and then over $[-\sqrt{T},0]$ using $|\partial_\omega\psi| < C\lambda^{1/18}$).

Finally, since $r(t/\beta,s,\omega) = \beta p(t,s,\omega)$, the statements $a)$–$f)$ in the lemma follows from the estimates $(i)-(vii)$. □

7.4. **Some formulas.** Before we proceed with the analyse of the the intersections, we shall derive some formulas needed to measure the geometry of the boundary of the sets:

Derivatives: Let $u(t,\theta,\omega)$ be the solution of

$$-u''(t,\theta,\omega)+\lambda^2 V(t,\theta+\omega t)u(t,\theta,\omega) = Eu(t,\theta,\omega), \tag{7.8}$$

with the initial conditions

$$u(0,\theta,\omega)=1, u'(0,\theta,\omega)=const,$$

where $\partial_t = '$. As always, we let

$$r(t,\theta,\omega) = u'(t,\theta,\omega)/u(t,\theta,\omega).$$

Differentiating this w.r.t. θ, and using the notation $v = \partial_\theta u$, we get

$$\partial_\theta r(t,\theta) = \partial_\theta(u'/u) = \frac{v'u-u'v}{u^2},$$

which, by using the initial conditions on u and u', equals

$$\frac{\int_0^t (v''u-u''v)\mathrm{d}p}{u^2}.$$

Since u satisfies (7.8), we verify that this equals

$$\frac{\int_0^t u^2\lambda^2\partial_y V\mathrm{d}p}{u^2}.$$

Moreover, by the fact that

$$|u(t)/u(p)| = \exp\left(\int_p^t r(\xi)\mathrm{d}\xi\right),$$

the last expression can be written

$$\lambda^2 \int_0^t \partial_y V(p, \theta + p\omega) \exp\left(-2\int_p^t r(\xi)\mathrm{d}\xi\right) \mathrm{d}p.$$

We have hence showed that

$$\partial_\theta r(t,\theta,\omega) = \lambda^2 \int_0^t \partial_y V(p, \theta + p\omega) \exp\left(-2\int_p^t r(\xi)\mathrm{d}\xi\right) \mathrm{d}p. \tag{7.9}$$

By differentiating w.r.t. ω instead, we end up with

$$\partial_\omega r(t,\theta,\omega) = \lambda^2 \int_0^t p\partial_y V(p, \theta + p\omega) \exp\left(-2\int_p^t r(\xi)\mathrm{d}\xi\right) \mathrm{d}p. \tag{7.10}$$

Contraction: We also need to measure the contraction. Let $r(t)$ and $q(t)$ be two solutions to equation (7.8). By a similar calculation as the above one, we find that

$$|r(t) - q(t)| = |r(0) - q(0)| \exp\left(-\int_0^t (r(\xi) + q(\xi))\mathrm{d}\xi\right). \tag{7.11}$$

7.5. **The first crossing.** We shall now begin the study of the first intersection, i.e. the crossing $\mathcal{P} \circ \Phi^{M_0}(A_0) \cap \mathcal{P} \circ \Phi^{-M_0}(B_0)$.

Lemma 7.5. *There exists a $\lambda_5 > 0$ such that for all $\lambda > \lambda_5$ and $E \in \mathcal{E}$, we have for each $\omega \in \mathcal{F}_0$*

$$\mathcal{P} \circ \Phi^{M_0}(A_0) = \{(1/2 - s\omega, 1/2 + s, r) : s \in S_0,\ \varphi_0^-(s,\omega) < r < \varphi_0^+(s,\omega)\}$$

and

$$\mathcal{P} \circ \Phi^{-M_0}(B_0) = \{(1/2 - s\omega, 1/2 + s, r) : s \in S_0,\ \psi_0^-(s,\omega) < r < \psi_0^+(s,\omega)\}$$

where $\varphi_0^\pm, \psi^\pm : S_0 \times \mathcal{F}_0$ are C^1-functions satisfying the following:

$$\varphi_0^\pm(s,\omega) < 0, \quad \psi_0^\pm(s,\omega) > 0, \quad |s| \le 1/\lambda^{2/3}, \tag{7.12}$$

$$\varphi_0^\pm(s,\omega) > 0, \quad \psi_0^\pm(s,\omega) < 0, \quad |s| \ge 1/(2\sqrt{\lambda}), \tag{7.13}$$

$$\begin{cases} -\lambda^{3/2} < \partial_s \varphi_0^\pm(s,\omega) < -\lambda^{3/4} \\ \lambda^{3/4} < \partial_s \psi_0^\pm(s,\omega) < \lambda^{3/2} \end{cases} \quad \textit{for } s \in S_0^1, \tag{7.14}$$

$$\begin{cases} -\lambda^{3/2} < \partial_s \psi_0^\pm(s,\omega) < -\lambda^{3/4} \\ \lambda^{3/4} < \partial_s \varphi_0^\pm(s,\omega) < \lambda^{3/2} \end{cases} \quad \textit{for } s \in S_0^2, \tag{7.15}$$

$$|\partial_\omega \varphi_0^\pm(s,\omega)|, |\partial_\omega \psi_0^\pm(s,\omega)| < \lambda^{2/3}, \quad \textit{for } s \in S_0^1 \cup S_0^2, \tag{7.16}$$

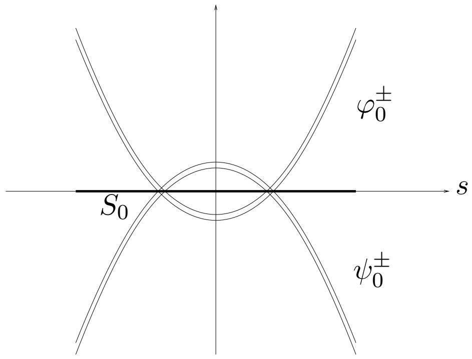

FIGURE 2. Picture of the first intersection

$$\exp(-5\lambda M_0) < \varphi_0^+ - \varphi_0^-, \psi_0^+ - \psi_0^- < \exp(-\delta_0(M_0 - 1)\lambda). \tag{7.17}$$

Remark 5. Recall the definition of S_0 and S_0^ι (definition 7.2). Note that (7.12) and (7.13) show that the graphs of $\varphi_0^\pm$ and $\psi_0^\pm$ must intersect over S_0^1 and S_0^2 and that they do not intersect outside $S_0^1 \cup S_0^2$ (see fig. 2).

Proof. Assume that $\lambda > 0$ is large. Let $r^-(0, \theta, \omega) = \delta_0\lambda$ and $r^+(0, \theta, \omega) = 2\lambda$, for ω in $\mathcal{F}_0$ and $\theta \in I_0 - M_0\omega$, be the horizontal boundaries of A_0. From proposition 5.1 it follows that

$$\int_T^{M_0} r^\pm(t)dt \geq \delta_0\lambda(M_0 - T), \quad 0 \leq T \leq M_0, \tag{7.18}$$

and

$$r^\pm(M_0, \theta, \omega) \in [\delta_0\lambda, 2\lambda]. \tag{7.19}$$

By letting

$$f^\pm(\theta, \omega) = r^\pm(M_0, \theta - M_0\omega, \omega), \quad \theta \in I_0, \omega \in \mathcal{F}_0,$$

we clearly have

$$\Phi^{M_0}(A_0) = \{(0, \theta, r) : \theta \in I_0, f^-(\theta, \omega) < r < f^+(\theta, \omega)\}.$$

Using the formulas (7.9), (7.10) and (7.11), combined with (7.18), we obtain

$$|\partial_\theta f^\pm(\theta, \omega)| \leq \lambda^2 \|V\|_{C^1} \int_0^{M_0} \exp\left(-2\int_t^{M_0} r^\pm(\xi)d\xi\right) dt < \lambda^2/10, \tag{7.20}$$

(7.21)
$$\begin{aligned} |\partial_\omega f^\pm(\theta, \omega)| =& |\partial_\omega r^\pm(M_0, \theta - M_0\omega, \omega) - M_0\partial_\theta r^\pm(M_0, \theta - M_0\omega, \omega)| \\ \leq& \lambda^2 \|V\|_{C^1} \int_0^{M_0} (t - M_0) \exp\left(-2\int_t^{M_0} r^\pm(\xi)d\xi\right) dt < \lambda^2/10 \end{aligned}$$

and, for the contraction,

$$f^+(\theta, \omega) - f^-(\theta, \omega) = \tag{7.22}$$

$$=(r^+(0,\theta-M_0\omega,\omega)-r^-(0,\theta-M_0\omega,\omega))\exp\left(-\int_0^{M_0}(r^+(t)+r^-(t))dt\right)$$
$$< 2\lambda\exp(-\delta_0 M_0\lambda).$$

Next we study the equation (we leave out the sup-scripts $\pm$)

$$p'(t,s,\omega)=-p(t,s,\omega)^2+\lambda^2 V(1/2-s\omega+t,1/2+s+\omega t)-E \tag{7.23}$$

(which is the equation studied in subsection 7.3) with the initial condition

$$p(-\tau,s,\omega)=f(\theta,\omega). \tag{7.24}$$

Note that then $p(0,s,\omega)=\varphi_0(s,\omega)$ (recall the definition of $\mathcal{P}$ in subsection 7.2). Since $\tau=(1-2s\omega)/2$ and $\theta=(1-\omega)/2+s(1+\omega^2)$, we get by differentiating the initial condition (7.24):

$$\partial_s p(-\tau,s,\omega)=\partial_\theta f(\theta,\omega)\cdot(1+\omega^2)-\partial_t p(-\tau,s,\omega)\cdot\omega$$

and

$$\partial_\omega p(-\tau,s,\omega)=\partial_\theta f(\theta,\omega)\cdot(2s\omega-1/2)+\partial_\omega f(\theta,\omega)-s\partial_t p(-\tau,s,\omega).$$

Using the estimates (7.19), (7.20) and (7.21), and the fact that p satisfies equation (7.23), we obtain

$$|\partial_s p(-\tau,s,\omega)|,|\partial_\omega p(-\tau,s,\omega)|<\lambda^2,\quad s\in S_0,\omega\in\mathcal{F}_0. \tag{7.25}$$

By the fact that V has a unique minimum, of the form

$$V(x,y)=(x-1/2)^2+(y-1/2)^2+h.o.t.$$

and since

$$\lambda^2V(x,y)-E<\lambda \iff V(x,y)<(\lambda+E)/\lambda^2$$

and $E<1.2\lambda$, we have that $\lambda^2V(x,y)-E\geq\lambda$ outside the ball

$$\{((x-1/2)^2+(y-1/2)^2)^{1/2}<2/\sqrt{\lambda}\}.$$

Letting $\varepsilon=\lambda^{1/9}/\sqrt{\lambda}$, it now follows that

$$\lambda^2V(1/2-s\omega+t,1/2+s+\omega t)-E\geq\lambda,\quad -\tau\leq t\leq-\varepsilon, s\in S_0.$$

Since also (7.19) holds, i.e. $\delta_0\lambda<p(-\tau,s,\omega)<2\lambda$, and since $p(t,s,\omega)$ satisfies (7.23), we must have

$$\sqrt{\lambda}\leq p(t,s,\omega)\leq 2\lambda,\quad \text{for } -\tau\leq t\leq-\varepsilon, s\in S_0. \tag{7.26}$$

(The upper bound follows since $\lambda^2V-E<\lambda^2$.)

By differentiating equation (7.23) w.r.t. s resp. ω, and making use of the estimates (7.25) and (7.26), one verifies that

$$|\partial_s p(-\varepsilon,s,\omega)|,|\partial_\omega p(-\varepsilon,s,\omega)|<\lambda^2,\quad s\in S_0,\omega\in\mathcal{F}_0.$$

Hence we can apply lemma 7.4 to get the wanted estimates of $p(0,s,\omega)=\varphi_0(s,\omega)$. Note that by (7.22), (7.26) and by *c*) in lemma 7.4, we obtain

$$p^+(0,s,\omega)-p^-(0,s,\omega)<\exp(-\delta_0M_0\lambda).$$

To get the lower bound, we use lemma 4.2. Since $|r^{\pm}(0)| \leq 2\lambda$ we get (again using formula (7.11))

$$\begin{aligned} p^{+}(0,s,\omega) - p^{-}(0,s,\omega) =& (2\lambda - \delta_0\lambda)\exp\left(-\int_0^{M_0+\tau}(r^{+}(t)+r^{-}(t))dt\right) \\ >& \lambda\exp(-4\lambda(M_0+\tau)) > \exp(-5\lambda M_0), \end{aligned}$$

since $\delta_0 < 1$ and $\tau < 1$.

For the backward case, we first apply proposition 5.1 to get estimates on $\Phi^{-M_0+1}(B_0)$, and the we proceed as above, again making use of lemma 7.4 (recall remark 4).

□

7.6. **The induction step.** We can now formulate the induction which shall give use the information about the critical sets S_n (and hence I_n). It is for the reason to get the formulation of the inductive lemma more uniform, that we have defined S_0^1 and S_0^2 as we did.

Lemma 7.6. *There exists a $\lambda_6 > 0$ such that for all $\lambda_6 > \lambda$, $E \in \mathcal{E}$ and all sequences $\{M_k\}_{k=0}^{\infty}$ satisfying $M_0 > 10$ and $M_{k+1} > 10M_k/\delta_0$, the following hold for all $n \geq 0$:*

Assume that

$$(\mathcal{A}1)_n \qquad \omega \in \mathcal{F}_{n-1} \Rightarrow \begin{cases} \mathrm{comp(S_n)} = \begin{cases} 1, & \textit{if } n = 0 \\ 2, & \textit{if } n > 0 \end{cases} \\ |S_n^{\iota}| > 0, \quad \iota = 1,2 \\ |\partial_\omega S_n^{\iota}(\omega)| < \infty \end{cases}$$

and that for $\omega \in \mathcal{F}_n$
$(\mathcal{A}2)_n$

$$\mathcal{P}\circ\Phi^{M_n}(A_n) = \left\{\left(\frac{1}{2} - s\omega, \frac{1}{2}+s, r\right) : s \in S_n,\ \varphi_n^{-}(s,\omega) < r < \varphi_n^{+}(s,\omega)\right\}$$

$$\mathcal{P}\circ\Phi^{-M_n}(B_n) = \left\{\left(\frac{1}{2} - s\omega, \frac{1}{2}+s, r\right) : s \in S_n,\ \psi_n^{-}(s,\omega) < r < \psi_n^{+}(s,\omega)\right\}$$

where $\psi_n^{\pm}, \varphi_n^{\pm} : \{(s,\omega) : \omega \in \mathcal{F}_n, s \in S_n(\omega)\} \to \mathbb{R}$ are C^1-functions satisfying the following: For fixed $\omega \in \mathcal{F}_n$, we have on each interval $S_n^{\iota}(\omega)$ $(\iota = 1,2)$
$(i)_n$

$$\begin{cases} -\lambda^{3/2} < \partial_s\varphi_n^{\pm}(s,\omega) < -\lambda^{3/4} \\ \lambda^{3/4} < \partial_s\psi_n^{\pm}(s,\omega) < \lambda^{3/2} \end{cases} \quad \textit{or} \quad \begin{cases} -\lambda^{3/2} < \partial_s\psi_n^{\pm}(s,\omega) < -\lambda^{3/4} \\ \lambda^{3/4} < \partial_s\varphi_n^{\pm}(s,\omega) < \lambda^{3/2} \end{cases}$$

$$(ii)_n \qquad |\partial_\omega\varphi_n^{\pm}(s,\omega)|, |\partial_\omega\psi_n^{\pm}(s,\omega)| < \lambda^{2/3},$$

$$(iii)_n \qquad \exp(-5M_n\lambda) < \varphi_n^{+} - \varphi_n^{-}, \psi_n^{+} - \psi_n^{-} < \exp(-\delta_0(M_n-1)\lambda)$$

$$(iv)_n \qquad \exists s_{\alpha,\beta}^{\iota} \in S_n^{\iota} \textit{ such that } \varphi_n^{\alpha}(s_{\alpha,\beta}^{\iota}) = \psi_n^{\beta}(s_{\alpha,\beta}^{\iota}), \quad \alpha,\beta = \pm,\pm.$$

$(v)_0$ *The graphs of* $\varphi_0^{\pm}$ *and* $\psi_0^{\pm}$ *do not intersect outside* $S_0^1 \cup S_0^2$.

Then $(\mathcal{A}1)_{n+1}$ *and* $(\mathcal{A}2)_{n+1}$ *hold. Moreover, for all* $\omega \in \mathcal{F}_n$

$$(I1)_{n+1} \qquad \overline{S_{n+1}^{\iota}} \subset S_n^{\iota},$$

$$(I2)_{n+1} \qquad 0 < |S_{n+1}^{\iota}| < \exp(\delta_0(M_0 - 1)\lambda)$$

and

$$(I3)_{n+1} \qquad |\partial_\omega S_{n+1}^{\iota}(\omega)| < 1/\lambda^{1/20}.$$

Furthermore, for all $\omega \in \mathcal{F}_{n+1}$ *we have*

$(I4)_{n+1}$ *If* $X \subset A_{n+1}$ *and* $\pi_2(X)$ *is* ε*-dense in* $I_{n+1}^{\iota} - M_{n+1}\omega$ $(\iota = 1, 2)$ *then* $\pi_3(\Phi^{M_{n+1}+M_n}(X))$ *is* $\max\{\varepsilon \exp(4\lambda M_n), \exp(-\lambda M_n)\}$*-dense in the interval* $(-2\lambda, -\delta_0\lambda)$.

$(I5)_{n+1}$ *If* $X \subset B_{n+1}$ *and* $\pi_2(X)$ *is* ε*-dense in* $I_{n+1}^{\iota} + M_{n+1}\omega$ $(\iota = 1, 2)$ *then* $\pi_3(\Phi^{-M_{n+1}-M_n}(X))$ *is* $\max\{\varepsilon \exp(4\lambda M_n), \exp(-\lambda M_n)\}$*-dense in the interval* $(\delta_0\lambda, 2\lambda)$.

Proof. From now on we assume that $\lambda > 0$ is sufficiently large, so that all the estimates below hold true and so that the application of previous lemmas and propositions is possible.

Assume that $(\mathcal{A}1-2)_n$ hold and fix $\omega \in \mathcal{F}_n$. We shall study the intersection over S_n^2, i.e. we assume that the second case of $(i)_n$ hold. The other case is symmetric.

We shall denote by A_n^2 and B_n^2 the sets we get when changing I_n to I_n^2 in the definition of A_n and B_n, respectively.

Just for simplicity we shall in what follows make some abuse of notation and write $S_n = S_n^2$ and $A_n = A_n^2$. Moreover, we always skip the index $\iota = 2$.

Proof that $(\mathcal{A}_1)_{n+1}$ *holds*: Since $(i)_n$ (the second case by the assumption above) hold, the four intersection points $s_{\pm,\pm}$, given by $(v)_n$, are unique. Let a_{n+1} be the intersection point between φ_n^+ and ψ_n^-, and let b_{n+1} be the intersection point between φ_n^- and ψ_n^+, so

$$(7.27) \qquad \begin{aligned} \varphi_n^-(a_{n+1}) < \varphi_n^+(a_{n+1}) = \psi_n^-(a_{n+1}) < \psi_n^+(a_{n+1}) \\ \varphi_n^+(b_{n+1}) > \varphi_n^-(b_{n+1}) = \psi_n^+(b_{n+1}) > \psi_n^-(b_{n+1}). \end{aligned}$$

Then we notice that (see fig.3)

$$S_{n+1} = (a_{n+1}, b_{n+1}).$$

In the case $n = 0$ we use $(v)_0$ to exclude any intersection outside S_0^1 and S_0^2. This shows $(I1)_{n+1}$. Moreover, an easy calculation, using $(i)_n$ and $(iii)_n$ yields

$$(7.28) \qquad b_{n+1} - a_{n+1} < \frac{2\exp(-\delta_0(M_0 - 1)\lambda)}{\lambda^{3/4} + \lambda^{3/4}} < \exp(-\delta_0(M_0 - 1)\lambda),$$

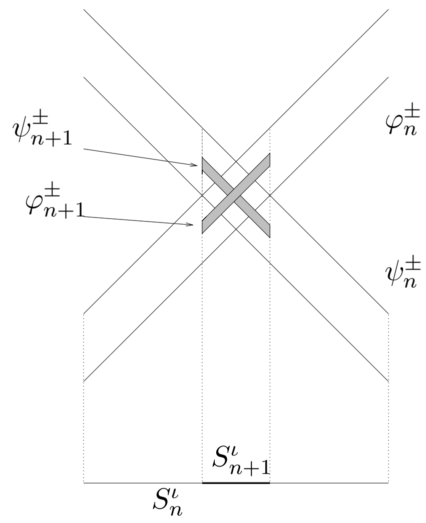

FIGURE 3. One of the two intersections.

which is the estimate in $(I1)_{n+1}$. Also, by the implicit function theorem, applied to the identity

$$\varphi_n^+(a_{n+1},\omega)=\psi_n^-(a_{n+1},\omega),$$

(see (7.27)) we get that $a_{n+1}(\omega)$ is differentiable in ω and that

$$a'_{n+1}(\omega)=\frac{\partial_\omega\psi_n^- - \partial_\omega\varphi_n^+}{\partial_\theta\varphi_n^+ - \partial_\theta\psi_n^-}.$$

From the estimates $(i)_n$ and $(ii)_n$ we now get

$$|a'_{n+1}(\omega)|<1/\lambda^{1/20}.$$

Since the same holds for the right endpoint, this gives $(I2)_{n+1}$.

We have hence verified that $(\mathcal{A}1)_{n+1}$ holds.

Proof that $(\mathcal{A}_2)_{n+1}$ *holds*: The existence of the functions $\varphi^{\pm}_{n+1}$ and $\psi^{\pm}_{n+1}$, as well as the estimates $(i)_{n+1}-(iii)_{n+1}$ is proved exactly as in the proof of lemma 7.5 (recall that $S_n\subset S_0^1\cup S_0^2$, where S_0^ι is defined in 7.2). We now proceed with the proof of $(iv)_{n+1}$.

By corollary 5.2 and $(I1)_{n+1}$, we see that $\mathcal{P}\circ\Phi^{M_{n+1}}(A_{n+1})$ is strictly included in $\mathcal{P}\circ\Phi^{M_n}(A_n)$ and that $\mathcal{P}\circ\Phi^{-M_{n+1}}(B_{n+1})$ is strictly included in $\mathcal{P}\circ\Phi^{-M_n}(B_n)$ (see fig 3). Combining this with (7.27) yields

$$\begin{aligned}\varphi^+_{n+1}(a_{n+1}+)&<\varphi_n^+(a_{n+1})=\psi_n^-(a_{n+1})<\psi^-_{n+1}(a_{n+1})\\ \varphi^-_{n+1}(b_{n+1}-)&>\varphi_n^-(b_{n+1})=\psi_n^+(b_{n+1})>\psi^+_{n+1}(b_{n+1}-)\end{aligned}\tag{7.29}$$

(when evaluating the functions $\varphi^{\pm}_{n+1}$, $\psi^{\pm}_{n+1}$ at the endpoints we use the left (right) limit, which exists by the derivative estimates). This shows that $(iv)_{n+1}$ holds and finishes the proof of $(\mathcal{A}2)_{n+1}$.

Proof of the density statement $(I4)_{n+1}$: Now we proceed with the proof of $(I4)_{n+1}$. Fix $\omega \in \mathcal{F}_{n+1}$.

From (7.29) and the derivative estimates $(i)_n$ and $(i)_{n+1}$ we see that the graphs of φ^{α}_{n+1} and ψ^{β}_n must intersect at a unique point in S_{n+1} ($\alpha = \pm, \beta = \pm$) (see fig. 3 and 4). Let $p, q \in S_{n+1}$ be such that

$$\varphi^-_{n+1}(p) = \psi^-_n(p), \quad \varphi^+_{n+1}(q) = \psi^+_n(q). \tag{7.30}$$

Using the estimates $(i)_n, (i)_{n+1}$ and $(iii)_n, (iii)_{n+1}$, in addition with the assumption $10M_n/\delta_0 \leq M_{n+1}$, one easily verifies

$$q - p > \frac{\exp(-5\lambda M_n) - \exp(-\delta_0(M_{n+1}-1)\lambda)}{2\lambda^{3/2}} > 0 \tag{7.31}$$

The important thing is not the lower bound, but the fact that we get a non-empty interval $(p, q) \subset S_{n+1}$.

We now let

$$\Delta = \{\theta : s(\theta) \in (p, q) \subset S_{n+1}\} \subset I_{n+1}$$

and

$$\hat{A}_{n+1} := \{(0, \theta, r) : \theta \in \Delta - M_{n+1}\omega, \delta_0\lambda < r < 2\lambda\} \subset A_{n+1}.$$

From the choice of p and q we then have that (see fig. 4)

$$\mathcal{P} \circ \Phi^{M_{n+1}}(\hat{A}_{n+1}) =$$
$$= \{(1/2 - s\omega, 1/2 + s, r) : s \in (p, q), \varphi^-_{n+1}(s) < r < \varphi^+_{n+1}(s)\}$$

is a subset of $\mathcal{P} \circ \Phi^{-M_n}(B_n)$. Hence

$$\Phi^{M_{n+1}+M_n}(\hat{A}_{n+1}) =$$
$$= \{(0, \theta, r) : \theta \in \Delta + M_n\omega, \phi^-(\theta) < r < \phi^+(\theta)\}$$

is a subset of B_n, where $\phi^\pm$ are two C^1−functions defined in an open neighborhood of the closure of $\Delta + M_n\omega$. We note that (7.30) implies

$$\phi^-(p') = -2\lambda, \quad \phi^+(q') = -\delta_0\lambda, \quad \text{where } (p', q') = \Delta + M_n\omega. \tag{7.32}$$

Now we shall estimate $\phi^\pm$. Let $r^+(0, \theta) = 2\lambda$ and $r^-(0, \theta) = \delta_0\lambda$ for $\theta \in \Delta - M_{n+1}\omega$ be the horizontal boundaries of $\hat{A}_{n+1}$. Then, by definition, we have $\phi^\pm(\theta) = r^\pm(M_{n+1} + M_n, \theta - M_{n+1} - M_n)$ for $\theta \in \Delta + M_n\omega$. Since $\phi^\pm(\theta) \in [-2\lambda, -\delta_0]$, we can apply lemma 4.2 to obtain the bound

$$\int_T^{M_{n+1}+M_n} r^\pm(t, \theta)dt \geq -2\lambda(M_{n+1} + M_n - T), \quad T \leq M_{n+1} + M_n. \tag{7.33}$$

Moreover, by proposition 5.1, we have

$$\int_T^{M_{n+1}} r(t, \theta)dt \geq \delta_0(M_{n+1} - T)\lambda/2, \quad 0 \leq T \leq M_{n+1}. \tag{7.34}$$

From formula (7.9) we get

$$|\partial_\theta \phi^\pm(\theta)| \leq \|V\|_{C^1} \int_0^{M_{n+1}+M_n} \exp\left(-2\int_t^{M_{n+1}+M_n} r^\pm(\xi, \theta)d\xi\right) dt =$$

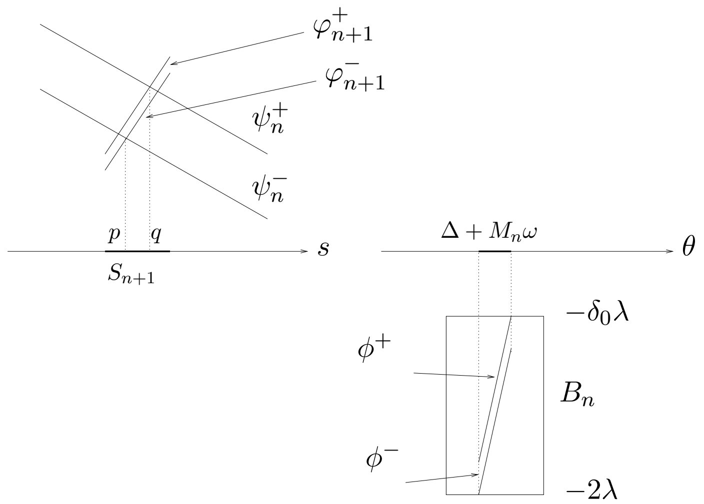

FIGURE 4. The "spreading out"

$$\Bigg[\int_0^{M_{n+1}} \exp\left(-2\int_t^{M_{n+1}} r^{\pm}(\xi,\theta)d\xi\right)\exp\left(-2\int_{M_{n+1}}^{M_{n+1}+M_n} r^{\pm}(\xi,\theta)d\xi\right)dt$$
$$+\int_{M_{n+1}}^{M_{n+1}+M_n}\exp\left(-2\int_t^{M_{n+1}+M_n} r^{\pm}(\xi,\theta)d\xi\right)dt\Bigg]\cdot\|V\|_{C^1}$$

which by the estimates (7.33) and (7.34) gives

$$|\partial_\theta\phi^{\pm}(\theta)| \leq \exp(4\lambda M_n)/2. \tag{7.35}$$

Furthermore, from (7.11) we obtain

$$\phi^+ - \phi^- \leq 2\lambda\exp\left(-\int_0^{M_{n+1}+M_n}(r^+(t,\theta)+r^-(t,\theta))dt\right)$$

which, again from (7.33) and (7.34), together with the assumption $M_{n+1} > 10M_n/\delta_0$, is bounded from above by

$$\exp(-\delta_0 M_{n+1}\lambda + 4\lambda M_n) < \exp(-\lambda M_n)/2. \tag{7.36}$$

Combining (7.32), (7.35) and (7.36) with the assumptions on X, and recalling the definition of $\hat{A}_{n+1}$ and $\phi^{\pm}$, now yields the statement in $(I4)_{n+1}$.

Finally, the proof of $(I5)_{n+1}$ is symmetric.

□

Summing up, lemma 7.1, 7.5 and 7.6 now gives

Proposition 7.7. *There exists a $\lambda_7 > 0$ such that for all $\lambda > \lambda_7$, all energies $E \in \mathcal{E}$ and all sequences $\{M_k\}_{k=0}^\infty$ satisfying $M_0 > 10$ and $M_{k+1} > 10M_k/\delta_0$,* *the following hold for $n \geq 0$:*

$$\omega \in \mathcal{F}_{-1} \Longrightarrow \begin{cases} \mathrm{comp(I_0)} = 1 \\ |I_0| < 10/\sqrt{\lambda} \\ |\partial_\omega(I_0^\iota + \omega/2)| < 1/4 \end{cases}$$

and

$$\omega \in \mathcal{F}_n \Longrightarrow \begin{cases} \mathrm{comp(I_j)} = 2 \\ \overline{I_j} \subset I_{j-1} \\ 0 < |I_j^\iota| < \exp(-\delta_0 M_{j-1}\lambda/2) \\ |\partial_\omega(I_j^\iota + \omega/2)| < 1/4 \end{cases} \qquad j = 1, \ldots, n+1.$$

Moreover, if $\omega \in \mathcal{F}_n, n \geq 1$, then $(I4-5)_j$ hold, $j = 1, \ldots, n$.

8. Good frequencies

Definition 8.1 (The choice of the sequence $\{N_n\}$)**.** For all $\lambda > 0$ we define

$$N_{n+1} = \lambda^{N_n/4}, \quad N_0 = \lambda^{1/4}.$$

From now on N_n shall always be as above.

Remark 6. Note that the sequence $\{N_k\}$ growth superexponentially fast.

Proposition 8.2. *There exists a $\lambda_8 > 0$ such that for all $\lambda > \lambda_8$ and all $E \in \mathcal{E}$ there is a measurable set $\Omega_E \subset \mathcal{F}_{-1} \setminus \mathbb{Q}$,*

$$(\mathcal{S}1) \qquad |\Omega_E| \geq 1/4 - \frac{1}{\lambda^{1/4}},$$

with the following properties: For all $\omega \in \Omega_E$ there is an infinite sequence $\{M_n\}_{n=0}^{\infty}$ of positive integers ($N_n \leq M_n \leq 2N_n$) such that:

$$(\mathcal{S}2) \qquad \omega \in \bigcap_{n \geq 0} \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E).$$

The key part for the proof of this proposition is the following lemma:

Lemma 8.3. *There exists a $\lambda_9 > 0$ such that for all $\lambda > \lambda_9$ and all $E \in \mathcal{E}$ the following holds for $n \geq 0$:*

If $M_j \in [N_j, 2N_j]$ for $j = 0, \ldots, n$, and if $\Omega \subset \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ is an interval, then there is a set $\Omega_+ \subset \Omega$ satisfying

$$(8.1) \qquad |\Omega_+| \geq |\Omega| - 1/N_{n+1}^{3/2},$$

$$(8.2) \qquad comp(\Omega_+) \leq N_{n+1}^4,$$

and to each component Ω_+^ι in Ω_+ there is an integer $M_{n+1} \in [N_{n+1}, 2N_{n+1}]$ so that $\Omega_+^\iota \subset \mathcal{F}_{n+1}(M_0, \ldots, M_n, M_{n+1}, \lambda, E)$.

Remark 7. The reason for choosing different M_{n+1} at each component in Ω_+ and not just one, is that condition $(\mathcal{F}2)_{n+1}$ in the definition of $\mathcal{F}_{n+1}$ otherwise will cause us troubles.

Before we start proving lemma 8.3, we need a result concerning the geometry of a no-fast-return-set:

Lemma 8.4. *Assume that for all $\omega \in \mathbb{T}$ there are intervals $I^1, \dots, I^{\mathcal{N}}$, each of length $\leq \delta$ and satisfying $|\partial_\omega I^\iota| \leq C < 1/2$. Then, for $M > 0$ and $\varepsilon > 0$, the set*

$$\left\{\omega \in \mathbb{T} : dist\left(\cup I^i, \bigcup_{m=1}^{M} (\cup I^i + m\omega)\right) < \varepsilon\right\}$$

has measure $\leq 2\mathcal{N}^2 M \frac{\delta+\varepsilon}{1-2C}$ and consists of at most $\mathcal{N}^2 M^2$ components.

Proof. By standard analysis one easily verifies that if $J_1 = J_1(\omega)$ and $J_2 = J_2(\omega)$ are two intervals in $\mathbb{T}$ such that for all $\omega \in \mathbb{T}$ we have $|\partial_\omega J_i(\omega)| \leq C < 1/2$ and $|J_i(\omega)| \leq \delta_i, i = 1, 2$, then for $m \neq 0$ the set

$$\{\omega \in \mathbb{T} : \mathrm{dist}(J_1, J_2 + m\omega) < \varepsilon\}$$

consists of at most $|m|$ components, each of length

$$\leq \frac{\delta_1 + \delta_2 + 2\varepsilon}{|m| - 2C}.$$

Let now W denote the set in the lemma. Clearly

$$W = \bigcup_{1 \leq i,j \leq \mathcal{N}} \bigcup_{m=1}^{M} \{\omega \in \mathbb{T} : \mathrm{dist}(I^i, I^j + m) < \varepsilon\},$$

and hence if follows from the above analysis that the measure of W is bounded by

$$\sum_{1 \leq i,j \leq \mathcal{N}} \sum_{m=1}^{M} |\{\omega \in \mathbb{T} : \mathrm{dist}(I^i, I^j + m) < \varepsilon\}|$$

$$\leq \mathcal{N}^2 \sum_{m=1}^{M} \frac{2\delta + 2\varepsilon}{m - 2C} \leq 2\mathcal{N}^2 M \frac{\delta + \varepsilon}{1 - 2C}$$

and that the number of components

$$\leq \mathcal{N}^2 \sum_{m=1}^{M} |m| \leq \mathcal{N}^2 M^2.$$

□

Now we are ready for the proof of lemma 8.3.

Proof of lemma 8.3. We shall in what follows assume that λ is sufficiently large. Note that we for each $\omega \in \Omega$ can apply proposition 7.7, since we clearly have $M_0 > 10$ and $10M_k/\delta_0 < M_{k+1}$ if λ is large (recall that the sequence $\{N_n\}$ grows superexponentially fast). In the proof we shall use the notation

$$\varepsilon_j = 1/\lambda^{M_{j-1}}, \quad j = 0, \dots, n.$$

From proposition 7.7 we get

$$\begin{cases} \operatorname{comp}(I_j) = 2(= 1 \text{ if } j = 0), \\ |I_j^\iota| < \exp(-\delta_0 M_{j-1}\lambda/2) < \varepsilon_j, \qquad j = 0, \dots, n+1. \\ |\partial_\omega(I_j^\iota + \omega_2)| < 1/4 \end{cases} \tag{8.3}$$

Search for M_{n+1} candidates: We now claim that there always is "room" for the intervals $I_{n+1} \pm M\omega$, for some $N_{n+1} \le M \le 2N_{n+1}$.

Claim: For every $\omega \in \Omega$ there is an integer $M \in [N_{n+1}, 2N_{n+1}]$ such that

$$\operatorname{dist}\left(I_{n+1} \pm M\omega, \bigcup_{l=0}^{n} \bigcup_{|m|\le 2M_l} (I_l + m\omega)\right) > \varepsilon_n.$$

Proof of claim. Fix $\omega \in \Omega$. We write $N = N_{n+1}$. Recall that $M_{-1} = 3/4$. Since $\omega \in \Omega \subset \mathcal{F}_n$, $(\mathcal{F}1)_n$ holds, i.e.

$$\operatorname{dist}\left(I_j, \bigcup_{m=1}^{2^{j+6}M_j/\delta_0} (I_j + m\omega)\right) > 3\varepsilon_j, \quad j = 0, \dots, n. \tag{8.4}$$

For each $j = 0, \dots, n$, if we take $\iota, k \in \{1, 2\}$ ($\iota = k = 1$ if $j = 0$) and an integer $L \in \mathbb{Z}$, it follows from (8.3) and (8.4) that there is at most one $p \in J := [L, L + 2^{j+6}M_j/\delta_0 - 1]$ such that $\operatorname{dist}(I_j^\iota + p\omega, I_j^k) \le \varepsilon_j$, and at most one $q \in J$ such that $\operatorname{dist}(I_j^\iota - q\omega, I_j^k) \le \varepsilon_j$. Hence there are at most $2(4M_j + 1)$ p:s in J such that

$$\operatorname{dist}\left(I_j^\iota \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j^k + m\omega)\right) \le \varepsilon_j.$$

Consequently, there is at most $2(4M_j + 1) \cdot 2^2$ different $p \in J$ satisfying

$$\operatorname{dist}\left(I_j \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_j.$$

Now, for each $j = 0, \dots, n$, we divide the interval $[N, N + 2^{n+6}M_n/\delta_0 - 1]$ into subintervals of length $2^{j+6}M_j/\delta_0$. Then we get $[2^{n-j}M_n/M_j]$ intervals, plus maybe one extra, shorter. By the above analysis, and by the fact that $I_{n+1} \subset I_j$ and $\varepsilon_n \le \varepsilon_j$, we get that on each of these subintervals there is at most $2(4M_j + 1) \cdot 2^2$ different p:s satisfying

$$\operatorname{dist}\left(I_{n+1} \pm p\omega, \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_n. \tag{8.5}$$

So, there are at most

$$2(4M_j + 1) \cdot 2^2([2^{n-j}M_n/M_j] + 1) \le 80 \cdot 2^{n-j}M_n$$

p:s in the interval $[N, N + 2^{n+6}M_n/\delta_0 - 1]$ such that (8.5) holds. From this we derive that there is at most

$$80(1 + 2 + \ldots + 2^n)M_n < 80 \cdot 2^{n+1}M_n < 2^{n+6}M_n/\delta_0$$

different p:s in $[N, N + 2^{n+6}M_n/\delta_0 - 1]$ such that

$$\operatorname{dist}\left(I_{n+1} \pm p\omega, \bigcup_{j=0}^{n} \bigcup_{|m|\le 2M_j} (I_j + m\omega)\right) \le \varepsilon_n,$$

(note that $\delta_0 < 1/5$).

From this the statement follows, since

$$2^{n+6}M_n/\delta_0 \le 2^{n+6}(2N_n)/\delta_0 < N_{n+1} = N$$

if λ is large, independent of n. □

Construction of Ω_+*:* We will now construct the set $\Omega_+ \subset \Omega$ satisfying (8.1) and (8.2). First we let

$$\delta := \frac{\varepsilon_n}{12N_{n+1}}.$$

Divide the interval Ω into $T-1$ disjoint intervals $\Omega^1, \ldots, \Omega^{T-1}$ of length δ, and one, Ω^T, of length $< \delta$, i.e.

$$\Omega = \bigcup_{i=1}^{T} \Omega^i, \quad |\Omega^i| = \delta, i = 1, \ldots, T-1, \quad |\Omega^T| < \delta.$$

Since $\Omega \subset (0, 1/4)$, we clearly have the trivial estimate $T < 1/\delta$.

If we now for each interval Ω^i take the midpoint ω_i and apply the above claim, we get an integer $M^i_{n+1} \in [N_{n+1}, 2N_{n+1}]$ so that

$$\operatorname{dist}\left(I_{n+1}(\omega_i) \pm M^i_{n+1}\omega_i, \bigcup_{l=0}^{n} \bigcup_{|m|\le 2M_l} (I_l(\omega_i) + m\omega_i)\right) > \varepsilon_n.$$

Moreover, since $|\Omega^i| \le \delta$, since $M_0 < \ldots < M_n < M^\iota_{n+1} \in [N_{n+1}, 2N_{n+1}]$ and since $|\partial_\omega I^\iota_j| < 1/4$ (8.3) for $j = 0, \ldots, n+1$, it is easily seen that

$$\operatorname{dist}\left(I_{n+1}(\omega) \pm M^i_{n+1}\omega, \bigcup_{l=0}^{n} \bigcup_{|m|\le 2M_l} (I_l(\omega) + m\omega)\right) > 0 \quad \forall \omega \in \Omega^i. \tag{8.6}$$

Hence all $\omega \in \Omega^i$ satisfy $(\mathcal{F}2)_{n+1}$). So now we only have to remove those ω:s which do not satisfy condition $(\mathcal{F}1)_{n+1}$.

Let Ω^i_+ be the set of ω:s in Ω^i which satisfies

$$\operatorname{dist}\left(I_{n+1}(\omega), \bigcup_{m=1}^{2^{(n+1)+6}M^i_{n+1}/\delta_0} (I_{n+1}(\omega) + m\omega)\right) > 3\varepsilon_{n+1}. \tag{8.7}$$

This is clearly equivalent with that ω satisfies

$$\operatorname{dist}\left(I_{n+1}(\omega)+\omega/2, \bigcup_{m=1}^{2^{(n+1)+6}M^i_{n+1}/\delta_0}((I_{n+1}(\omega)+\omega/2)+m\omega)\right) > 3\varepsilon_{n+1}.$$

To estimate the size and numbers of intervals in Ω^i_+ we apply lemma 8.4 and think that all "bad" ω are concentrated in Ω^i_+. This is of course a very rough estimate, but it is enough for our purpose. We hence get, making use of the above estimates of $(I_{n+1}(\omega)+\omega/2)$ (proposition 7.7) and the fact that $N_{n+1} \le M^i_{n+1} \le 2N_{n+1}$ and $M_n \ge N_n$,

$$|\Omega^i_+| \ge |\Omega^i| - 2\mathcal{N}^2(2^{n+7}M^i_{n+1}/\delta_0)\cdot\frac{4\varepsilon_{n+1}}{1/2} \ge |\Omega^i| - \frac{c\cdot 2^n}{N^3_{n+1}}, \tag{8.8}$$

and

$$\operatorname{comp}(\Omega^i_+) \le 2^2(2^{n+7}M^i_{n+1}/\delta_0)^2 \le c4^nN^2_{n+1}. \tag{8.9}$$

Finally we define

$$\Omega_+ = \bigcup_{i=1}^{T}\Omega^i_+.$$

Then, since $T \le 1/\delta = 12\cdot N_{n+1}/\varepsilon_n \le 12N_{n+1}N^8_n$, it follows from (8.8) and (8.9) that

$$|\Omega_+| \ge \sum_{i=1}^{T}|\Omega^i| - \frac{12c2^nN_{n+1}N^8_n}{N^3_{n+1}} \ge |\Omega| - \frac{1}{N^{3/2}_{n+1}}$$

and

$$\operatorname{comp}(\Omega_+) \le \sum_{i=1}^{T}\operatorname{comp}(\Omega^i_+) \le N^4_{n+1},$$

provided λ is sufficiently large (independent of n). □

Proof of proposition 8.2. We assume that λ is sufficiently large.

Let us fix $E\in\mathcal{E}$. First we let $M_0 := N_0$ and $\Omega_0 = \Omega_0(E)$ be the set of $\omega\in\mathcal{F}_{-1}$ such that (recall that $M_{-1} = 3/4$)

$$\operatorname{dist}\left(I_0, \bigcup_{m=1}^{2^6M_0/\delta_0}(I_0+m\omega)\right) > \frac{3}{\lambda^{M_{-1}}}. \tag{8.10}$$

(Note that $\Omega_0 = \mathcal{F}_0(M_0,\lambda,E)$.) Since $\operatorname{comp}(I_0) = 1$, $|I_0| < 1/\lambda^{3/4}$ and $|\partial_\omega(I_0+\omega/2)| < 1/4$ (lemma 7.1), we have

$$|\Omega_0| \ge 1/4 - 2(2^6M_0/\delta_0)\frac{4/\lambda^{3/4}}{1/2} \ge 1/4 - \frac{1}{N^{3/2}_0} \tag{8.11}$$

and

$$\operatorname{comp}(\Omega_0) \le (2^6M_0/\delta_0)^2 \le N^4_0. \tag{8.12}$$

Assume now that we inductively have gotten sets $\Omega_n \subset \ldots \subset \Omega_0$, where each Ω_j satisfies

$(O1)_j$ $\text{comp}(\Omega_j) \leq N_0^4 \cdots N_j^4$.

$(O2)_j$ $|\Omega_j| \geq 1/4 - \frac{1}{N_0^{3/2}} - \sum_{i=1}^{j} \frac{1}{N_i}$

$(O3)_j$ For each $1 \leq \iota \leq \text{comp}(\Omega_j)$ there is a sequence $(M_0, \ldots, M_j)$, $N_i \leq M_i \leq 2N_i$, such that $\Omega_j^\iota \subset \mathcal{F}_j(M_0, \ldots, M_j, \lambda, E)$.

Now we want to construct Ω_{n+1}. To each component in Ω_n we can apply proposition 8.3. Hence, for each Ω_n^ι in Ω_n we get a set $\Omega_{n+1}^\iota \in \Omega_n^\iota$ which contains at most N_{n+1}^4 components, which satisfies

$$|\Omega_n^\iota| - |\Omega_{n+1}^\iota| \leq \frac{1}{N_{n+1}^{3/2}},$$

and where each component satisfies $(O3)_{n+1}$. Consequently, letting

$$\Omega_{n+1} = \bigcup_{\iota=1}^{\text{comp}(\Omega_n)} \Omega_{n+1}^\iota$$

gives us

$$\text{comp}(\Omega_{n+1}) \leq N_{n+1}^4 \text{comp}(\Omega_n) \leq N_0^4 \cdots N_{n+1}^4$$

and

$$|\Omega_n| - |\Omega_{n+1}| \leq \frac{\text{comp}(\Omega_n)}{N_{n+1}^{3/2}} \leq \frac{N_0^4 \cdots N_n^4}{N_{n+1}^{3/2}} \leq \frac{1}{N_{n+1}}.$$

Hence Ω_{n+1} satisfies $(O1-3)_{n+1}$.

We now let

$$\Omega_E := \bigcap_n \Omega_n \setminus \mathbb{Q}.$$

From the above estimates if then follows that Ω_E is measurable and satisfies

$$|\Omega_E| \geq 1/4 - \frac{1}{N_0^{3/2}} - \sum_{i=1}^{\infty} \frac{1}{N_i} \geq 1/4 - 1/\lambda^{1/4}.$$

Moreover, for each $(E, \omega) \in \mathcal{S}$ there is an infinite sequence $(M_0, M_1, \ldots)$, where $N_i \leq M_i \leq 2N_i$, such that $\omega \in \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E)$ for all n. $\square$

9. Proof of theorem 1 and 2

In this final section we finish the proof of Theorem 1 and 2. Hence forward, we assume that λ is sufficiently large, so that we can apply previous lemmas and propositions.

First we fix $E \in \mathcal{E}$ and let Ω_E be the set given by 8.2. Take any $\omega \in \Omega_E$. Then we get a sequence $\{M_k\}_{k=1}^{\infty}$, satisfying

$$N_k \leq M_k \leq 2N_k \quad \text{for all } k \geq 0, \tag{9.1}$$

such that

$$\omega \in (\mathbb{R} \setminus \mathbb{Q}) \cap \bigcap_n \mathcal{F}_n(M_0, \ldots, M_n, \lambda, E).$$

From now on E, ω and $\{M_k\}$ shall be fixed as above.

From proposition 7.7 we get

$$\text{comp}(I_n) = 2(= 1 \text{ if } n = 0), \quad 0 < |I_n^\iota| \leq 1/\lambda^{M_{n-1}}, \quad n \geq 0. \tag{9.2}$$

(Note that $\exp(-\delta_0 M_n \lambda/2) << 1/\lambda^{M_n}$, and recall that $M_{-1} = 3/4$.) This implies that the set

$$\Theta_\infty := \bigcap_n \Theta_n = \mathbb{T} \setminus \bigcup_{l=0}^{\infty} \bigcup_{|m| \leq M_l - 1} (I_l + m\omega), \tag{9.3}$$

which is closed, satisfies

$$|\Theta_\infty| \geq 1 - \sum_{n=0}^{\infty} (2M_n - 1)|I_n| \geq 1 - 8 \sum_{n=0}^{\infty} N_n/\lambda^{N_{n-1}} \overset{\lambda \to \infty}{\to} 1.$$

Hence we have $|\Theta_\infty| > 0$ for large λ. (Here we used the estimates (9.1) and (9.2) together with the definition of the N_n:s.)

We can now prove that $(*)$ and $(**)$ in Theorem 1 hold:

Proof of $(*)$. Fix $\theta_0 \in \Theta_\infty$ and take any $r(0) \in (\delta_0\lambda, 2\lambda)$. Let $0 < T_0 \leq T_1 \leq T_2 \leq \ldots$ be the first times such that

$$\theta_{T_n} \in I_n,$$

i.e. $N = T_n$ is the smallest positive integer such that $\theta_N \in I_n$. These times exists since $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and since $|I_n| > 0$. Moreover, since $\theta_0 \in \Theta_\infty$, we have $T_n \geq M_n$ for all n. From proposition 5.1 it now follows that

$$\int_0^{T_n} r(t)dt \geq \delta_0 T_n \lambda/2. \tag{9.4}$$

Recalling that (in view of the Schrödinger equation (1.1))

$$r(0) = u'(0)/u(0) \quad \text{and} \quad \exp\left(\int_0^{T_n} r(t)dt\right) = |u(T_n)/u(0)|,$$

we see that(9.4) implies

$$\limsup_{t\to\infty} \frac{1}{2t} \log(u(t)^2 + u'(t)^2) \geq \limsup_{n\to\infty} \frac{1}{T_n}(\delta_0 T_n \lambda/2) = \delta_0\lambda/2.$$

Since Θ_∞ has positive measure, this implies that

$$\gamma(E) \geq \frac{\delta_0\lambda}{2}.$$

□

Proof of $(**)$. Take

$$(0,\theta_0,r(0))\in\bigcap_n\left(\Phi^{M_n}(A_n)\cap\Phi^{-M_n}(B_n)\right)\neq\emptyset.$$

The intersection is non-empty by proposition 7.7 and corollary 5.2. Applying proposition 5.1 to each of the points

$$(0,\theta_{-M_n},r(-M_n))\in A_n,\quad (0,\theta_{M_n},r(M_n))\in B_n$$

gives

$$\log|u(0)/u(-T)|=\int_{-T}^{0}r(t)dt>\delta_0 T\lambda,\quad \forall T\geq 0$$

and

$$\log|u(T)/u(0)|=\int_{0}^{T}r(t)dt<-\delta_0 T\lambda,\quad \forall T\geq 0.$$

□

We now continue with the proof of the statement in Theorem 2:

Proof of the minimality. Let us fix $\iota\in\{1,2\}$ an work with the non-empty intervals

$$I_0^\iota\supset I_1^\iota\supset I_2^\iota\supset\cdots$$

We recall (9.2).

Forward iterations: Take any $\theta_0\in\Theta_\infty$ and $r(0)\in(\delta_0\lambda,2\lambda)$. We shall now prove that the forward orbit $\{\Phi^k(0,\theta_0,r(0))\}_{k=0}^\infty$ is dense in $\{0\}\times\mathbb{T}\times\hat{\mathbb{R}}$.

Firstly, the points $\{\theta_k\}_{k=0}^\infty$ are dense in $\mathbb{T}$ since ω is irrational. Since $(C2)_n$ in proposition 5.1 hold for each n, we have

$$r(k)\notin(\delta_0\lambda,2\lambda)\Longrightarrow\theta_k\in\bigcup_{l=0}^{\infty}\bigcup_{m=1}^{M_l}(I_l+m\omega),\quad k\geq 0. \tag{9.5}$$

From $(\mathcal{F}1-2)_n$ we derive

$$(I_n-M_n\omega)\cap\bigcup_{l=0}^{n}\bigcup_{m=1}^{M_l}(I_l+m\omega)=\emptyset.$$

Combining this with (9.5) yields

$$r(k)\notin(\delta_0\lambda,2\lambda)\text{ and }\theta_k\in I_n^\iota-M_n\omega\Longrightarrow\theta_k\in\bigcup_{l=n+1}^{\infty}\bigcup_{m=1}^{M_l}(I_l+m\omega),\quad k\geq 0. \tag{9.6}$$

Now, for $n\geq 1$ we let

$$X_n=\{(0,\theta_k,r(k)):\theta_k\in I_n^\iota-M_n\omega,r(k)\in(\delta_0\lambda,2\lambda),k\geq 0\}\subset A_n.$$

Since

$$\left|\bigcup_{l=n+1}^{\infty}\bigcup_{m=1}^{M_l}(I_l+m\omega)\right|\leq 2\mathcal{N}\sum_{l=n+1}\underbrace{N_l/\lambda^{N_{l-1}}}_{1/\lambda^{3N_{l-1}/4}}<1/\lambda^{N_n/2},$$

it follows from (9.6) that $\pi_2(X_n)$ is $1/\lambda^{N_n/2}$-dense in $I_n^\iota - M_n\omega$. Since also

$$\max\{\exp(4\lambda M_{n-1})/\lambda^{N_n/2}, \exp(-\lambda M_{n-1})\} \le \exp(-\lambda N_{n-1}),$$

(recall (9.1)) it follows from $(I4)_n$ in proposition 7.7 that

$$\pi_3(\Phi^{M_n+M_{n-1}}(X_n)) \text{ is } \exp(-\lambda N_{n-1})\text{-dense in } (-2\lambda, -\delta_0\lambda). \tag{9.7}$$

We now note that

$$I_n^\iota + M_{n-1}\omega \subset I_{n-1} + M_{n-1}\omega \subset \Theta_{n-1} \tag{9.8}$$

and that $(\mathcal{F}1-2)_{n-1}$ gives

$$I_{n-1} + (M_{n-1}+1)\omega \subset \Theta_{n-1}. \tag{9.9}$$

Take any sequence $\{p_n\}_{n=1}^\infty$ such that $p_n \in I_n^\iota + M_{n-1}\omega$. Then, by (9.8) and (9.9) we have

$$p_n, p_n + \omega \in \Theta_{n-1}, \quad n \ge 1. \tag{9.10}$$

By the compactness of $\mathbb{T}$, there exists a subsequence $\{n_j\}$ and a $p \in \mathbb{T}$ such that

$$p_{n_j} \to p \text{ as } n \to \infty.$$

Since the sets $\Theta_{n_j} \subset \Theta_{n_{j-1}}$ are closed, it follows from (9.10) that

$$p, p + \omega \in \Theta_\infty. \tag{9.11}$$

Since also $|I_n^\iota| \to 0$ as $n \to \infty$, we must have

$$I_{n_j}^\iota + M_{n_j-1}\omega \to \{p\}$$

in the Hausdorff metric. From the fact that

$$\pi_2\left(\Phi^{M_n+M_{n-1}}(X_n)\right) \subset I_n^\iota + M_{n-1}\omega$$

it hence follows from (9.7) that

$$\overline{\{\Phi^k(0,\theta_0,r(0))\}_{k=0}^\infty} \supset \{0\} \times \{p\} \times [-2\lambda, -\delta_0\lambda] := \Gamma. \tag{9.12}$$

We shall now iterate the vertical line-segment Γ. Since $I_0 \cap \Theta_\infty = \emptyset$, lemma 4.1 implies

$$\Phi(\Gamma) = \{0\} \times \{p+\omega\} \times (\hat{\mathbb{R}} \setminus (2\delta_0\lambda, 3\lambda/2))$$

Now, if we take $(0, \theta_0^-, r_0^-), (0, \theta_0^+, r_0^+) \in \{0\} \times \{p+\omega\} \times (2\delta_0\lambda, 3\lambda/2)$, we can proceed as in the proof of $(*)$ (recall (9.11)), making use of formula (7.11) to show that the length of the vertical line-segments $\Phi^k(\{0\} \times \{p+\omega\} \times (2\delta_0\lambda, 3\lambda/2))$ goes to zero as $k \to \infty$. From this we conclude that

$$\overline{\bigcup_{k=0}^\infty \Phi^k(\Gamma)} = \mathbb{T} \times \hat{\mathbb{R}}.$$

This, together with (9.12) now yields

$$\overline{\{\Phi^k(0,\theta_0,r(0))\}_{k=0}^\infty} = \{0\} \times \mathbb{T} \times \hat{\mathbb{R}}.$$

Backward iterations: Similarly one shows that if $\theta_0 \in \Theta_\infty$ and $r(0) \in (-2\lambda, -\delta_0\lambda)$, then the backward orbit $\{\Phi^{-k}(0, \theta_0, r(0))\}_{k=0}^{\infty}$ is dense in $\{0\} \times \mathbb{T} \times \hat{\mathbb{R}}$.

The minimality: To finish the proof we do as follows: Let

$$\Theta'_\infty := \mathbb{T} \setminus \bigcup_{l=0}^{\infty} \bigcup_{|m| \le M_l} (I_l + m\omega)$$

(compare with Θ_∞, (9.3)). Then we have

$$\theta \in \Theta'_\infty \quad \Longrightarrow \quad \theta, \theta - \omega \in \Theta_\infty.$$

Note also that $|\Theta'_\infty| > 0$.

Now, take any $\theta_0 \in \Theta'_\infty$ and any $r(0) \in \hat{\mathbb{R}}$. If $r(0) \in (\delta_0\lambda, 2\lambda)$, then it follows from the forward iteration analysis above that $\{(\theta_k, r(k))\}_{k=0}^{\infty}$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$.

If $r(0) \notin (\delta_0\lambda, 2\lambda)$, then, by lemma 4.1, $r(-1) \in (-2\lambda, -\delta_0\lambda)$. Since $\theta_{-1} \in \Theta_\infty$, we get that $\{(\theta_{-k}, r(-k))\}_{k=0}^{\infty}$ is dense in $\mathbb{T} \times \hat{\mathbb{R}}$.

From this we conclude that the two graphs, given by the directions in the Oseledet's theorem, are dense in $\mathbb{T} \times \hat{\mathbb{R}}$, and hence that time-one map

$$(\theta, r) \mapsto \pi_{23}\Phi^1(0, \theta, r)$$

is minimal in $\mathbb{T} \times \hat{\mathbb{R}}$. This implies that Φ it self is minimal. □

Selection of the set $\mathcal{S}$: Now it only remains to find a measurable set $\mathcal{S}$ satisfying the statement in Theorem 1. To do this we define the three sets

$$A := \{(E, \omega) \in \mathcal{E} \times \mathbb{T} : (*) \text{ hold}\},$$

$$B := \{(E, \omega) \in \mathcal{E} \times \mathbb{T} : (**) \text{ hold}\},$$

$$C := \left\{(E, \omega) \in \mathcal{E} \times \mathbb{T} : \Phi^t_{E,\omega} \text{ is minimal}\right\}.$$

It is standard to verify that each of these sets are measurable. We define

$$\mathcal{S} = A \cap B \cap C.$$

By the above analysis, we have that for each $E \in \mathcal{E}$, the set Ω_E, given by proposition 8.2, satisfies

$$\{E\} \times \Omega_E \subset \mathcal{S}.$$

Since $|\Omega_E| > 1/4 - 1/\lambda^{1/4}$, it hence follows that

$$|\mathcal{S}| \ge 0.1\lambda \left(1/4 - 1/\lambda^{1/4}\right).$$

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