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Combinatorial complexes, Bruhat intervals and reflection distances

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Abstract

The various results presented in this thesis are naturally subdivided into three different topics, namely **combinatorial complexes**, **Bruhat intervals** and **expected reflection distances**. Each topic is made up of one or several of the altogether six papers that constitute the thesis. The following are some of our results, listed by topic:

Combinatorial complexes:

- Using a shellability argument, we compute the cohomology groups of the complements of polygraph arrangements. These are the subspace arrangements that were exploited by Mark Haiman in his proof of the $n!$ theorem. We also extend these results to Dowling generalizations of polygraph arrangements.
- We consider certain B- and D-analogues of the quotient complex $\Delta(\Pi_n)/S_n$, i.e. the order complex of the partition lattice modulo the symmetric group, and some related complexes. Applying discrete Morse theory and an improved version of known lexicographic shellability techniques, we determine their homotopy types.
- Given a directed graph G, we study the complex of acyclic subgraphs of G as well as the complex of not strongly connected subgraphs of G. Known results in the case of G being the complete graph are generalized.

Bruhat intervals:
We list the (isomorphism classes of) posets that appear as intervals of length 4 in the Bruhat order on some Weyl group. In the special case of symmetric groups, we list all occuring intervals of lengths 4 and 5.

Expected reflection distances:
Consider a random walk in the Cayley graph of the complex reflection group $G(r, 1, n)$ with respect to the generating set of reflections. We determine the expected distance from the starting point after t steps. The symmetric group case ($r = 1$) has bearing on the biologist's problem of computing evolutionary distances between different genomes. More precisely, it is a good approximation of the expected reversal distance between a genome and the genome with t random reversals applied to it.

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Preface

If you are a mathematician, then you know all too well the following question: *"But, what does a mathematician really do?"*. Luckily, thanks to Pál Erdős, we all know the proper answer: mathematicians convert coffee into theorems. However, some people are rather persistent and keep going: *"But, what is a theorem anyway?"*. The answer is, of course, that a theorem is a tautology, a way to rephrase something. It doesn't say anything new! Thus, mathematics is basically the art of converting coffee into nothing, which is, clearly, to violate the first law of thermodynamics. This, of course, must never be told to the people that fund us.

During my graduate studies, a range of helpful people have guided and aided my attempts to violate the first law of thermodynamics. Among them are the following:

- Anders Björner has been a superb advisor. Being an enormously rich source of advice and hints, of answers and questions, of ideas and suggestions, he has provided the best possible support a graduate student can hope for.

- Kimmo Eriksson helped me excellently through the first part of my graduate studies, while Anders was ill. He also carefully read the article which has become Paper 5 of this thesis.

- After Kimmo had left for Mälardalen University, Dmitry Kozlov was kind enough to take his place. Apart from being ever helpful, he guided me through writing my first paper, which forms Paper 2 of this thesis.

- Through a large portion of my time as a graduate student, Niklas Eriksen was the only other combinatorics student at the department. We had many interesting discussions, and we wrote two articles together. They form Papers 5 and 6 of this thesis. Niklas also filled the important position as organizer of the department's weekly floorball matches.

- Niklas and I tried for some time to prove a crucial result (Theorem 5.5.1) without success. We then asked Richard Stanley of MIT for advice. He replied with the solution virtually before we had even asked the question.

- The entire staff of the department has provided a helpful and friendly environment suitable to work in.

- Finally, the people who have helped me the most (although the majority couldn't care less about mathematics!), and without whom I would never have had the opportunity nor the energy to work on this thesis, are my ever-supportive family and friends. You know who you are.

Contents

Introduction

This thesis consists of six different papers. There are ways to tie all of them together. For instance, the symmetric group and its relatives play important roles in all of the papers. However, a less artificial and more natural point of view is probably to divide the papers into three different categories:

- **Combinatorial complexes.** This category includes the first three papers, which all deal with topological properties of combinatorially defined cell complexes.
- **Bruhat intervals.** Paper 4 and a complement to it are the only members of this category. They address the problem of characterizing those posets that occur as intervals in Bruhat orders of Weyl groups.
- **Expected reflection distances.** The remainder of the thesis, Papers 5 and 6, concern expected distances after random walks on Cayley graphs of reflection groups.

Any of the above categories is mostly independent of the others. Below, we thus give an independent introduction to each. We will assume some knowledge of combinatorics, topology, Coxeter groups and complex reflection groups. We refer to [69], [62], [50] and [67], respectively, for more on these subjects.

0.1 Combinatorial complexes

Among the various classes of cell complexes, the most fundamental and important is probably the class of simplicial complexes. Viewing a simplicial complex as an abstract complex, i.e. as a collection of sets which is closed under taking subsets, it is clear that simplicial complexes are indeed very combinatorial by nature. On the other hand, viewing a complex as its geometric realization, it has a distinct topological flavour. Exploiting this connection between topology and combinatorics, the following question has often turned out to be highly interesting: *"What are the topological properties of this combinatorially defined simplicial complex?"*.

A fundamental construction in this area is the *order complex* $\Delta(P)$ of a poset P. It is the simplicial complex whose faces are the chains in P. Thus, to any

poset we may associate topological properties. Moreover, the topological properties are closely connected to the combinatorics of the poset. The Möbius function of an interval, for instance, coincides with the reduced Euler characteristic of the order complex of the interval with $\hat{0}$ and $\hat{1}$ removed. (As is customary, we use the convention that the unique maximal and minimal elements of a poset are denoted $\hat{0}$ and $\hat{1}$, respectively, if they exist.)

0.1.1 Subspace arrangements and shellability

One nice application of order complexes belongs to the theory of subspace arrangements. A (real) *subspace arrangement* is a collection of linear subspaces of Euclidean space $\mathbb{R}^n$. To an arrangement $\mathcal{A}$, one associates the *intersection lattice* $L(\mathcal{A})$. It consists of all possible intersections of subsets of $\mathcal{A}$ ordered by reverse inclusion. An interesting topological space related to $\mathcal{A}$ is its *complement* $\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n \setminus \cup\mathcal{A}$. Goresky and MacPherson [39] showed that the cohomology groups of $\mathcal{M}_{\mathcal{A}}$ can be computed from the homology groups of the open lower intervals of $L(\mathcal{A})$:

Theorem 0.1.1 (The Goresky-MacPherson formula). *For all i,*

$$\widetilde{H}^i(\mathcal{M}_{\mathcal{A}};\mathbb{Z}) \cong \bigoplus_{x \in L(\mathcal{A})\setminus\{\hat{0}\}} \widetilde{H}_{codim_{\mathbb{R}}(x)-i-2}(\Delta((\hat{0},x));\mathbb{Z})$$

In order to apply Theorem 0.1.1, one clearly has to get a grip on the homology of the order complexes of various posets, namely the open lower intervals in the intersection lattice. If this lattice is well-behaved enough, it may be worthwhile to try one of the various *lexicographic shellability* techniques. We now describe the most commonly used of them, namely *EL-shellability*. It was introduced in [6] for graded posets and generalized to non-graded posets in [16].

A *bounded* poset is one equipped with $\hat{0}$ and $\hat{1}$. We also use the notation $\overline{P} = P \setminus \{\hat{0},\hat{1}\}$ to denote the *proper part* of P if P is bounded. Similarly, if P is any poset, then $\hat{P}$ denotes the bounded poset obtained by adjoining a top and a bottom element, $\hat{0}$ and $\hat{1}$, to P.

We write $x \to y$ if y covers x in P, and let $R(P)$ denote the covering relation, which can be thought of as the set of edges in the Hasse diagram of P. Therefore, a map $\lambda : R(P) \to \Lambda$ is called an *edge labelling* of P. Here, Λ is a poset whose elements we think of as labels.

Given an edge labelling λ, a saturated chain $x = (x_0 \to x_1 \to \cdots \to x_k)$ is *rising* if $\lambda(x_{i-1} \to x_i) < \lambda(x_i \to x_{i+1})$ for all $i \in [k-1]$. It is *falling* if, instead, $\lambda(x_{i-1} \to x_i) \not< \lambda(x_i \to x_{i+1})$ for all i. Letting $\lambda(x)$ denote the sequence $(\lambda(x_0 \to x_1), \ldots, \lambda(x_{k-1} \to x_k))$, we compare two saturated chains x and y with common endpoints using the lexicographic partial order induced by Λ on the sequences $\lambda(x)$ and $\lambda(y)$.

Definition 0.1.2. *Suppose that P is bounded. An edge labelling $\lambda : R(P) \to \Lambda$ is called an* EL-labelling *if the following two conditions hold for all pairs $p < q$ in P.*

- *There is a unique rising saturated chain from p to q.*
- *The unique rising chain is lexicographically smaller than every other saturated chain from p to q.*

If P has an EL-labelling, then P is called EL-shellable.

If P is EL-shellable, then so is, clearly, every interval in P. The following theorem thus provides the information needed to apply the Goresky-MacPherson formula if the intersection lattice is EL-shellable.

Theorem 0.1.3 (see [6, 16]). *If $\hat{P}$ is EL-shellable, then $\Delta(P)$ is homotopy equivalent to a wedge of spheres. The number of i-dimensional spheres in the wedge equals the number of maximal chains in $\hat{P}$ that have exactly $i+3$ elements.*

In [42], Haiman introduced a class of subspace arrangements, called *polygraph arrangements*, whose algebraic properties he exploited in his proof of the celebrated $n!$ theorem. In Paper 1, we apply the Goresky-MacPherson formula to compute the cohomology of the complement of these and some more general arrangements. We achieve this by showing that their intersection lattices are EL-shellable.

0.1.2 Boolean cell complexes

To a regular CW complex Δ, we associate the *face poset* $P(\Delta)$. It consists of the faces of Δ ordered by inclusion. If Δ is simplicial, then every interval in $P(\Delta)$ is a boolean algebra. In fact, a poset, equipped with $\hat{0}$, whose every interval is a boolean algebra is called a *simplicial poset*. The class of complexes with simplicial posets as face posets is, however, a little broader; such complexes are called *cell complexes of boolean type*, or *boolean cell complexes* for short, and were introduced by Björner [8] and by Garsia and Stanton [36]. Note that in a boolean cell complex, several cells may share the same set of vertices.

An interesting way to construct boolean cell complexes is the following. Suppose that G is a group which acts order-preservingly on a poset P. Then, we can construct the quotient complex $\Delta(P)/G$. It is a boolean cell complex whose cells are the G-orbits of faces in $\Delta(P)$. One reason to be interested in this complex is its connection to representation theory. More precisely, it follows e.g. from Bredon [19], that the multiplicity of the trivial character in the induced G-representation on the ith homology group of $\Delta(P)$ equals the rank of the ith homology group of $\Delta(P)/G$ (unless we have coefficients in a field of "bad" positive characteristic).

Kozlov [56] studied the quotient construction for the case of the symmetric group S_n acting on the (proper part of the) partition lattice Π_n as well as on more general lattices. He showed that the resulting quotient complex is collapsible. In Coxeter group terminology, this quotient complex can be thought of as a type A construction. In Paper 2, we obtain B- and D-analogues of Kozlov's results. In the process, we describe and use a new lexicographic shellability technique for a class of boolean cell complexes. It generalizes the non-pure CL-shellability of Björner

and Wachs [16] beyond posets. It is very closely related to the technique for pure boolean cell complexes developed by Hersh [45].

0.1.3 Graph complexes

So far, the "combinatoriality" of the combinatorial complexes that we have considered has been inherited from some underlying poset. A different class of constructions has to do with graph properties. A property of (directed or undirected) graphs is called *monotone* if it is preserved under deletion of edges. For instance, the property "forest" is monotone, whereas the property "tree" is not. Given $n \in \mathbb{N}$ and a monotone graph property P, we may define a simplicial complex Δ_n^P. Its simplices are the edge sets of graphs on vertex set $[n]$ that satisfy P. We refer to Δ_n^P as *the complex of graphs satisfying* P.

In a recent upsurge of activity in this area, this construction has been studied for numerous properties P. Examples include complexes of not i-connected graphs [2, 53, 74], i-colorable graphs [58], directed forests [18, 55], matchings [75], graphs of bounded degree [24, 75], and so on.

Fixing P and a graph G, a more general construction than Δ_n^P is obtained by letting Δ_G^P be the complex of all subgraphs of G that satisfy P. In Paper 3, we study directed subgraph complexes of this type using the properties acyclicity and strong non-connectivity. Björner and Welker [18] did this in the case of all graphs, and basically we adapt their techniques in order to generalize their results.

0.2 Bruhat intervals

The *Bruhat order* is a partial order defined on any Coxeter group. It arises in a multitude of contexts; for example, it records the incidences between closed Schubert cells for a reductive algebraic group over an algebraically closed field. It may be defined, though, in a highly combinatorial manner.

Let (W, S) be a Coxeter system with reflection set T and length function ℓ.

Definition 0.2.1. *The* Bruhat order *on W is the partial order defined by*

$$u \leq w \text{ iff there exist } t_1, \ldots, t_j \in T \text{ such that } w = ut_1 \ldots t_j \\ \text{and } \ell(ut_1 \ldots t_i) < \ell(ut_1 \ldots t_{i+1}) \text{ for all } i.$$

Although not obvious from Definition 0.2.1, the Bruhat order is a graded poset with ℓ as rank function. It is well-known (see e.g. [12]) that there are infinitely many ranked posets of length 3 that appear as intervals in Bruhat orders on infinite Coxeter groups. In [7], on the other hand, Björner asked if only finitely many posets of each fixed length are Bruhat intervals in finite Coxeter groups. Dyer [28] answered Björner's question affirmatively. Thus, the following problem is natural: "*List the posets, of each fixed length, that are Bruhat intervals in finite Coxeter groups.*". The answers for lengths 2 and 3 follow e.g. from [15] together with any

table of Kazhdan-Lusztig polynomials, for instance [38]. In length 2, the "diamond-shaped" poset is the only possibility, whereas in length 3, the face posets of k-gons, $k \in \{2, 3, 4, 5\}$, are the four available options (see Figure 4.3).

In Paper 4, we list the Bruhat intervals of length 4 that appear in Weyl groups, and in a complement to Paper 4 we list the Bruhat intervals of length 5 that appear in symmetric groups. Our method is to find the intervals that occur in Weyl groups of rank at most 4 through a combination of computer calculations and theoretical arguments. We also do the same thing using an independent, completely computerized, approach, which we (in the aforementioned complement) also apply to the symmetric group A_5. This suffices since, by a result of Dyer [28] and our Lemma 4.2.6, if a poset is a Bruhat interval of length i in a particular class of Coxeter groups, then it must occur already in a group of rank at most i in the same class. The applicable classes include finite Coxeter groups, Weyl groups, simply-laced Weyl groups and direct products of symmetric groups.

0.3 Expected reflection distances

Consider the Cayley graph of the symmetric group S_n with respect to the generating set of transpositions. A random walker starts walking from the identity. At each step, he moves along one of the edges incident to his current position, randomly chosen with equal probability. How far away from the identity do we expect him to be after t steps?

One motivation to study this problem comes from computational biology. One model for evolution among biologists is that genomes are subject to *reversals*, mutations that modify genomes in a prescribed fashion. Thus, applying random reversals to a genome is thought to simulate evolution. As we justify in Paper 5, a random walk in the Cayley graph described above is for many purposes a good approximation.

Together with Niklas Eriksen, we present a solution to our problem in the paper cited above. In other words, we give a formula for the expected transposition distance of a product of t random transpositions in S_n. It turns out that, using results of Ito [51] and some manipulations, our problem is equivalent to finding the explicit decomposition of the class function ℓ (which sends an integer partition to its number of parts) into irreducible S_n-characters. Generously aided by Richard Stanley, we apply tools from the theory of symmetric functions to determine this decomposition.

The symmetric group S_n is the $r = 1$ member of the family $G(r, 1, n)$ of complex reflection groups. (Another way to view $G(r, 1, n)$ is as the wreath product $\mathbb{Z}_r \wr S_n$, where $\mathbb{Z}_r$ is the cyclic group on r elements.) The transpositions in S_n are then replaced by the (complex) reflections in $G(r, 1, n)$. Again together with Niklas Eriksen, we generalize the mathematical results of Paper 5 from S_n to $G(r, 1, n)$ in Paper 6. In particular, when $r = 2$, we provide B-analogues. Our approach is

basically the same as in Paper 5, although the representation theory of $G(r,1,n)$, when $r > 1$, provides some additional obstacles.

0.4 Overview of the papers

Here, we give a brief overview of the six papers that constitute this thesis.

1. **Polygraph arrangements.**
   (Based on [47], published in *European Journal of Combinatorics*.)

   A class of subspace arrangements, $Z(n,m)$, known as polygraph arrangements was exploited by Haiman in order to prove the $n!$ theorem. By showing that their intersection lattices, $L(Z(n,m))$, are EL-shellable, we determine the cohomology groups of the complements of the arrangements. Moreover, we generalize the shellability results to a class of lattices which deserve to be called Dowling generalizations of $L(Z(n,m))$. As a consequence, we obtain the cohomology groups of the complements of certain Dowling analogues of polygraph arrangements.

2. **Quotient complexes and lexicographic shellability.**
   (Based on [46], published in *Journal of Algebraic Combinatorics*.)

   Let $\Pi_{n,k,k}$ and $\Pi_{n,k,h}$, $h < k$, denote the intersection lattices of the k-equal subspace arrangement of type $\mathcal{D}_n$ and the k,h-equal subspace arrangement of type $\mathcal{B}_n$ respectively. Denote by S_n^B the group of signed permutations. We show that $\Delta(\Pi_{n,k,k})/S_n^B$ is collapsible. For $\Delta(\Pi_{n,k,h})/S_n^B$, $h < k$, we show the following. If $n \equiv 0 \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $\frac{2n}{k}-2$. If $n \equiv h \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k}-1$. Otherwise, it is contractible. Immediate consequences for the multiplicity of the trivial characters in the representations of S_n^B on the homology groups of $\Delta(\Pi_{n,k,k})$ and $\Delta(\Pi_{n,k,h})$ are stated.

   The collapsibility of $\Delta(\Pi_{n,k,k})/S_n^B$ is established using a discrete Morse function. The same method is used to show that $\Delta(\Pi_{n,k,h})/S_n^B$, $h < k$, is homotopy equivalent to a certain subcomplex. The homotopy type of this subcomplex is calculated by showing that it is shellable. To do this, we are led to introduce a lexicographic shelling condition for balanced cell complexes of boolean type. This extends to the non-pure case work of Hersh [45] and specializes to the CL-shellability of Björner and Wachs [16] when the cell complex is an order complex of a poset.

3. **Directed subgraph complexes.**
   (Based on [48], unpublished.)

Let G be a directed graph, and let Δ_G^{ACY} be the simplicial complex whose simplices are the edge sets of acyclic subgraphs of G. Similarly, we define Δ_G^{NSC} to be the simplicial complex with the edge sets of not strongly connected subgraphs of G as simplices. We show that Δ_G^{ACY} is homotopy equivalent to the $(n-1-k)$-dimensional sphere if G is a disjoint union of k strongly connected graphs. Otherwise, it is contractible. If G belongs to a certain class of graphs, the homotopy type of Δ_G^{NSC} is shown to be a wedge of $(2n-4)$-dimensional spheres. The number of spheres can easily be read off the chromatic polynomial of a certain associated undirected graph.

4. **Bruhat intervals of length 4 in Weyl groups.**
(Based on [49], published in *Journal of Combinatorial Theory, Series A.*)

   We determine all isomorphism classes of intervals of length 4 in the Bruhat order on the Weyl groups A_4, B_4, D_4 and F_4. It turns out that there are 24 of them (some of which are dual to each other). Work of Dyer allows us to conclude that these are the only intervals of length 4 that can occur in the Bruhat order on any Weyl group. We also determine the intervals that arise already in the smaller classes of simply-laced Weyl groups and symmetric groups.

   Our method combines theoretical arguments and computer calculations. We also present an independent, completely computerized, approach.

   Complement. (Not included in [49].) The completely computerized approach is applied to find all intervals of length 5 in A_5. The 25 posets that are found are the only intervals of length 5 that appear in Bruhat orders on symmetric groups.

5. **Estimating the expected reversal distance after t random reversals.**
(With N. Eriksen, based on [31], accepted for publication in *Advances in Applied Mathematics.*)

   We address the problem of computing the expected reversal distance of a genome with n genes obtained by applying t random reversals to the identity genome. A good approximation is the expected transposition distance of a product of t random transpositions in S_n. Computing the latter turns out to be equivalent to computing the coefficients of the length function (i.e. the class function returning the number of parts in an integer partition) when written as a linear combination of the irreducible characters of S_n. Using symmetric functions theory, we compute these coefficients, thus obtaining a formula for the expected transposition distance. We also briefly sketch how to compute the variance.

6. **Expected reflection distance in $G(r,1,n)$ after t random reflections.** (With N. Eriksen, based on [32], unpublished.)

   Extending to $r > 1$ the formula for the expected transposition distance mentioned above, we compute the expected reflection distance of a product of t random reflections in the complex reflection group $G(r,1,n)$. The result relies on an explicit decomposition of the reflection distance function into irreducible $G(r,1,n)$-characters and on the eigenvalues of certain adjacency matrices.

Part I

Combinatorial complexes

Paper 1.

Polygraph arrangements

1.1 Introduction

Macdonald [59] introduced a family of polynomials known as *Macdonald polynomials*. They constitute a basis of the algebra of symmetric functions in the variables $x_1, x_2, \ldots$ with coefficients in the field of fractions of $\mathbb{Q}[y, z]$. Transformation to the basis of Schur functions gives rise to transition coefficients that are called *Kostka-Macdonald coefficients*. Until recently, the conjecture that the Kostka-Macdonald coefficients in fact are polynomials in y and z with nonnegative integer coefficients was open. This conjecture was known as the *Macdonald positivity conjecture*.

Garsia and Haiman [35] conjectured that the Kostka-Macdonald coefficients are multiplicities of graded characters of certain S_n-modules. An equivalent (see Haiman [42]) formulation of this has become known as the $n!$ *conjecture*, since it asserts that the said modules are of dimension $n!$. This implies the positivity conjecture.

Recently, Haiman [41] proved the $n!$ conjecture. The proof relies on the fact that a class of subspace arrangements in $(\mathbb{C}^2)^{n+m}$, called *polygraph arrangements*, have coordinate rings that are free modules over the polynomial ring in one coordinate set on $(\mathbb{C}^2)^n$.

In this paper, we show that certain lattices, which deserve to be called Dowling generalizations of the intersection lattices of the polygraph arrangements, are EL-shellable. Via the Goresky-MacPherson formula, this allows us to determine the cohomology groups of the complements of the polygraph arrangements as well as of Dowling analogues of these arrangements. In particular, it turns out that the cohomology is free and vanishing in "most" dimensions.

The structure of this paper is as follows. After briefly reviewing basic definitions and tools in Section 1.2, we deal with the case of ordinary polygraph arrangements in Sections 1.3 and 1.4. In Section 1.5, we give Dowling generalizations of the results in Section 1.4.

The proofs of Section 1.5 certainly specialize to proofs of the theorems in Section 1.4. However, they do not quite specialize to the proofs given in Section 1.4; the latter are simpler and more transparent. This is the reason why we treat ordinary polygraph arrangements and their Dowling generalizations separately.

Acknowledgement. The author is grateful to his advisor Anders Björner who suggested the study of polygraph arrangements. Moreover, his careful reading and remarks have led to substantial improvements in the paper.

1.2 Tools for investigation of subspace arrangements

We give a brief survey of the techniques that are used in this paper. For basic combinatorial and topological concepts, the reader is referred to the textbooks by Stanley [68] and Munkres [62]. For more on subspace arrangements, see e.g. Björner's survey article [9].

A *subspace arrangement* is a collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of affine subspaces of $\mathbf{k}^m$, where $\mathbf{k}$ is some field. In case $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}\}$, one is often interested in the topological features of the complement $\mathcal{M}_{\mathcal{A}} := \mathbf{k}^m \setminus (\bigcup_{i=1}^n A_i)$.

1.2.1 The Goresky-MacPherson formula

To any poset P, we associate the *order complex* $\Delta(P)$. This is the simplicial complex having the chains of P as simplices. If P has a bottom and/or a top element, then the symbols $\hat{0}$ and $\hat{1}$ will be used to denote them, respectively. The *proper part* $\overline{P}$ is the poset $P \setminus \{\hat{0}, \hat{1}\}$.

The *intersection semi-lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is the meet semi-lattice of all nonempty intersections of subsets of $\mathcal{A}$ ordered by reverse inclusion. It is a lattice iff $\bigcap_{i=1}^n A_i \neq \emptyset$. In case $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}\}$, the following result of Goresky and MacPherson [39] relates the reduced cohomology groups of $\mathcal{M}_{\mathcal{A}}$ and the reduced homology of the lower intervals of $L(\mathcal{A})$:

Theorem 1.2.1. (The Goresky-MacPherson formula) *Let $\mathcal{A}$ be a real subspace arrangement (i.e. $\mathbf{k} = \mathbb{R}$). Then, for all i,*

$$\widetilde{H}^i(\mathcal{M}_{\mathcal{A}}; \mathbb{Z}) \cong \bigoplus_{x \in L(\mathcal{A}) \setminus \{\hat{0}\}} \widetilde{H}_{codim_{\mathbb{R}}(x)-i-2}(\Delta(\overline{[\hat{0}, x]}); \mathbb{Z}).$$

Note that we can apply Theorem 1.2.1 to complex arrangements by identifying $\mathbb{C}$ with $\mathbb{R}^2$. Then $codim_{\mathbb{R}}(\cdot)$ is replaced by $2codim_{\mathbb{C}}(\cdot)$.

1.2.2 Lexicographic shellings

If we are interested in the cohomology of $\mathcal{M}_{\mathcal{A}}$, then Theorem 1.2.1 leaves us with the task of determining the homology of $L(\mathcal{A})$ and its lower intervals. To this end, the technique of EL-shellability described below will be useful to us. It was introduced for ranked posets by Björner [6] and later extended to arbitrary posets by Björner and Wachs [16].

For a ranked poset P, let $\hat{P}$ denote the poset obtained by adding an additional top element $\hat{1}$ and an additional bottom element $\hat{0}$ to P. Let $R(\hat{P}) \subset \hat{P}^2$ denote the covering relation of $\hat{P}$. We write $x \to y$ if y covers x. An *edge-labelling* of P is a map $\lambda : R(\hat{P}) \to \Lambda$, where Λ is some poset of labels. A saturated chain $c = \{c_1 \to \ldots \to c_t\} \subseteq \hat{P}$ is *rising* if $\lambda(c_1 \to c_2) < \cdots < \lambda(c_{t-1} \to c_t)$. The chain c is *falling* if, instead, $\lambda(c_1 \to c_2) \not< \cdots \not< \lambda(c_{t-1} \to c_t)$. Given an interval $[x, y] \subseteq \hat{P}$, we compare two saturated chains $c = \{x = c_1 \to \ldots \to c_t = y\}$ and $d = \{x = d_1 \to \ldots \to d_t = y\}$ using the lexicographic order induced by Λ on the sequences $\lambda(c_1 \to c_2), \ldots, \lambda(c_{t-1} \to c_t)$ and $\lambda(d_1 \to d_2), \ldots, \lambda(d_{t-1} \to d_t)$.

Definition 1.2.2. $\lambda : R(\hat{P}) \to \Lambda$ *is an* EL-labelling *of P if every $\hat{P}$-interval contains a unique rising saturated chain and this chain is lexicographically smaller than every other saturated chain in the interval. If P admits an EL-labelling, then P is called* EL-shellable.

Clearly, if P is EL-shellable, then so is every interval of P. Moreover, the homotopy type of $\Delta(P)$ can be read off the labelling:

Theorem 1.2.3. *Suppose P is ranked of length i. If P has an EL-labelling, then $\Delta(P)$ is homotopy equivalent to a wedge of $(i-2)$-dimensional spheres. The spheres in the wedge are indexed by the falling maximal chains.*

1.3 Polygraph arrangements

Now we describe our main objects of study. Let V be a d-dimensional vector space over $\mathbf{k}$. For $m, n \in \mathbb{N}$ and a function $f : [m] \to [n]$, we let

$$W_f = \{(x_{f(1)}, \ldots, x_{f(m)}, x_1, \ldots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}.$$

This is a linear subspace of V^{m+n}. The *polygraph arrangement*, $Z_V(n, m)$, is the collection of all such subspaces:

$$Z_V(n, m) := \{W_f \mid f : [m] \to [n]\}.$$

Often, the choice of V (and $\mathbf{k}$) is not important. Therefore, we frequently write $Z(n, m)$ instead of $Z_V(n, m)$. To avoid confusion, we mention that Haiman [41] lets $Z(n, m)$ denote the union, not just the collection, of all W_f.

Note that $Z_V(n, m)$ is an arrangement of n^m subspaces of dimension nd in the $(m + n)d$-dimensional vector space V^{m+n}. The dimensions of all intersections of such subspaces are divisible by d.

We need a combinatorial description of the intersection lattice $L(Z(n,m))$. From now on, we let $P = \{p_1 < \cdots < p_m\}$ and $Q = \{q_1 < \cdots < q_n\}$ be fixed disjoint ordered sets. For a subset $S \subseteq P \cup Q$, we use the notation $S^P := S \cap P$ and $S^Q := S \cap Q$, so that $S = S^P \cup S^Q$. Consider the following lattice, which is a join-subsemilattice of the partition lattice $\Pi_{P\cup Q}$.

$$L(Q,P) := \{\pi_1|\ldots|\pi_t \in \Pi_{P\cup Q} \mid \pi_i^P = \emptyset \Rightarrow |\pi_i^Q| = 1 \text{ and } \pi_i^P \neq \emptyset \Rightarrow \pi_i^Q \neq \emptyset\ \forall i \in [t]\} \cup \{\hat{0}\}.$$

Proposition 1.3.1. *$L(Z(n,m))$ and $L(Q,P)$ are isomorphic.*

Proof. Pick a subset $F \subseteq \{f : [m] \to [n]\}$. The element $\bigcap_{f\in F} W_f \in L(Z(n,m))$ can be represented by a bipartite graph $G_F = (P \cup Q, E)$, where $\{p_i, q_j\} \in E$ iff $f(i) = j$ for some $f \in F$. Two such graphs represent the same subspace of V^{m+n} precisely if they have the same connected components. Clearly, all bipartite graphs on $P \cup Q$ in which $deg(p_i) \geq 1$ for all $i \in [m]$ occur in this way. Thus, $L(Z(n,m))$ is isomorphic to the lattice of partitions of $P \cup Q$ whose non-minimal elements correspond to connected components in bipartite graphs on $P \cup Q$ with no isolated vertices in P. This is precisely $L(Q,P)$. □

Corollary 1.3.2. *The intersection lattice $L(Z(n,m))$ is ranked of length n.*

1.4 An EL-labelling of $\overline{L(Q,P)}$

We will give an edge-labelling λ of $\overline{L(Q,P)}$. It will turn out to be an EL-labelling. Our poset Λ of labels consists of four different types of labels: A_x-, B_x- and C_x-labels are ordered internally with respect to their indices. The fourth type is the set $[n]^m$ which is ordered lexicographically. The order on the different types is $A < B < [n]^m < C$. More explicitly, we define

$$\Lambda = \{A_2 < \cdots < A_n < B_1 < \cdots < B_n < \underbrace{11\ldots11}_{m} < \underbrace{11\ldots12}_{m} < \cdots < \underbrace{nn\ldots n}_{m} < C_2 < \cdots < C_n\}.$$

The labelling $\lambda : R(L(Q,P)) \to \Lambda$ is defined by:

- $\lambda(\pi \to \tau) = A_x$ if two non-singleton blocks, π_i and π_j, in π are merged in τ and $q_x = max(\pi_i^Q \cup \pi_j^Q)$.
- $\lambda(\pi \to \tau) = B_x$ if a singleton, q_x, and a non-singleton block, π_i, in π are merged in τ and $q_x < max(\pi_i^Q)$.
- $\lambda(\pi \to \tau) = C_x$ if a singleton, q_x, and a non-singleton block, π_i, in π are merged in τ and $q_x > max(\pi_i^Q)$.

- $\lambda(\hat{0} \to \tau) = f(1)f(2)\dots f(m)$ (juxtapositioning) if τ corresponds to the subspace W_f.

Theorem 1.4.1. *The labelling λ is an EL-labelling of $\overline{L(Q,P)}$.*

Proof. Pick an interval $I = [\pi, \tau] \subseteq L(Q,P)$. We must verify that I contains a unique rising chain and that this chain is lexicographically least in I.

Suppose, to begin with, that $\pi \neq \hat{0}$. If there are non-singleton blocks in π which are merged in τ, then any rising chain must begin with the merging of these blocks. This gives rise to A-labels, and for the subscripts of these labels to form a rising sequence, the order in which to merge the blocks is unique. Next, all singletons q_i that are to be merged with non-singleton blocks containing some $q_j > q_i$ must be so. This gives rise to B-labels, and again there is a unique way to make their subscripts form a rising sequence. Finally, the remaining singletons that are larger than all Q-elements in their blocks in τ are to be merged, this giving rise to C-labels. Once again, there is a unique order in which to do this within a rising chain. Thus, we have constructed the unique rising chain in I. Note that if we had replaced the word *rising* with the phrase *lexicographically least*, then the above construction would give us the unique lexicographically least chain. Hence, it coincides with the rising chain.

Now, suppose that $\pi = \hat{0}$. Note that I contains exactly one atom a_f, corresponding to the subspace W_f, with the property that $[a_f, \tau]$ contains a chain with only C-labels. The function f sends $i \in [m]$ to the least $j \in [n]$ such that p_i and q_j are in the same block in τ. As before, the C-labels occur with rising subscripts in exactly one chain in $[a_f, \tau]$. Note that $\lambda(\hat{0} \to a_f) < \lambda(\hat{0} \to a)$ for all atoms $a \in I \setminus \{a_f\}$. Hence, the rising chain is again lexicographically least in I. □

Remark. Let $r = (r_1, \dots, r_m) \in [n]^m$. Haiman [43] has considered the subarrangement $Z(n,m,r) \subseteq Z(n,m)$ which consists of those W_f that satisfy $f(i) \leq r_i$ for all $i \in [m]$. It is not difficult to see that, with straightforward modifications, the proof of Theorem 1.4.1 goes through for the appropriate subsemilattice $L(Q,P,r)$ of $L(Q,P)$.

Theorem 1.2.3 tells us that $\Delta(\overline{L(Q,P)})$ is homotopy equivalent to a wedge of spheres in top dimension, the spheres being indexed by the falling chains in $L(Q,P)$ under the labelling λ. In order to calculate the number of spheres in the wedge, we define an easily counted set of combinatorial objects which is in 1-1 correspondence with the set of falling chains in $L(Q,P)$.

Consider the set $C(Q,P)$ of ordered partitions $(\pi_1, \dots, \pi_k)$ of $P \cup Q$ such that $q_n \in \pi_1$, $\pi_i^P \neq \emptyset$ and $\pi_i^Q \neq \emptyset$ for all $i \in [k]$, k arbitrary. Define $\Gamma(n,m) := |C(Q,P)|$. Clearly,

$$\Gamma(n,m) = \sum_{k=1}^{min(n,m)} S(n,k)S(m,k)k!(k-1)!,$$

where the $S(i,j)$ are Stirling numbers of the second kind.

Now we establish the bijection mentioned above.

Theorem 1.4.2. *$\Delta(\overline{L(Q,P)})$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres. The number of spheres in the wedge is*

$$\sum_{k=1}^{min(n,m)} S(n,k)S(m,k)k!(k-1)!.$$

Proof. We construct a bijection $\phi : \{\text{falling chains in } L(Q,P)\} \to C(Q,P)$ as follows. Let $c = \{\hat{0} \to c_1 \to \ldots \to c_n = \hat{1}\} \subseteq L(Q,P)$ be a falling chain. Then, for some j, all c_i, $i < j$, contain singleton blocks whereas all c_i, $i \geq j$, do not. The blocks in c_j are the blocks in $\phi(c)$. Let π_1 be the block in c_j which contains q_n. Since c is falling, c_{j+1} is obtained by merging π_1 with some other block which we call π_2. Then c_{j+2} is obtained by merging $\pi_1 \cup \pi_2$ with a block which we denote π_3 and so on, until finally $\hat{1}$ is obtained from c_{n-1} by merging $\pi_1 \cup \cdots \cup \pi_{n-j}$ with the only other block, which is then given the name π_{n-j+1}. We define $\phi(c) := (\pi_1, \ldots, \pi_{n-j+1})$.

To check injectivity of ϕ, consider two distinct falling chains $c = \{\hat{0} \to c_1 \to \ldots \to c_n = \hat{1}\}$ and $d = \{\hat{0} \to d_1 \to \ldots \to d_n = \hat{1}\}$ in $L(Q,P)$. Let j be the smallest index such that $c_j \neq d_j$.

If $j = 1$, then c_1 and d_1 correspond to different functions $f_c, f_d : [m] \to [n]$. Note that $\{q_i \in Q \mid i \in f_c([m])\}$ is the set of maximal Q-elements in blocks in $\phi(c)$, since no falling chain possesses C-labels. An analogous statement holds for f_d. Therefore, if $f_c([m]) \neq f_d([m])$, then $\phi(c) \neq \phi(d)$. If, on the other hand, $f_c([m]) = f_d([m])$, then we can pick $i \in [m]$ such that $f_c(i) \neq f_d(i)$ and both $f_c(i)$ and $f_d(i)$ are maximal Q-elements in blocks in both $\phi(c)$ and $\phi(d)$. Therefore, p_i and $q_{f_c(i)}$ belong to the same block in $\phi(c)$ but to different blocks in $\phi(d)$. Hence, $\phi(c) \neq \phi(d)$.

Now suppose $j > 1$. If $\lambda(c_{j-1} \to c_j) = B_x$, for some x, then $\lambda(d_{j-1} \to d_j) = B_x$, too. Since $c_j \neq d_j$, this means that the block containing x in $\phi(c)$ is different from the block containing x in $\phi(d)$. This implies $\phi(c) \neq \phi(d)$.

The only case left is $\lambda(c_{j-1} \to c_j) = \lambda(d_{j-1} \to d_j) = A_n$. This implies that the set of blocks in $\phi(c)$ is equal to the set of blocks in $\phi(d)$. Since $c \neq d$, $\phi(c)$ must differ from $\phi(d)$ in the order of the blocks. Hence $\phi(c) \neq \phi(d)$ in this case too, and ϕ is injective.

To establish surjectivity, choose $\pi = (\pi_1, \ldots, \pi_k) \in C(Q,P)$. We will construct a falling chain $c \subseteq L(Q,P)$ such that $\phi(c) = \pi$. Let $\hat{f} : P \to Q$ be the function mapping all elements in π_i^P to $max(\pi_i^Q)$ for all $i \in [k]$. Define $f : [m] \to [n]$ to be the corresponding function on the indices, i.e. by requiring that $\hat{f}(p_i) = q_{f(i)}$ for all $i \in [m]$. The atom of $L(Q,P)$ corresponding to W_f is c_1. The chain $\{\hat{0} \to c_1 \to \ldots \to c_{n-k+1} = \pi_1|\ldots|\pi_k\}$ is produced by merging the singletons in c_1 one by one with their corresponding non-singleton blocks in the only possible way which ends with $\pi_1|\ldots|\pi_k$ while giving rise to a falling sequence of B-labels. For $l \in [k-1]$, let $c_{n-k+l} = \pi_1 \cup \cdots \cup \pi_l|\pi_{l+1}|\ldots|\pi_k$. Now, $c = \{\hat{0} \to c_1 \to \ldots \to c_{n-1} \to \hat{1}\}$ is mapped to π by ϕ, so ϕ is surjective. □

Note that $\Gamma(n,m) = \Gamma(m,n)$. This implies an unsuspected numerical relationship between the combinatorially very distinct arrangements $Z(n,m)$ and $Z(m,n)$.

The cohomology groups of the complement $\mathcal{M}_{Z_{\mathbb{R}^d}(n,m)}$ are determined by the Goresky-MacPherson formula (Theorem 1.2.1) and the following corollary:

Corollary 1.4.3. *Let* $\pi = \pi_1|\dots|\pi_k \in L(Q,P)$. *Then* $\Delta(\overline{[\hat{0},\pi]})$ *is homotopy equivalent to a wedge of* $(n-1-k)$*-dimensional spheres. The number of spheres in the wedge is* $\prod_{j=1}^{k} \Gamma(|\pi_j^Q|, |\pi_j^P|)$.

Proof. $\overline{[\hat{0},\pi]}$ is ranked, and it is EL-shellable since $\overline{L(Q,P)}$ is. Hence, by Theorem 1.2.3, $\Delta(\overline{[\hat{0},\pi]})$ is homotopy equivalent to a wedge of spheres in top dimension. The number of spheres in the wedge is the absolute value $|\mu(\hat{0},\pi)|$ of the Möbius function. Note that $[\hat{0},\pi] \cong L(\pi_1^Q,\pi_1^P) \times \cdots \times L(\pi_k^Q,\pi_k^P)$. The Möbius function is multiplicative, so $\mu(\hat{0},\pi) = \prod_{j=1}^{k} \mu_j(\hat{0},\pi_j)$, where μ_j is the Möbius function of $L(\pi_j^Q,\pi_j^P)$. The corollary now follows from Theorem 1.4.2. □

In general dimension, the expression for the cohomology of the complement, although determined by Corollary 1.4.3, is not pretty. In the following theorem we restrict ourselves to weaker, readable, information. As before, the complex case is obtained by identifying $\mathbb{C}$ and $\mathbb{R}^2$.

Theorem 1.4.4. *For all* i, $\widetilde{H}^i(\mathcal{M}_{Z_{\mathbb{R}^d}(n,m)};\mathbb{Z})$ *is free. Let* $\widetilde{\beta}^i$ *denote its rank. We have*

1. $\widetilde{\beta}^i = 0$, *unless* $i = d(m-1) + j(d-1)$ *for some* $j \in [n]$.
2. $\widetilde{\beta}^{dm-1} = n^m$, *if* $d \geq 2$.
3. $\widetilde{\beta}^{d(m+n-1)-n} = \Gamma(n,m)$, *if* $d \geq 2$.

Proof. Each cohomology group is free since, by Theorem 1.2.3 and the Goresky-MacPherson formula, it is the direct sum of free groups.

For $\pi \in L(P,Q)$, let $codim(\pi)$ denote the real codimension of the corresponding element in the intersection lattice of $Z_{\mathbb{R}^d}(n,m)$. Then $codim(\pi) = d(m+n-b(\pi))$, where $b(\cdot)$ denotes the number of blocks. Since $\Delta(\overline{[\hat{0},\pi]})$ is homotopy equivalent to a wedge of $(n-1-b(\pi))$-spheres, π gives a contribution to $\widetilde{\beta}^i$ only if $codim(\pi)-i-2 = n-1-b(\pi)$, i.e. if $i = (d-1)(n-b(\pi)+1) + d(m-1)$. Hence,

$$\widetilde{\beta}^i = \sum_{\pi_1|\dots|\pi_{n-j+1} \in L(Q,P)} \prod_{k=1}^{n-j+1} \Gamma(|\pi_i^Q|, |\pi_i^P|),$$

if $i = j(d-1) + d(m-1)$ for $j \in [n]$. Otherwise, $\widetilde{\beta}^i = 0$. This shows the first assertion. For the second and third, let $j = 1$ and $j = n$, respectively. □

Remark. Unlike its complement, the union $\cup\mathcal{A}$ of an arrangement of linear subspaces is topologically not very exciting; it is a cone with apex in the origin. A more interesting object is the *link*, $lk(\mathcal{A}) := S^{l-1} \cap (\cup\mathcal{A})$, where l is the dimension of the space in which the arrangement is embedded. From Ziegler and Živaljević [79, Thm. 2.4], it follows that the link of a real linear subspace arrangement with shellable intersection lattice has the homotopy type of a wedge of spheres. In particular, this applies to the polygraph arrangements $Z_{\mathbb{R}^d}(n, m)$.

1.5 A Dowling generalization

1.5.1 Dowling lattices

Let G be a finite group and n a positive integer. G acts on the set $([n] \times G) \cup \{0\}$ by $0g := 0$ and $(i, h)g := (i, hg)$ for $i \in [n]$ and $g, h \in G$. For a subset $S \subseteq ([n] \times G) \cup \{0\}$, we define $Sg := \{xg \mid x \in S\}$. A partition $\pi = \pi_1|\dots|\pi_t$ of $([n] \times G) \cup \{0\}$ is *G-symmetric* if $\pi_i g \in \pi$ for all $g \in G$ and $i \in [t]$. The block π_i is called *g-symmetric*, for $g \in G$, if $\pi_i g = \pi_i$. If the identity element is the only $g \in G$ which makes π_i g-symmetric, then π_i is called *simple*. Note that if π is G-symmetric, then the block containing 0 is necessarily g-symmetric for all $g \in G$.

Definition 1.5.1. *Let G be a finite group. The* Dowling lattice Π_n^G *is the lattice of all G-symmetric partitions π of $([n] \times G) \cup \{0\}$ such that all blocks not containing 0 are simple. The block containing 0 is called the* null block *of π.*

Note that $\Pi_n^{\{e\}} \cong \Pi_{n+1}$. Thus, Dowling lattices constitute a generalization of the partition lattice. They were first introduced by Dowling [26]. Two more special cases are worth mentioning. The lattice $\Pi_n^{\mathbb{Z}_2}$ is isomorphic to the partition lattice of type B, i.e. the intersection lattice of the arrangement of reflecting hyperplanes of the Coxeter group B_n. This is a special case of $\Pi_n^{\mathbb{Z}_r}$, which is isomorphic to the intersection lattice of the *Dowling arrangement*, i.e. the arrangement in $\mathbb{C}^n$ of complex hyperplanes given by the equations $x_i = \zeta^k x_j$ and $x_l = 0$, where $i < j \in [n]$, $k \in [r]$, $l \in [n]$ and ζ is a primitive r:th root of unity.

For obvious reasons, the notation tends to get horrible when dealing with Dowling lattices. We agree on some conventions to simplify it. We write $i^g := (i, g)$ for $i \in [n]$ and $g \in G$. The G-orbit of a simple block in $\pi \in \Pi_n^G$ has cardinality $|G|$ and is of course completely determined by any representative. When we write out π, we therefore often omit all but one (arbitrary) block in every orbit of a simple block. Thus, $\pi = \pi_1|\dots|\pi_t \in \Pi_n^G$ should be interpreted as an element with t G-orbits of blocks; hence with $(t-1)|G|+1$ blocks (since the null block is alone in its orbit). When the G-elements in the superscripts are irrelevant, namely in the null block and in singletons, we omit them, too. For example, we write $02|4|1^0 3^1$ for the element $0(2,0)(2,1)(2,2)|(4,0)|(4,1)|(4,2)|(1,0)(3,1)|(1,1)(3,2)|(1,2)(3,0)$ in $\Pi_4^{\mathbb{Z}_3}$.

We view an element in Π_n^G as a "signed" partition of $[n] \cup \{0\}$, where G is the group of "signs". Sometimes we wish to disregard the signs. Therefore, for

$S \subseteq ([n] \times G) \cup \{0\}$, we define $\overline{S} := \{i \in [n] \mid i^g \in S \text{ for some } g \in G\} \cup S^0$, where $S^0 := \{0\}$ if $0 \in S$ and $S^0 := \emptyset$ otherwise. With this, we can define the *absolute value* $\overline{\pi} \in \Pi_{[n]\cup\{0\}}$ of $\pi = \pi_1|\dots|\pi_t \in \Pi_n^G$ by $\overline{\pi} := \overline{\pi_1}|\dots|\overline{\pi_t}$. If π_i is the null block of π, then we say that $\overline{\pi_i}$ is the null block of $\overline{\pi}$.

1.5.2 Dowling analogues of $L(Q, P)$

Recall that x is a *modular* element in a ranked lattice L if $rank(x) + rank(y) = rank(x\vee y)+rank(x\wedge y)$ for all $y \in L$. Björner [11] observed that the lattice $L(Q, P)$ can be constructed in the following way, which suggests possible generalizations of the results in Section 1.3. Consider the modular element $\pi = p_1|\dots|p_m|Q$ in the partition lattice $\Pi_{P\cup Q}$. Note that the set of complements $Co(\pi) := \{\tau \in \Pi_{P\cup Q} \mid \tau \wedge \pi = \hat{0} \text{ and } \tau \vee \pi = \hat{1}\}$ is precisely the set of atoms in $L(Q, P)$, so that $L(Q, P)$ is the lattice join-generated by $Co(\pi)$.

Now, let G be a finite group and consider the element $\pi = 0Q|p_1|\dots|p_m$ in the Dowling lattice $\Pi_{P\cup Q}^G$ (meaning that we replace $[n]$ with $P \cup Q$ in Definition 1.5.1). By [26, Theorem 4], π is modular. Let $L^G(Q, P)$ be the lattice which is join-generated by $Co(\pi)$. Note that $Co(\pi)$ consists of the elements in which every simple block contains exactly one $Q \times G$-element and the null block contains no $Q \times G$-elements. Therefore, $L^G(Q, P)$ consists of those elements in $\Pi_{P\cup Q}^G$ in which every singleton is either 0 or from $Q \times G$ and every non-singleton intersects $P \times G$. In other words, for $\pi \in \Pi_{P\cup Q}^G$, we have $\pi \in L^G(Q, P)$ iff $\overline{\pi} \in L(Q \cup \{0\}, P)$.

We have $L^{\{e\}}(Q, P) \cong L(Q \cup \{0\}, P)$. The cases $G = \mathbb{Z}_2$ and $G = \mathbb{Z}_r$ are also interesting. As before, let V be a vector space over $\mathbf{k}$, and let r be a positive integer.

Definition 1.5.2. *Suppose that* $char(\mathbf{k}) \neq 2$. *The* polygraph arrangement of type B, $Z_V^B(n, m)$, *is the collection of all subspaces of the form*

$$\{(\tau_1 x_{f(1)}, \dots, \tau_m x_{f(m)}, x_1, \dots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}$$

over all $f : [m] \to [n]$ *and* $(\tau_1, \dots, \tau_m) \in \{-1, 0, 1\}^m$.

Definition 1.5.3. *Suppose that* $\mathbf{k}$ *contains a primitive* r*:th root of unity. The* Dowling polygraph arrangement, $Z_V^r(n, m)$, *is the collection of all subspaces of the form*

$$\{(\tau_1 x_{f(1)}, \dots, \tau_m x_{f(m)}, x_1, \dots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}$$

over all $f : [m] \to [n]$ *and* $(\tau_1, \dots, \tau_m) \in \{0, \zeta, \zeta^2, \dots, \zeta^r = 1\}^m$, *where* ζ *is a primitive* r*th root of unity.*

As before, we frequently suppress the vector space in the subscript. It is clear that $L(Z^B(n, m)) \cong L^{\mathbb{Z}_2}(Q, P)$ and $L(Z^r(n, m)) \cong L^{\mathbb{Z}_r}(Q, P)$.

We will show that $L^G(Q, P)$ is EL-shellable. To this end, we define an edge-labelling $\omega : R(L^G(Q, P)) \to \Omega$. This time, Ω contains labels of six different types: α_x-, β_x-, A_x-, B_x- and C_x-types are ordered internally according to the indices.

The sixth type is the set $([n]\cup\{0\})^m$ which is ordered lexicographically. The order on the types is $\alpha < A < B < ([n]\cup\{0\})^m < \beta < C$. More explicitly, we have

$$\Omega = \{\alpha_1 < \cdots < \alpha_n < A_2 < \cdots < A_n < B_1 < \cdots < B_n < \underbrace{00\ldots00}_{m} <$$
$$\underbrace{00\ldots01}_{m} < \cdots < \underbrace{nn\ldots n}_{m} < \beta_1 < \cdots < \beta_n < C_2 < \cdots < C_n\}$$

To simplify notation, we agree that from now on, the term *block* means a block which is neither a singleton nor a null block. Bearing this in mind, we define ω as follows:

- $\omega(\pi \to \tau) = \alpha_x$ if a block, $\overline{\pi_i}$, and the null block in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x = max(\overline{\pi_i}^Q)$.
- $\omega(\pi \to \tau) = \beta_x$ if a singleton, q_x, and the null block in $\overline{\pi}$ are merged in $\overline{\tau}$.
- $\omega(\pi \to \tau) = A_x$ if two blocks, $\overline{\pi_i}$ and $\overline{\pi_j}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x = max(\overline{\pi_i}^Q \cup \overline{\pi_j}^Q)$.
- $\omega(\pi \to \tau) = B_x$ if a singleton, q_x, and a block, $\overline{\pi_i}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x < max(\overline{\pi_i}^Q)$.
- $\omega(\pi \to \tau) = C_x$ if a singleton, q_x, and a block, $\overline{\pi_i}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x > max(\overline{\pi_i}^Q)$.
- $\omega(\hat{0} \to \tau) = f(1)f(2)\ldots f(m)$ (juxtapositioning), where f is the function $f : [m] \to [n]\cup\{0\}$ which satisfies that $q_{f(i)}$ is the unique element in $Q\cup\{q_0\}$ sharing block (or null block) with p_i in $\overline{\tau}$. (Here, $q_0 := 0$.)

Given an atom $a \in L^G(Q,P)$, we define $f_a : [m] \to [n]\cup\{0\}$ by requiring that $f_a(1)\ldots f_a(m) = \omega(\hat{0} \to a)$.

The reader may wish just to read the statement of the following theorem; its proof is along the same lines as (although it does not specialize to) the proof of Theorem 1.4.1.

Theorem 1.5.4. *ω is an EL-labelling of $\overline{L^G(Q,P)}$.*

Proof. Consider the interval $I = [\pi,\tau] \subseteq L^G(Q,P)$. Suppose, to begin with, that $\pi \neq \hat{0}$. If π contains blocks that are merged with the null block in τ, then they must be merged (in unique order) in the beginning of any increasing chain, thereby producing α-labels. The argument which shows that the next segment of any increasing chain is a unique sequence which produces A- and B-labels can be recycled from the proof of Theorem 1.4.1. We are left with a partition which differs from τ only by containing some singletons q_i^g that are merged with blocks containing only smaller Q-elements (i.e. elements $q_j^{g'}$ with $j < i$) or with the null block in τ. An increasing chain must proceed by merging singletons with the null block in unique

order. This produces β-labels. Thereafter, the singletons that are to be merged with blocks must be so, again in unique order, and this process creates C-labels. As in the proof of Theorem 1.4.1, if we had replaced the word *rising* with the phrase *lexicographically least*, then the above construction would work just as well. Hence, the unique rising chain constructed above coincides with the lexicographically least chain.

If $\pi = \hat{0}$, then there is a unique atom $a \in I$ with the property that a differs from τ only by containing some singletons that are merged with blocks containing only smaller Q-elements or with the null block in τ. Specifically, a is the following atom: its null block contains 0 and precisely those p_i^g, $i \in [m]$, $g \in G$, that are in the null block of τ. If p_i^g is not in the null block, then the unique q_j^h sharing block with p_i^g is determined by $q_j = min\{q \in Q \mid q \text{ and } p_i \text{ share block in } \overline{\tau}\}$. (Note that h is unique, given that $a < \tau$.) Any increasing chain must contain a, and above it was shown that $[a, \tau]$ contains a unique increasing chain, so the same holds for $[\hat{0}, \tau]$. By definition of Ω, $\omega(\hat{0} \to a) < \omega(\hat{0} \to a')$ for all atoms $a' \in I \setminus \{a\}$. Hence, the rising chain in $[\hat{0}, \tau]$ is indeed lexicographically least there. □

As in Section 1.4, we may exploit the EL-labelling ω to calculate the homotopy type of $\Delta(\overline{L^G(Q,P)})$. We divide the problem of counting falling chains into parts that can be conquered separately.

Lemma 1.5.5. *Let R be the set of elements in $L^G(Q,P)$ that contain no non-zero singletons. Fix an atom $a \in L^G(Q,P)$. Define ϕ_a^G to be the number of ω-falling chains $c = \{\hat{0} \to a = c_1 \to \ldots \to c_{t+1}\}$ such that $c_{t+1} \in R$ and $\omega(c_j \to c_{j+1})$ is a B-label for all $j \in [t]$. Then,*

$$\phi_a^G = |G|^{n-|f_a([m])\setminus\{0\}|} \prod_{i \in [n]\setminus f_a([m])} |\{j \in f_a([m]) \mid j > i\}|.$$

Proof. If $a \in R$, then $f_a([m]) \setminus \{0\} = [n]$, and the assertion is clear. Otherwise, a contains some singletons $q_{i_1}^g, \ldots, q_{i_t}^g$ (and the rest of their G-orbits), where $i_j > i_{j+1}$ for $j \in [t-1]$. Let c be as in the statement of the lemma. Since $\omega(c_j \to c_{j+1})$ is a B-label, c_{j+1} must be obtained from c_j by merging $q_{i_j}^g$ with a block $\tau \ni q_{k_j}^{g'}$ for some $k_j > i_j$ (and, consequently, merging all $q_{i_j}^{gh}$ with τh for $h \in G$). There are $|G| \cdot |\{k \in f_a([m]) \mid k > i_j\}|$ such blocks. Therefore, ϕ_a^G is the product of the factors $|G| \cdot |\{k \in f_a([m]) \mid k > i_j\}|$ over all G-orbits of singletons $q_{i_j}^g$ in a. This proves the lemma. □

We define $\psi : 2^{[n]} \to \mathbb{N}$ by $\psi(S) = \prod_{i \in [n]\setminus S} |\{j \in S \mid j > i\}|$, so that $\phi_a^G = |G|^{n-|f_a([m])\setminus\{0\}|}\psi(f_a([m]) \setminus \{0\})$. The Stirling numbers are related to ψ in the following way:

Lemma 1.5.6. *We have:*

$$\sum_{S \in \binom{[n]}{k}} \psi(S) = S(n,k).$$

Proof. $\psi(S)$ counts all partitions $\pi = \pi_1|\dots|\pi_{|S|}$ of $[n]$ with the property that $S = \{max(\pi_i) \mid i \in [|S|]\}$. □

Lemma 1.5.7. *Let $S \subseteq [n]$ be fixed. Let Λ_S be the set of atoms $a \in L^G(Q,P)$ such that $f_a([m]) \setminus \{0\} = S$. Then*

$$|\Lambda_S| = |S|! \sum_{j=0}^{m-|S|} \binom{m}{j} S(m-j,|S|)|G|^{m-j}.$$

Proof. Suppose that $S = \{s_1,\dots,s_k\}$. For any $T \subseteq [m]$, define $P_T := \{p_i \mid i \in T\}$. We construct an atom a which satisfies the condition in the statement of the lemma and the additional condition that the null block of $\overline{a}$ contains exactly j P-elements by first choosing this j-subset of P in one of $\binom{m}{j}$ ways, then choosing the ordered partition $(P_{f_a^{-1}(s_1)},\dots,P_{f_a^{-1}(s_k)})$ of the remaining P-elements in $S(m-j,k)k!$ ways. After assigning a G-element to each of these $(m-j)$ P-elements, a is completely determined, and this assignment can be done in $|G|^{m-j}$ ways. Altogether, we have $\binom{m}{j}S(m-j,k)k!|G|^{m-j}$ atoms from which to pick a. Summing over j yields the desired result. □

Lemma 1.5.8. *Let R_k be the set of elements ρ in $L^G(Q,P)$ such that $\overline{\rho}$ consists of a null block, k blocks and no non-zero singletons. Define $\phi^G_\downarrow(k)$ to be the number of ω-falling chains $c = \{\hat{0} \to c_1 \to \dots \to c_{t+1}\}$ such that $c_{t+1} \in R_k$ and $\omega(c_i \to c_{i+1})$ is a B-label for all $i \in [t]$. Then*

$$\phi^G_\downarrow(k) = S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m+n-j-k}.$$

In particular, this number only depends on k.

Proof. Let c be as in the statement of the lemma. Since all $\overline{c_i}$, $i \in [t]$, have the same number of blocks, k, we obtain

$$\phi^G_\downarrow(k) = \textstyle\sum_{S\in\binom{[n]}{k}} \sum_{a\in\Lambda_S} \phi^G_a =$$

$$\overset{(1)}{=} \textstyle\sum_{S\in\binom{[n]}{k}} \sum_{a\in\Lambda_S} |G|^{n-k}\psi(S) =$$

$$= \textstyle\sum_{S\in\binom{[n]}{k}} |\Lambda_S| \cdot |G|^{n-k}\psi(S) =$$

$$\overset{(2)}{=} |G|^{n-k}S(n,k)|\Lambda_S| =$$

$$\overset{(3)}{=} |G|^{n-k}S(n,k)k! \textstyle\sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m-j}.$$

Here, (1) follows from Lemma 1.5.5, (2) follows from Lemma 1.5.6 and (3) follows from Lemma 1.5.7. □

Lemma 1.5.9. *Suppose $\rho \in R_k$. Then the number of ω-falling chains in $[\rho, \hat{1}]$ is $\phi_\uparrow^G(k)$, where*

$$\phi_\uparrow^G(k) := (1 + |G|)(1 + 2|G|) \dots (1 + (k-1)|G|).$$

In particular, this number only depends on k.

Proof. There is a natural isomorphism between $[\rho, \hat{1}]$ and the Dowling lattice Π_k^G obtained by identifying the blocks of ρ with the set $[k] \times G$ and the null block of ρ with 0. Hence, ω induces an EL-labelling of $\overline{\Pi_k^G}$. As then follows from Dowling's [26] computation of the Möbius function of Π_k^G, $\Delta(\overline{\Pi_k^G})$ is homotopy equivalent to a wedge of $\phi_\uparrow^G(k)$ spheres (of top dimension). Therefore, by Theorem 1.2.3, $\phi_\uparrow(k)$ must be the number of ω-falling chains in $[\rho, \hat{1}]$. □

Now, we are ready to count the falling chains in $L^G(Q, P)$.

Theorem 1.5.10. *$\Delta(\overline{L^G(Q,P)})$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. Let $\Gamma^G(n, m)$ denote the number of spheres in the wedge. Then,*

$$\Gamma^G(n,m) = \sum_{k=1}^{min(n,m)} S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m+n-j-k} \prod_{i=1}^{k-1}(1 + i|G|).$$

Proof. It is clear that the number of falling chains in $L^G(Q, P)$ under ω is

$$\sum_{k=0}^{min(n,m)} \phi_\downarrow^G(k)\phi_\uparrow^G(k).$$

The theorem now follows from Lemma 1.5.8, Lemma 1.5.9 and Theorem 1.2.3. □

Below, let $\alpha = min(n+1, m)$. We check that Theorem 1.5.10 indeed generalizes Theorem 1.4.2. Note that

$$\begin{aligned}
\Gamma^{\{e\}}(n,m) &= \textstyle\sum_{k=1}^{min(n,m)} S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)k! = \\
&= \textstyle\sum_{k=1}^{min(n,m)} S(n,k)S(m+1,k+1)(k!)^2 = \\
&= \textstyle\sum_{k=1}^{\alpha} S(n,k)S(m+1,k+1)(k!)^2 = \\
&= \textstyle\sum_{k=1}^{\alpha} \frac{S(n+1,k)-S(n,k-1)}{k}(S(m,k) + (k+1)S(m,k+1))(k!)^2 = \\
&= \Gamma(n+1,m) - \textstyle\sum_{k=1}^{\alpha} S(n,k-1)S(m,k)k!(k-1)! + \\
&\quad + \textstyle\sum_{k=1}^{\alpha} S(n,k)S(m,k+1)(k+1)!k! =
\end{aligned}$$

$$\stackrel{(*)}{=} \Gamma(n+1,m) - S(n,0)S(m,1)1!0! + S(n,\alpha)S(m,\alpha+1)(\alpha+1)!\alpha! =$$

$$= \Gamma(n+1,m),$$

as required. The identity $(*)$ follows from substituting $j = k-1$ in the first sum on the left hand side.

Corollary 1.5.11. *Pick $\pi \in L^G(Q,P)$. Suppose that $\overline{\pi} = 0\pi_0|\pi_1|\pi_2|\ldots|\pi_t$. Then, $\Delta(\overline{[\hat{0},\pi]})$ is homotopy equivalent to a wedge of top-dimensional spheres. The number of spheres in the wedge is*

$$\Gamma^G(|\pi_0^Q|,|\pi_0^P|)\cdot\prod_{i=1}^{t}\Gamma(|\pi_i^Q|,|\pi_i^P|).$$

Proof. Note that $[\hat{0},\pi] \cong L^G(\pi_0^Q,\pi_0^P)\times L(\pi_1^Q,\pi_1^P)\times\cdots\times L(\pi_t^Q,\pi_t^P)$. The rest of the proof is analogous to the proof of Corollary 1.4.3. □

As in Section 1.4, Corollary 1.5.11 provides the information needed to calculate the cohomology groups of $\mathcal{M}_{Z^r(n,m)}$ using the Goresky-MacPherson formula, thereby obtaining a generalization of Theorem 1.4.4. We omit the details.

Paper 2.

Quotient complexes and lexicographic shellability

2.1 Introduction

Kozlov [56] studied the complex $\Delta(\Pi_n)/S_n$, i.e. the order complex of the partition lattice modulo the symmetric group, and showed that it is collapsible. The partition lattice occurs in a variety of combinatorial subjects. Of interest here is that it is (isomorphic to) the intersection lattice $L(\mathcal{A}_n)$ of the braid arrangement. This is the arrangement of reflecting hyperplanes of a Coxeter group of type A_{n-1} (for Coxeter group terminology, see Humphreys [50]). In fact, Kozlov used a larger collection of subspace arrangements, including the k-equal braid arrangement $\mathcal{A}_{n,k}$. It seems natural to consider complexes originating from other Coxeter groups.

The aim of this paper is to determine the homotopy type of two families of quotient complexes $\Delta(L(\mathcal{H}))/G$, namely when $\mathcal{H} = \mathcal{D}_{n,k}$, the k-equal subspace arrangement of type D_n, and when $\mathcal{H} = \mathcal{B}_{n,k,h}$, the k,h-equal subspace arrangement of type B_n. In particular, the arrangements of reflecting hyperplanes of Coxeter groups of types B_n and D_n are special cases ($\mathcal{B}_{n,2,1}$ and $\mathcal{D}_{n,2}$ respectively). In our case, G will be the group of signed permutations, S_n^B, which has a natural action on these arrangements.

To establish our results we proceed in two steps. First, we apply discrete Morse theory to show that there is a sequence of elementary collapses leading from our original complexes to certain subcomplexes. In the type D case, these subcomplexes are just simplices, and the collapsibility result follows. This is very similar to what Kozlov did in [56]. In the type B case, however, the remaining subcomplexes are more difficult to understand. We determine their homotopy type by proving that they are shellable. To facilitate this, we introduce a lexicographic shellability condition for balanced (pure or non-pure) cell complexes of boolean type. This

technique generalizes to the non-pure case a method which recently was introduced by Hersh [45].

It should be pointed out that the original complexes themselves are in general not shellable; a construction of Hersh [45] can easily be modified to show this. Thus, we provide a model (in the type B case) for how one can use discrete Morse theory in conjunction with lexicographic shellability where it is not clear how to proceed solely by either method.

The material is organized as follows. After reviewing some necessary notation and tools in Section 2.2, we introduce the aforementioned lexicographic shelling condition in Section 2.3. In Section 2.4, we define the complexes we wish to study, and they are described using a combinatorial model in terms of trees in Section 2.5. This model is then used to establish the main results in Section 2.6; we determine the homotopy type of $\Delta(L(\mathcal{D}_{n,k}))/S_n^B$ and $\Delta(L(\mathcal{B}_{n,k,h}))/S_n^B$. Following the beaten track and work of e.g. Babson and Kozlov [3], Hersh [45] and Kozlov [56], we use these results to draw conclusions concerning representations of S_n^B.

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2.2 Basic definitions and notation

In this section we collect basic definitions and agree on notation. For anything not explained here, we refer to the standard textbooks by Stanley [69] (combinatorics) and Munkres [62] (topology).

2.2.1 Shelling cell complexes of boolean type

A *cell complex of boolean type*, or *boolean cell complex* for short, is a regular cell complex whose face poset is a *simplicial poset*, i.e. a poset, equipped with a minimal element, in which every interval is a boolean algebra. Hence, a boolean cell complex is almost a simplicial complex, except that several simplices may share the same vertex set. Cell complexes of boolean type were introduced by Björner [8] and by Garsia and Stanton [36]. Boolean cell complexes and simplicial posets have since received considerable attention e.g. from Stanley [68], Reiner [64], Duval [25] and Hersh [45].

A cell complex is *pure* if all its *facets*, i.e. inclusion-maximal cells, are equidimensional. Björner [8] defined shellability for pure regular cell complexes. The natural translation to non-pure complexes was given by Björner and Wachs [17]. Specializing to boolean cell complexes gives the following definition.

Definition 2.2.1. *An ordering $F_1, \ldots, F_t$ of the facets of a boolean cell complex Δ is a* shelling order *of Δ if $F_j \cap (\cup_{\alpha<j} F_\alpha)$ is pure of codimension 1 in F_j for all $2 \leq j \leq t$. If there exists a shelling order of Δ, then Δ is* shellable.

We think of a shelling order as a way of putting together Δ facet by facet. Therefore we say that F_j *attaches* over $F_j \cap (\cup_{\alpha<j} F_\alpha)$. If Δ is shellable, then it has the homotopy type of a wedge of spheres, the spheres of dimension i being indexed by the i-dimensional facets that attach over their entire boundary. In the pure case, this was proven by Björner [8], and the proof can easily be modified to the non-pure case.

2.2.2 Discrete Morse theory

Let Δ be a regular cell complex. A *matching* on the face poset $P(\Delta)$ is a partition of $P(\Delta)$ into three sets X, Y and Z, such that there exists a bijection $\phi : Y \to X$ with the property that y is covered by $\phi(y)$ for all $y \in Y$. The remaining set Z is called the *critical set* of the matching. The matching is *acyclic* if there exists no sequence $y_1, \ldots, y_q \in Y$ such that $y_q = y_1$, $y_i \neq y_{i+1}$ and $\phi(y_i)$ covers y_{i+1} for all $i \in [q-1]$.

From Forman's work [34], the next result follows. See also Kozlov [56] for a direct combinatorial proof. We formulate the result in terms of matchings rather than discrete Morse functions. The connection between the two points of view is given by Chari [22].

Theorem 2.2.2. *Suppose we have an acyclic matching on $P(\Delta)$ with critical set Z. If Z is a subcomplex of Δ, then Z can be obtained from Δ by a sequence of elementary collapses. In particular, Z and Δ are homotopy equivalent.* □

Remark. We wish to emphasize the requirement of Theorem 2.2.2 that Z be a *subcomplex* of Δ. This ensures that the incidences between the simplices in Z are left unchanged during the collapsing, and this is vital for our applications.

2.2.3 Quotient complexes

Throughout we will assume that all posets we consider are finite. We will not make any notational distinctions between a simplicial complex and its geometric realization. Given a poset P equipped with a maximal element $\hat{1}$ and a minimal element $\hat{0}$, we let $\overline{P}$ denote the *proper part* $P \setminus \{\hat{0}, \hat{1}\}$. The *order complex* $\Delta(P)$ is the simplicial complex having the chains of $\overline{P}$ as simplices. If G is a group acting on P in an order-preserving way, we may define $\Delta(P)/G$ as the boolean cell complex whose simplices are the G-orbits of simplices of $\Delta(P)$. In general, $\Delta(P)/G$ is not a simplicial complex, since there may be more than one simplex on the same set of vertices. Babson and Kozlov [3] give conditions under which $\Delta(P)/G \cong \Delta(P/G)$. Earlier, Welker [77] had given specific examples of posets and groups with this property.

2.3 Lexicographic shellability of balanced boolean cell complexes

A d-dimensional boolean cell complex Δ is *balanced* if there exists a colouring $f : vert(\Delta) \to [d+1] := \{1, \dots, d+1\}$ of the vertices of Δ whose restrictions to all simplices are injective. An order complex $\Delta(P)$ of a poset P is balanced (define $f(v)$ to be the maximal cardinality of a P-chain with v as maximal element). Furthermore, if a group G acts on P order-preservingly, then this balancing is inherited by $\Delta(P)/G$.

In [45], Hersh gave a lexicographic shelling condition for pure balanced boolean cell complexes. In this section we extend her work to the non-pure case.

Consider a simplex c in a balanced boolean cell complex with colouring f. Suppose that $f(vert(c)) = \{f_0 < \cdots < f_r\}$. To shorten notation, we let $c^{i\to j}$, $-1 \leq i < j \leq r+1$, denote the unique simplex contained in c with colours $\{f_0, \dots, f_i, f_j, \dots, f_r\}$. We also use the expressions $c^{i\to} := c^{i\to r+1}$, $c^{i} := c^{i-1\to i+1}$ and $c^{i_1\to j_1,\dots,i_m\to j_m} := (\dots(c^{i_m\to j_m})\dots)^{i_1\to j_1}$.

For a d-dimensional boolean cell complex Δ, let $\hat{\Delta}$ denote the complex whose facets are $\{F \cup \{\hat{0}, \hat{1}\} \mid F \text{ is a facet in } \Delta\}$. This is the *join* of Δ and the one-dimensional simplex on $\{\hat{0}, \hat{1}\}$ (see e.g. Björner [10, Section 9]). If Δ is balanced by $f : vert(\Delta) \to [d+1]$, we extend the balancing to $\hat{\Delta}$ by defining $f(\hat{0}) = 0$ and $f(\hat{1}) = d+2$.

Suppose that $\{F_1, \dots, F_t\}$ is the set of facets in $\hat{\Delta}$. A *root simplex* of $\hat{\Delta}$ is a simplex of the form $F_\alpha^{i\to}$ for some α, $i \geq 1$. (In particular, all facets are root simplices.) Furthermore, a *rooted interval* is determined by a simplex of the form $c = F_\alpha^{i\to j,j\to}$, $i+2 \leq j$. It consists of all minimal root simplices that contain c.

Let $R(\hat{\Delta})$ be the set of root simplices of $\hat{\Delta}$. A *chain labelling* of Δ is a map $\lambda : R(\hat{\Delta}) \to \Lambda$, where Λ is some poset of labels.

Pick a root simplex $c = F_\alpha^{r\to} \in R(\hat{\Delta})$. Given a chain labelling λ, we define the *descent set* of c to be $D(c) := \{i \in [r-1] \mid \lambda(c^{i\to}) \not\leq \lambda(c^{i+1\to})\}$. Consider the rooted interval given by $c^{i\to j,j\to}$ for some $i+2 \leq j < r$. We say that c is *falling* on this interval if $D(c) \supseteq \{i+1, \dots, j-1\}$. If, instead, $D(c) \cap \{i+1, \dots, j-1\} = \emptyset$, then c is *rising* on the interval.

If two distinct root simplices, b_1 and b_2 contain $c^{i\to j,j\to}$, then we compare them on the rooted interval of $c^{i\to j,j\to}$ using the lexicographic order, i.e. $b_1 <_{lex} b_2$ iff $\lambda(b_1^{t\to}) < \lambda(b_2^{t\to})$, where t is the smallest index such that $i < t \leq j$ and $\lambda(b_1^{t\to}) \neq \lambda(b_2^{t\to})$. If no such t exists, then b_1 and b_2 are incomparable on the interval.

Note that the the notions of rising and falling, as well as the lexicographic order, are defined in the context of a rooted interval. When we apply them to facets of $\hat{\Delta}$ without referring to a specific interval, we have the interval determined by $\{\hat{0}, \hat{1}\}$ in mind.

We now give a lexicographic shelling condition for balanced boolean cell complexes. Although stated differently, the most significant difference being in the

formulation of Condition 4 below, it implies the CL-version of [45, Definition 2.4] in the pure case.

Definition 2.3.1. *A balanced boolean cell complex Δ is* CL-shellable *if there exists a chain labelling of Δ such that the following four conditions are fulfilled:*

1. *Every rooted interval contains a unique simplex which is rising on the interval.*

2. *In every rooted interval, the rising simplex is lexicographically smaller than all other simplices.*

In 3 *and* 4*, let $F_1, \ldots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographic order.*

3. *Let c be a maximal simplex in $F_p \cap F_r$, for $p < r$. Write $c = F_r^{s_1 \to t_1, \ldots, s_m \to t_m}$, where $s_i \leq t_i - 2$ and $t_{i-1} \leq s_i$ for all i. (There is a unique way to do this.) Let $j \in [m]$ be maximal such that F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$, $i < j$. Then, for some $q < r$ and $i \leq j$, we have $F_r^{s_i \to t_i} \subseteq F_q \cap F_r$.*

4. *Let c be a maximal simplex in $F_q \cap (\cup_{\alpha<q} F_\alpha)$. Suppose $c = F_q^{s \to t}$ and that F_q is rising on the rooted interval given by $F_q^{s \to t, t \to}$. Then $codim_{F_q}(c) = 1$.*

Remark. If Δ is the (simplicial) order complex of a poset P, then a chain labelling of Δ is just a chain-edge labelling of $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ in the sense of Björner and Wachs [16]. In this case, Conditions 3 and 4 are trivially satisfied and Definition 2.3.1 defines ordinary CL-shellability (see [16, Definition 5.2]) for $\hat{P}$. Condition 3 is satisfied since if two maximal P-chains $c_1 <_{lex} c_2$ differ on several intervals, then one can find a chain $d <_{lex} c_2$ which only differs from c_2 on the first of those intervals (simply by replacing this interval in c_2 with the rising chain). Condition 4 follows since no such c can exist without Condition 1 to be violated.

Theorem 2.3.2. *If a balanced boolean cell complex Δ is CL-shellable, then it is shellable.*

Proof. Adjusted to fit our formulation of Definition 2.3.1, the proof of [45, Theorem 2.1] goes through in the non-pure case, too. We sketch it using our notation. Let the ordering $F_1, \ldots, F_t$ be as in Definition 2.3.1. Condition 3 of Definition 2.3.1 ensures that a maximal simplex c in $F_j \cap (\cup_{\alpha<j} F_\alpha)$ can be written $c = F_j^{l \to m}$. If F_j is rising on the rooted interval of $F_j^{l \to m, m \to}$, then $codim_{F_j}(c) = 1$ by Condition 4. Otherwise, by Conditions 1 and 2, there is a facet preceding F_j which contains $F_j^{p, p+1 \to}$ but not $F_j^{p+1 \to}$ for some $l < p < m$. By Condition 3, $F_j^p \subset F_i$, for some $i < j$. Hence $c = F_j^p$, and we are done. □

The following result is reminiscent of [45, Proposition 2.1]. It will be of use to us later.

Proposition 2.3.3. *Let G be a group acting on the poset P in an order preserving way. Then $\Delta = \Delta(P)/G$ is CL-shellable if and only if it has a chain labelling satisfying Conditions 1 and 2 of Definition 2.3.1 together with the following condition:*

3′. *Let $F_1, \ldots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographic order. Let c be a simplex in $F_r \cap (\cup_{\alpha<r} F_\alpha)$. Write $c = F_r^{s_1 \to t_1, \ldots, s_m \to t_m}$, where $s_i \leq t_i - 2$ and $t_{i-1} \leq s_i$ for all i. (Again, there is a unique way to do this.) Suppose F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$. Then $c \subseteq b \subseteq F_r \cap (\cup_{\alpha<r} F_\alpha)$ for some simplex b with $codim_{F_r}(b) = 1$.*

Proof. The only if direction follows immediately from Theorem 2.3.2 and the definition of shellability.

Now suppose that we have a chain labelling of Δ satisfying Conditions 1, 2 and 3′. Then Condition 4 is immediate. Let c and F_r be as in Condition 3. If F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$, then Condition 3 follows since c is contained in a codimension 1 simplex in $F_r \cap (\cup_{\alpha<r} F_\alpha)$. Otherwise, Condition 3 follows via an argument similar to Hersh's proof of [45, Proposition 2.1]. □

As with simplicial complexes, if a boolean cell complex Δ is shellable, then it is homotopy equivalent to a wedge of spheres. Just as in lexicographic shellings of posets, the falling facets correspond to simplices attached over their entire boundaries. However, unlike the ordinary case, there may exist other simplices that attach over their entire boundaries. Consider, e.g., two facets, F and G, on the same vertex set. Even if, say, $1 \notin D(F) \cup D(G)$, we may still have $F^1 = G^1$. Hence, the last attached of F and G will be attached over the boundary simplex $F^1 = G^1$ even though 1 is not a descent. For an example, see the proof of Theorem 2.6.5 and the illustration in Figure 2.2. This motivates the following definition:

Definition 2.3.4. *Let $F_1, \ldots, F_t$ be an ordering of the facets of Δ which is a linear extension of the lexicographic order. We say that F_j has a* topological descent *at i if $F_j^i \subset \cup_{\alpha<j} F_\alpha$. Otherwise, i is a* topological ascent.

The concepts *topologically falling* and *topologically rising* facets are defined in the obvious way. Our definitions are tailor-made for the following proposition to hold:

Proposition 2.3.5. *If a balanced boolean cell complex Δ is CL-shellable, then it is homotopy equivalent to a wedge of spheres. For all i, its (reduced) Betti numbers satisfy*

$$\widetilde{\beta}_i(\Delta) = \ \#\textit{topologically falling facets on } i+1 \textit{ vertices.}$$

□

Remark. If "rising" is replaced by "topologically rising" in Definition 2.3.1 and Theorem 2.3.2 (and Proposition 2.3.3), one obtains a shellability criterion which in

the pure case is virtually identical to the CC-shellability introduced by Hersh [45]. The reason for the name is that it is modelled after the CC-shellability for posets that was introduced by Kozlov [54]. It has the advantage of making Proposition 2.3.5 less artificial. For our applications, though, CL-shellability is sufficient.

2.4 The objects of study

Throughout the rest of the paper we will frequently encounter the triple (n, k, h). Whenever these integers appear, it will be assumed that $1 \leq h \leq k \leq n$ and that $k \geq 2$ if nothing else is explicitly stated.

Definition 2.4.1. *The* k-equal subspace arrangement of type D_n, $\mathcal{D}_{n,k}$, *is the collection of all linear subspaces of the form*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \tau_1 x_{i_1} = \cdots = \tau_k x_{i_k}\}$$

for $1 \leq i_1 < \cdots < i_k \leq n$ *and* $\tau_i \in \{-1, 1\}$ *for all* i.

Definition 2.4.2. *For* $h < k$, *we define* $\mathcal{B}_{n,k,h}$, the k, h-equal subspace arrangement of type B_n, *to be the union of* $\mathcal{D}_{n,k}$ *and the collection of linear subspaces of the form*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{i_1} = \cdots = x_{i_h} = 0\}$$

for $1 \leq i_1 < \cdots < i_h \leq n$.

These arrangements were introduced by Björner and Sagan [14]. The special cases $\mathcal{B}_n = \mathcal{B}_{n,2,1}$ and $\mathcal{D}_n = \mathcal{D}_{n,2}$ are the ordinary *hyperplane arrangements of types* B_n *and* D_n respectively.

Zaslavsky's work [78] provides a nice description of the intersection lattices $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$ in terms of lattices of signed graphs. We will, however, only briefly consider the structure of these lattices, so we settle for a more naive description of $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$. For more on subspace arrangements and intersection lattices we refer to Björner [9].

Definition 2.4.3. *Let* $\Pi_{n,k,h}$ *be the lattice of set partitions of the set* $\{-1, 1, -2, 2, \ldots, -n, n\}$ *such that the following conditions hold:*

1. *The partitions are sign-symmetric, i.e. if all plus and minus signs are interchanged, then the partition is unchanged.*
2. *There is at most one self-symmetric block, i.e. block containing both* $-a$ *and* a *for some* $a \in [n]$.
3. *The non-singleton non-self-symmetric blocks have size at least* k.
4. *The self-symmetric block has size at least* $2h$, *if it exists.*

With the obvious interpretation $\tau_s x_s = \tau_t x_t$ iff $\tau_s s$ and $\tau_t t$ are in the same block (in particular, $x_s = 0$ iff $\pm s$ belong to the self-symmetric block), we see that $\Pi_{n,k,h}$, $h < k$, is isomorphic to $L(\mathcal{B}_{n,k,h})$ and that $\Pi_{n,k,k}$ is isomorphic to $L(\mathcal{D}_{n,k})$.

Note that "most" of the lattices $\Pi_{n,k,h}$ are not graded. It is straightforward to check that when $k > 2$, $\Pi_{n,k,h}$ is graded if and only if $n < k + h$ and $k \in \{h, h+1\}$. In the hyperplane case $k = 2$, $\Pi_{n,k,h}$ is always graded.

To shorten notation, we introduce $\Delta_{n,k,h} := \Delta(\Pi_{n,k,h})/S_n^B$. These are the objects we will study. Here, S_n^B is the group of signed permutations, which acts in a natural way on $\Pi_{n,k,h}$. It is a Coxeter group of type B_n. For our purposes, it suffices to view S_n^B as the group of permutations π of the set $\{-1, 1, -2, 2, \ldots, -n, n\}$ such that $\pi(a) = -\pi(-a)$ for all $a \in [n]$.

A more thorough analysis of the simplex structure of $\Delta_{n,k,h}$ takes place in Section 2.5. Here we will only describe the vertices, $vert(\Delta_{n,k,h})$, of $\Delta_{n,k,h}$. Given two partitions $\pi, \tau \in \Pi_{n,k,h}$, it is clear that there exists a $g \in S_n^B$ such that $g\tau = \pi$ if and only if there is a bijection between the blocks of τ and the blocks of π which respects block size and commutes with the operation of interchanging all plus and minus signs.

To avoid confusion, the elements of an integer partition will be denoted *parts*, as opposed to the *blocks* of a set partition. Let N_n denote the set of integer partitions of n in which we allow at most one part, the *null* part, to be distinguished. Define the *nullity*, $null(\lambda)$, of λ to be the size of the null part of λ, or zero, if λ has no null part. Order N_n by the rule: $\lambda \leq \kappa$ if λ refines κ as an integer partition and $null(\lambda) \leq null(\kappa)$.

Definition 2.4.4. *We let $N_{n,k,h}$ denote the subposet of N_n induced by the elements λ with $null(\lambda) \notin [h-1]$ in which all non-null parts are either singletons or have size at least k.*

Consider the map $type : vert(\Delta_{n,k,h}) \to \overline{N_{n,k,h}}$ defined as follows. Pick $v \in vert(\Delta_{n,k,h})$ and $\tau_v \in \Pi_{n,k,h}$ such that $orb(\tau_v) = v$, where $orb(\cdot)$ denotes S_n^B-orbit. A size $2s$ self-symmetric block in τ_v gives rise to a size s null part in $type(v)$. The other blocks in τ_v occur in sign-reflected pairs. Each such pair of blocks of size s gives rise to a size s (non-null) part in $type(v)$. For example, with $(n, k, h) = (4, 2, 1)$, we have $type(orb(\{\{-1, 1\}, \{-2, 3\}, \{4\}, \{2, -3\}, \{-4\}\})) = \{\overline{1}, 2, 1\}$, where the bar indicates null part. Obviously, in the light of the discussion above, *type* is well-defined and bijective. Hereafter, we will consider the vertices of $\Delta_{n,k,h}$ to be elements of $N_{n,k,h}$.

2.5 Describing $\Delta_{n,k,h}$ using trees

In this section we give a description of the simplices of $\Delta_{n,k,h}$ in terms of a certain kind of trees. With modifications, we follow Kozlov's [57, Section 4] description of some complexes related to the order complex of the partition lattice modulo the symmetric group.

2.5.1 The trees

In the following, we will suppose that all trees are finite. Given a tree T, let $V(T)$ denote the vertex set of T. For a rooted tree T, let $l_i(T)$ be the number of vertices at distance i from the root. A rooted tree T is called a *graded tree of rank r* if the distance from an arbitrary leaf to the root is $r+1$ and $1 = l_0(T) < l_1(T) < \cdots < l_{r+1}(T)$. In such a tree, the *depth* of a vertex v is the distance from the root to v.

Let $\langle\overline{n}\rangle$ denote the set $\{\overline{0}, \overline{1}, \ldots, \overline{n}\}$. Define $\overline{i} + j := \overline{i+j}$ for integers i and j, and extend the definition by associativity and commutativity. (Sums of the type $\overline{i} + \overline{j}$ are not defined.)

Definition 2.5.1. *An (n,k,h)-tree of rank r is a pair (T,η), where T is a graded tree of rank r and $\eta : V(T) \to \{\overline{0}, \overline{h}, \overline{h+1}, \ldots, \overline{n}\} \cup \{1, k, k+1, \ldots, n\}$ is a labelling of the vertices of T such that*

1. *$\eta(\rho) = \overline{n}$, where ρ is the root of T.*
2. *For all non-leaf vertices $v \in V(T)$, we have $\eta(v) = \sum \eta(w)$ (sum over all children w of v).*
3. *On every depth, there is at least one non-trivial vertex v, i.e. $\eta(v) \notin \{\overline{0}, 1\}$.*

We often abuse notation and write T for (T,η).

Remark. In particular, the second condition implies that if $\eta(v) \in \langle\overline{n}\rangle$ for a non-leaf v, then $\eta(w) \in \langle\overline{n}\rangle$ for exactly one child w of v.

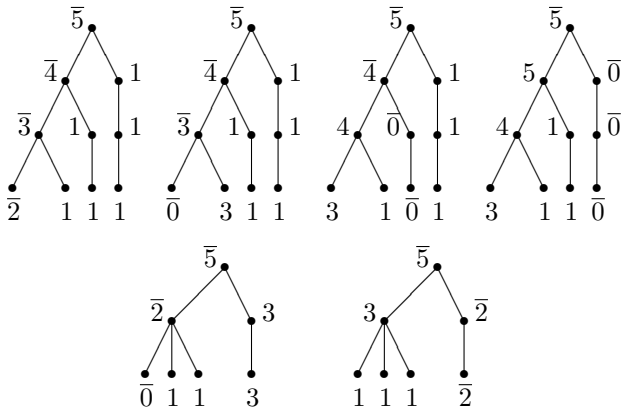


Figure 2.1. *All maximal $(5,3,2)$-trees. The upper ones are of rank 2 and the lower of rank 1.*

Let $T^r_{n,k,h}$ denote the set of (n,k,h)-trees of rank r (we will not distinguish between isomorphic trees), and let $T_{n,k,h}$ be the set of all (n,k,h)-trees so that, in particular, $T_{n,k,h} = \cup_r T^r_{n,k,h}$.

2.5.2 The deletion operator

For an integer i, $0 < i \leq r+1$, we define the deletion operator $\delta^i : T^r_{n,k,h} \to T^{r-1}_{n,k,h}$ by $\delta^i((T,\eta)) := (T^i, \eta^i)$, where T^i is the tree obtained from T by deleting all vertices of depth i and letting the grandchildren of the depth $i-1$ vertices be their new children. The labelling η^i is the restriction of η to $V(T^i)$. For convenience, we introduce $\delta^{i\to j} := \delta^{i+1} \circ \delta^{i+2} \circ \cdots \circ \delta^{j-1}$ for $i < j+1$. We will also use the notation $\delta^{i\to} := \delta^{i+1} \circ \delta^{i+2} \circ \cdots \circ \delta^{r+1}$.

2.5.3 A description of $\Delta_{n,k,h}$

Recall from Section 2.4 the description of the vertices of $\Delta_{n,k,h}$ in terms of the elements of $N_{n,k,h}$. Note that each level, i.e. set of vertices of some fixed depth, of $T \in T_{n,k,h}$ can be viewed as an element $\lambda \in N_{n,k,h}$. The (labelled) vertices of T correspond to parts in λ. A vertex v such that $\eta(v) \in \langle\overline{n}\rangle$ corresponds to a null part. (We interpret $\eta(v) = \overline{0}$ as the non-existence of a null part.)

Let $\Delta^r_{n,k,h}$ be the set of r-dimensional simplices in $\Delta_{n,k,h}$. We describe a mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ as follows:

Take a simplex $c \in \Delta^r_{n,k,h}$ and let the chain $\pi = \{\pi_{r+1} < \pi_r \cdots < \pi_1\} \subseteq \Pi_{n,k,h}$ be a representative of c. We construct $\psi(c)$ level by level. The (labelled) vertices at depth i of $\psi(c)$ are the parts in $orb(\pi_i) \in N_{n,k,h}$. We put an edge between two vertices a and b, of depths i and $i+1$ respectively, if the block in π_{i+1} corresponding to b is contained in the block in π_i corresponding to a. Finally, we add the root $\overline{n}$ and put edges from it to all vertices of depth 1. It follows immediately from the construction that ψ is well-defined, i.e. $\psi(c)$ does not depend on the choice of representative of c.

The following result is analogous to [57, Theorem 4.4], and the same straightforward proof applies with obvious modifications. It allows us to use $T_{n,k,h}$ as a model of $\Delta_{n,k,h}$.

Proposition 2.5.2. *The mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ is bijective. Furthermore, under ψ, inclusion in $\Delta_{n,k,h}$ corresponds to deletion in $T_{n,k,h}$. In other words, for two simplices $c_1, c_2 \in \Delta_{n,k,h}$ we have $c_1 \subseteq c_2$ iff $\psi(c_1) = \delta^{i_1} \circ \cdots \circ \delta^{i_t}(\psi(c_2))$ for some $i_1, \ldots, i_t$.* □

2.6 The homotopy type of $\Delta_{n,k,h}$

In the following we will need two projection maps $\mu, \nu : N_{n,k,h} \to N_{n,k,h}$.

Definition 2.6.1. *Let $\lambda \in N_{n,k,h}$ be given. Recall that $null(\lambda)$ is the size of the null part of λ.*

1. *Define $\mu(\lambda)$ to be the maximal element $\mu \leq \lambda$ such that $null(\mu) = 0$ and all non-singleton parts of μ have size k.*
2. *Define $\nu(\lambda)$ to be the maximal element $\nu \leq \lambda$ such that $null(\nu) = null(\lambda)$ and all non-singleton non-null parts of ν have size k.*
3. *We say that λ is μ-like (ν-like) if λ is fixed by μ (ν).*

For example, with $n = 7$ and $k = 2$, $\mu(\{\overline{3}, 3, 1\}) = \{2, 2, 1, 1, 1\}$, which is a μ-like partition, and $\nu(\{\overline{3}, 3, 1\}) = \{\overline{3}, 2, 1, 1\}$, which is ν-like.

Note that $\lambda \in N_{n,k,h}$ is μ-like iff $null(\lambda) = 0$ and all non-singleton parts of λ have size k. Similarly, λ is ν-like iff all non-singleton non-null parts of λ have size k.

2.6.1 The $\mathcal{D}_{n,k}$ case

Theorem 2.6.2. *The complex $\Delta_{n,k,k}$ is collapsible.*

Proof. The idea is the same as in Kozlov's proof of [56, Theorem 4.1]. We will give an acyclic matching on the face poset $P(\Delta_{n,k,k})$. In view of Proposition 2.5.2, we will consider the elements of $P(\Delta_{n,k,k})$ to be (n, k, k)-trees. Recall from Section 2.5.3 that the levels of a tree $T \in P(\Delta_{n,k,k})$ are viewed as elements of $N_{n,k,k}$. Let $\lambda_i(T) \in N_{n,k,k}$ denote the element obtained from depth i in T.

$P(\Delta_{n,k,k})$ is partitioned into the sets X, Y and Z in the following way. Pick a tree $T \in P(\Delta_{n,k,k})$. If $\lambda_i(T)$ is μ-like for all i, then T belongs to Z. Otherwise, let i be the largest index such that $\lambda_i(T)$ is not μ-like. If $\mu(\lambda_i(T)) = \lambda_{i+1}(T)$, then T belongs to X. If, on the other hand, $\mu(\lambda_i(T)) \neq \lambda_{i+1}(T)$, then T belongs to Y.

Let $T \in Y$ be given, and let i be the largest index such that $\lambda_i(T)$ is not μ-like. It is easily seen that there is a unique way (up to tree-isomorphisms) to insert a new level corresponding to $\mu(\lambda_i(T))$ directly beneath level i in T. The tree $\tilde{T}$ thus obtained covers T (since $\delta^{i+1}(\tilde{T}) = T$) and belongs to X. Conversely, given $\tilde{T} \in X$, we construct $T \in Y$ uniquely by deleting the level below the deepest not μ-like level in $\tilde{T}$. Hence we have a bijection $\tilde{}: Y \to X$ such that $\tilde{T}$ covers T.

It remains to show that our matching is acyclic. To this end, suppose $T_1, T_2 \in Y$, $T_1 \neq T_2$ and that $\tilde{T_1}$ covers T_2. Since $T_2 \neq T_1$, T_2 must be obtained from $\tilde{T_1}$ by deleting some other level than the one below the deepest not μ-like level. This other level must in fact be the deepest not μ-like level; otherwise we would have $T_2 \in X$. Hence the number of μ-like levels in T_2 is one larger than in T_1. Repeating this argument shows the non-existence of a sequence $T_1, \ldots, T_t \in Y$ such that $T_1 = T_t$, $T_i \neq T_{i+1}$ and $\tilde{T_i}$ covers T_{i+1} for $i \in [t-1]$. Hence the matching is acyclic.

Since the property that all levels are μ-like is preserved under deletion of levels, Z is a subcomplex of $\Delta_{n,k,k}$. Theorem 2.2.2 shows that there is a sequence of elementary collapses transforming $\Delta_{n,k,k}$ to Z. This subcomplex consists of the

(n,k,k)-trees in which every level is μ-like. There is obviously exactly one maximal such tree. Hence, the subcomplex given by Z is a simplicial complex generated by one simplex and thus collapsible. □

2.6.2 The $\mathcal{B}_{n,k,h}$ case

When $h < k$, the proof of Theorem 2.6.2 fails. The reason is that for $h \leq i < k$, we have $\mu(\{\bar{i}, 1, 1, \ldots, 1\}) = \{1, 1, \ldots, 1\}$ which is not in the proper part of $N_{n,k,h}$. To overcome this, we will use ν instead of μ. As before, we will use simplices of $\Delta_{n,k,h}$ and their tree representations interchangeably.

Definition 2.6.3. *Let $U_{n,k,h}$ be the subcomplex of $\Delta_{n,k,h}$ consisting of the (n,k,h)-trees in which every level is ν-like.*

Lemma 2.6.4. *The complexes $\Delta_{n,k,h}$ and $U_{n,k,h}$ are homotopy equivalent.*

Proof. Except for the last two sentences, the proof of Theorem 2.6.2 goes through with obvious modifications if we replace μ with ν. □

Theorem 2.6.5. *Let $h < k$. If $n \equiv 0 \pmod{k}$, then $\Delta_{n,k,h}$ is homotopy equivalent to a sphere of dimension $\frac{2n}{k} - 2$. If $n \equiv h \pmod{k}$, then $\Delta_{n,k,h}$ is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k} - 1$. Otherwise, $\Delta_{n,k,h}$ is contractible.*

Proof. In the light of Lemma 2.6.4, we restrict our attention to $U_{n,k,h}$. We let P be the set of ν-like elements in $N_{n,k,h}$ and we choose to order them in *reverse fashion* to $N_{n,k,h}$. That is, $\kappa \leq_P \lambda$ if λ refines κ as number partitions and $null(\lambda) \leq null(\kappa)$. The map $f : vert(U_{n,k,h}) \to [dim(U_{n,k,h})+1]$, where $f(v)$ is the maximal cardinality of a P-chain with v as largest element clearly balances $U_{n,k,h}$. We identify $\hat{0} = \{\overline{n}\}$ and $\hat{1} = \{1, 1, \ldots, 1\}$ in $\hat{U}_{n,k,h}$.

We give a chain-labelling of $U_{n,k,h}$ which induces a CL-shelling. Our poset of labels is $\Lambda = \{A < B < C_1 < \cdots < C_{n-1} < D\}$. The label of a root simplex c in $\hat{U}_{n,k,h}$ is determined by the refinement taking place between level $r = rank(T_c)$ and the leaves of its corresponding tree $T_c = \psi(c \setminus \{\hat{0}\})$. (If c contains $\hat{1}$, and hence is a facet, we let T_c be the tree $\psi(c \setminus \{\hat{0}, \hat{1}\})$ to which we have attached a leaf-level corresponding to $\hat{1}$ in the obvious way.) Let $x = \lambda_r(T_c)$ and $y = \lambda_{r+1}(T_c)$ (notation as in the proof of Theorem 2.6.2). Define the labelling $\omega : R(\hat{U}_{n,k,h}) \to \Lambda$ by:

- $\omega(c) = A$, if $null(x) = null(y) + 1$, i.e. if y is obtained from x by cutting a singleton off the null part.
- $\omega(c) = B$, if $h > 1$, $null(x) = h$ and $null(y) = 0$, i.e. if the null part in x is of size h and is split into singletons in y.
- $\omega(c) = C_i$, if a non-null k-part in x is split into singletons in y and the k-part first appeared at depth i in T_c. By this we mean that the k-part was cut off the null part in the refinement process between depth $i-1$ and depth i in T_c.

- $\omega(c) = D$, if $null(x) = null(y) + k$, i.e. if y is obtained from x by cutting a k-part off the null part.

Remark. If $h = 1$, then label B is never used. However, A plays its role in this case.

The simplices determining root intervals in $\hat{U}_{n,k,h}$ correspond to trees which are saturated except between some level r and the leaves. It is not hard to see, that in every rooted interval there is a unique rising simplex and that this simplex is lexicographically least on the interval. Hence, the first and second conditions of Definition 2.3.1 are fulfilled.

In the following, we will not make distinctions between simplices and their corresponding trees. Let $F_1, \ldots, F_t$ be the lexicographic ordering of the facets of $\hat{U}_{n,k,h}$. To verify Condition 3′ of Proposition 2.3.3, pick c in $F_j \cap (\cup_{\alpha<j} F_\alpha)$. Write $c = \delta^{a_1 \to b_1} \circ \cdots \circ \delta^{a_m \to b_m}(F_j)$ for some appropriate a_x, b_x with $a_x + 2 \leq b_x$ and $b_{x-1} \leq a_x$. Choose F_i, $i < j$, such that $c = \delta^{a'_1 \to b'_1} \circ \cdots \circ \delta^{a'_m \to b'_m}(F_i)$ for some a'_x, b'_x such that $a'_x + 2 \leq b'_x$ and $b'_{x-1} \leq a'_x$. (Since $\hat{U}_{n,k,h}$ possibly is nonpure, we do not necessarily have $a_x = a'_x$ and $b_x = b'_x$.) Suppose that F_j is rising on every rooted interval given by $\delta^{a_x \to b_x} \circ \delta^{b_x \to}(F_j)$. We must show that c is contained in some simplex $b \subseteq F_j \cap (\cup_{\alpha<j} F_\alpha)$ with $codim_{F_j}(b) = 1$.

Let a be minimal such that $\omega(\delta^{a \to}(F_j)) \neq \omega(\delta^{a \to}(F_i))$. We cannot have $a_x < a < b_x$ for any x, because this would imply $\delta^{a_x \to}(F_j) = \delta^{a_x \to}(F_i)$ and $\delta^{b_x \to}(F_j) \neq \delta^{b'_x \to}(F_i)$, which is impossible since $i < j$ and F_j is rising on the rooted interval of $\delta^{a_x \to b_x} \circ \delta^{b_x \to}(F_j) = \delta^{a_x \to b'_x} \circ \delta^{b'_x \to}(F_i)$. Thus, the refinement process which determines $\omega(\delta^{a \to}(F_j))$ occurs within $c \subseteq F_i \cap F_j$. By the definition of ω, this implies $\omega(\delta^{a \to}(F_j)) = C_{x_j}$ and $\omega(\delta^{a \to}(F_i)) = C_{x_i}$ for some $x_i < x_j$. Moreover, we know that $a_x < x_i < x_j < b_x$ for some x and that $\omega(\delta^{x_i \to}(F_j)) = \omega(\delta^{x_i+1 \to}(F_j)) = \cdots = \omega(\delta^{x_j \to}(F_j)) = D$; otherwise C_{x_i} and C_{x_j} would correspond to different refinement processes within c. The label C_{x_j} precedes C_{x_i} in the sequence of labels of F_j, so there is some x_s such that $x_i \leq x_s < x_s + 1 \leq x_j$ and C_{x_s+1} precedes C_{x_s} in this sequence. Now create the facet F_s which has every ω-label in common with F_j except that C_{x_s} and C_{x_s+1} are interchanged. Then $s < j$ and $\delta^{x_s}(F_s) = \delta^{x_s}(F_j) \supseteq c$. Thus c is contained in a simplex of codimension 1. Hence $U_{n,k,h}$ is CL-shellable.

In order to determine the homotopy type of $U_{n,k,h}$, we will identify the topologically falling facets in the described shelling. The homotopy type is then given by Proposition 2.3.5. To this end, consider a facet F and its boundary simplex $\delta^i(F)$. Let ω_i and ω_{i+1} denote the labels of the root simplices $\delta^{i \to}(F)$ and $\delta^{i+1 \to}(F)$ respectively. If i is a descent, i.e. $\omega_i > \omega_{i+1}$ (since Λ is a total ordering), then it is also a topological descent. Now suppose $\omega_i \leq \omega_{i+1}$. If $\omega_i \neq D$, then no facet preceding F contains $\delta^i(F)$. This is because all labels below depth i depend only on refinement processes within $\delta^i(F)$ in this case. The only case left to check is thus $\omega_i = \omega_{i+1} = D$. As above, it is seen that $\delta^i(F)$ is contained in some facet preceding F if and only if the label C_{i+1} precedes the label C_i, or, in other words, if the (unique) root simplex contained in F with label C_{i+1} is contained in the one

labelled C_i. Hence, the only possible facets attaching over $\delta^i(F)$ for all i must be labelled either $DD\dots DC_tC_{t-1}\dots C_1$ or $DD\dots DC_tC_{t-1}\dots C_1B$ (B is replaced by A if $h=1$), where t is equal to the number of D:s in the sequence. For an example, see Figure 2.2. Clearly, such a facet exists (and is unique) if and only if $n \equiv 0 \pmod k$ (the first kind) or $n \equiv h \pmod k$ (the second kind). When it exists, it contains $\frac{2n}{k} - 1$ vertices in the former case and $2\frac{n-h}{k}$ vertices in the latter. Hence the theorem.

□

Remark. The hypothesis $h < k$ is necessary (hence Theorem 2.6.5 does not contradict Theorem 2.6.2). To see this, note that to get rid of a null part in the $h = k$ case, one must at some place let a D-label precede a C_x-label, thereby violating Condition 1 of Definition 2.3.1.

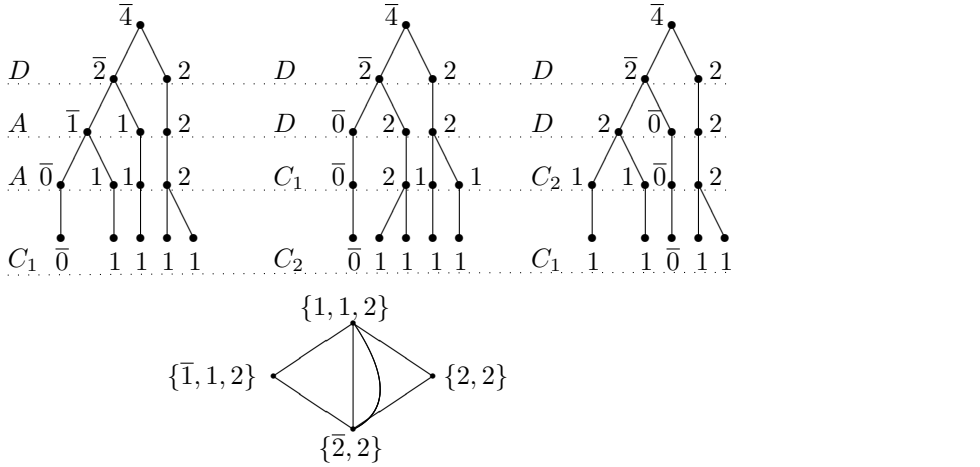


Figure 2.2. *A subcomplex of $U_{4,2,1}$ containing three of the ten facets (bottom) and the corresponding facets of $\hat{U}_{4,2,1}$ seen as trees (top). The labelling ω on the successive rooted subsimplices is indicated. Note that the third facet attaches over its entire boundary (hence is topologically falling) even though it is not falling.*

We now state some immediate consequences of Theorems 2.6.2 and 2.6.5 concerning S_n^B-representations. The key is the following well-known fact, which follows, e.g., from Bredon [19, Theorem 2.4, page 120].

Lemma 2.6.6. *Let K be a field. If Δ is a finite simplicial complex acted upon by a finite group G, where $\mathrm{char}(K)$ does not divide $|G|$, then the multiplicity of the trivial character of the induced representation of G on $\widetilde{H}_i(\Delta, K)$ equals $\widetilde{\beta}_i(\Delta/G, K)$.* □

Corollary 2.6.7. *Let K be a field with $\mathrm{char}(K) \notin [n]$. Then*

1. *The trivial character of the induced representation of S_n^B on the vector space $\widetilde{H}_i(\Delta(\Pi_{n,k,k}), K)$ has multiplicity zero for all i.*

2. *Let $h < k$. The trivial character of the induced representation of S_n^B on the vector space $\widetilde{H}_i(\Delta(\Pi_{n,k,h}), K)$ has multiplicity one if $n \equiv 0 \pmod{k}$ and $i = \frac{2n}{k} - 2$ or if $n \equiv h \pmod{k}$ and $i = 2\frac{n-h}{k} - 1$. Otherwise it has multiplicity zero.*

Proof. Note that $|S_n^B| = 2^n n!$. The corollary now follows from Theorem 2.6.2, Theorem 2.6.5 and Lemma 2.6.6. □

Paper 3.

Directed subgraph complexes

3.1 Introduction

A *monotone* property of a (directed or undirected) graph is one which is preserved under deletion of edges. Hence, the set of all graphs on a particular vertex set, $[n]$ say, that satisfy a monotone property form a simplicial complex whose vertex set is the set of edges of the graphs. In numerous recent papers, see e.g. [2, 18, 24, 53, 55, 74, 75], the topological properties of such complexes of graphs have been studied. Although most papers have dealt with complexes of all graphs having a particular property P, it is indeed natural to study the complex of all subgraphs of a given graph that satisfy P. The purpose of this paper is to study directed graph complexes of this type. The properties that we focus on are acyclicity and strong non-connectivity. Both were studied by Björner and Welker [18] in the case of all graphs. We adapt their techniques in order to generalize their results. We also consider some consequences related to topics such as finite topologies and hyperplane arrangements.

In Section 3.3 we study acyclic graphs. The homotopy type of the complex of acyclic subgraphs of any given directed graph is determined. It is either a homotopy sphere or contractible. Thereafter, in Section 3.4, we focus on not strongly connected graphs. More precisely, we compute the homotopy type of the complex of not strongly connected subgraphs of a directed graph, if the graph belongs to a particular class, which we call 2-*dense* graphs.

We begin, however, with a brief survey in the next section of the more or less standard tools from topological combinatorics that will be made use of later.

Acknowledgement. The author is grateful to his advisor Anders Björner for suggesting the study of subgraph complexes.

3.2 Basic topological combinatorics

Here, we briefly review the parts of the topological combinatorics machinery that we will use later. For more details we refer to the survey [10].

To any poset P, we associate the *order complex* $\Delta(P)$. It is the simplicial complex whose faces are the chains of P. Similarly, to any simplicial complex Σ, we associate its *face poset* $P(\Sigma)$ which consists of the nonempty faces of Σ ordered by inclusion. The complex $\Delta(P(\Sigma))$ is the barycentric subdivision of Σ, hence it is homeomorphic to Σ. (We do not distinguish notationally between a complex and its underlying topological space.)

Our first two tools are due to Quillen [63]. In a poset P with an element $x \in P$, we write $P_{\leq x} = \{y \in P \mid y \leq x\}$.

Lemma 3.2.1 (Quillen Fiber Lemma). *Let P and Q be posets, and suppose we have an order-preserving map $f : P \to Q$ such that $\Delta(f^{-1}(Q_{\leq q}))$ is contractible for all $q \in Q$. Then $\Delta(P)$ is homotopy equivalent to $\Delta(Q)$.*

Lemma 3.2.2 (Closure Lemma). *Let P be a poset, and suppose that $f : P \to P$ is a closure operator (i.e. $f(p) \geq p$ and $f^2(p) = f(p)$ for all $p \in P$). Then $\Delta(P)$ is homotopy equivalent to $\Delta(f(P))$.*

If a poset P has unique minimal and maximal elements, denoted by $\hat{0}$ and $\hat{1}$, respectively, then its *proper part* is $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$.

The next result can be found e.g. in [10].

Lemma 3.2.3. *Let L be a lattice. Form a simplicial complex Σ whose faces are those subsets of the atoms of L whose joins are not the top element. Then $\Delta(\overline{L})$ and Σ are homotopy equivalent.*

For simplicial complexes Δ_1 and Δ_2, let $\Delta_1 * \Delta_2$ denote their join. The following lemma is well-known.

Lemma 3.2.4. *Suppose Δ_i is homotopy equivalent to a wedge of n_i spheres of dimension d_i, $i = 1, 2$. Then $\Delta_1 * \Delta_2$ is homotopy equivalent to a wedge of $n_1 n_2$ spheres of dimension $d_1 + d_2 + 1$.*

Finally, we need a convenient collapsibility lemma stated by Björner and Welker [18]. For a simplicial complex Δ on the vertex set $[n]$, a face $F \in \Delta$ and a vertex $i \in [n]$, define a map $\Delta \to \Delta$ by

$$F \mapsto F \pm i = \begin{cases} F \cup \{i\} & \text{if } i \notin F, \\ F \setminus \{i\} & \text{if } i \in F. \end{cases}$$

Lemma 3.2.5. *If $\Delta_1 \subseteq \Delta_2$ are simplicial complexes on the vertex set $[n]$ and there exist vertices $i, j \in [n]$ such that $F \pm i$ maps $\Delta_2 \setminus \Delta_1$ to itself and $F \pm j$ maps Δ_1 to itself, then Δ_2 is contractible (and so is Δ_1).*

3.3 Acyclic subgraphs

From now on, let G be a fixed directed graph on vertex set $[n]$. Like all graphs in this paper (directed and undirected), G will be assumed to have no loops or multiple edges. Our first object of study is the complex Δ_G^{ACY} of all acyclic subgraphs of G. More precisely, with $E(G)$ denoting the edge set of G, we define

$$\Delta_G^{ACY} = \{F \subseteq E(G) \mid ([n], F) \text{ has no directed cycle}\}.$$

Let $\mathrm{Tr}(\cdot)$ denote transitive closure. Define Pos_G to be the following subset of all partial orders on $[n]$: a poset belongs to Pos_G iff its comparability graph is $\mathrm{Tr}(H)$ for some subgraph H of G. Under inclusion, Pos_G is a poset. We denote its unique minimal element, the empty relation, by $\hat{0}$. In the following lemma, the case of G being the complete graph is [18, Lemma 2.1].

Lemma 3.3.1. *The complexes $\Delta(\mathrm{Pos}_G \setminus \{\hat{0}\})$ and Δ_G^{ACY} are homotopy equivalent.*

Proof. The map $H \mapsto \mathrm{Tr}(H) \cap G$ is a closure operator on $P(\Delta_G^{ACY})$. We claim that its image is isomorphic to $\mathrm{Pos}_G \setminus \{\hat{0}\}$. To show this, it suffices to check that $\mathrm{Tr}(H)$ can be reconstructed from $\mathrm{Tr}(H) \cap G$; it can, since $\mathrm{Tr}(H) = \mathrm{Tr}(\mathrm{Tr}(H) \cap (G))$. Thus, by Lemma 3.2.2, the barycentric subdivision of Δ_G^{ACY} is homotopy equivalent to $\Delta(\mathrm{Pos}_G \setminus \{\hat{0}\})$. □

Recall that the vertices of any directed graph can be partitioned into *strongly connected components*: x and y belong to the same component iff there exist directed paths from x to y and from y to x. If every vertex belongs to the same component, then the graph is *strongly connected*.

Björner and Welker stated the following theorem in the case of G being the complete graph only. However, it is straightforward to check that their proof goes through in the more general case, too.

Theorem 3.3.2 (See Theorem 2.2 in [18]). *If G is strongly connected, then $\Delta(\mathrm{Pos}_G \setminus \{\hat{0}\})$ is homotopy equivalent to the $(n-2)$-sphere.*

It is now straightforward to prove the main result of this section.

Theorem 3.3.3. *If G is a disjoint union of k strongly connected components, then Δ_G^{ACY} is homotopy equivalent to the $(n-1-k)$-sphere. Otherwise, Δ_G^{ACY} is contractible.*

Proof. If G is not a disjoint union of strongly connected components, then G has an edge e which is not included in any cycle. Thus, Δ_G^{ACY} is a cone with apex e.

Now suppose that G is a disjoint union of k strongly connected components. If $k = 1$, then we are done by Theorem 3.3.2 and Lemma 3.3.1. Otherwise, Δ_G^{ACY} is a

join of k complexes of this type. Applying Lemma 3.2.4 $(k-1)$ times, we conclude that Δ_G^{ACY} is homotopy equivalent to the sphere of dimension

$$\sum_{i=1}^{k}(a_i - 2) + k - 1 = n - k - 1,$$

where a_i is the number of vertices in the ith component of G. □

Recall that a *quasiorder* is a reflexive and transitive relation. The poset (actually a lattice) of quasiorders on $[n]$ is a well-studied object (see e.g. [33]), mainly since quasiorders on $[n]$ correspond in a 1-1 fashion to topologies on $[n]$. The subposet Pos_n of partial orders on $[n]$ then corresponds to the topologies that satisfy the T_0 separation axiom. Thus, the next corollary can be thought of as a statement about finite topologies.

For a quasiorder R on $[n]$, let Pos_n^R be the poset of all posets that are contained in R.

Corollary 3.3.4. *Let R be a quasiorder on $[n]$. If R is in fact an equivalence relation with k equivalence classes, then $\Delta(\mathrm{Pos}_n^R \setminus \{\hat{0}\})$ is homotopy equivalent to the $(n-1-k)$-sphere. Otherwise, $\Delta(\mathrm{Pos}_n^R \setminus \{\hat{0}\})$ is contractible.*

Proof. There is an obvious correspondence between quasiorders and transitively closed directed graphs. Applying Theorem 3.3.3 with the graph corresponding to R yields the result via Lemma 3.3.1. □

3.4 Not strongly connected subgraphs

In this section we turn our attention to Δ_G^{NSC}, the complex of subgraphs of G that are not strongly connected. More precisely,

$$\Delta_G^{NSC} = \{F \subseteq E(G) \mid ([n], F) \text{ is not strongly connected}\}.$$

Again, the case of G being the complete graph was analysed in [18].

Let Π_G be the subposet of the partition lattice Π_n consisting of the possible partitions into strongly connected components of subgraphs of G. The partition corresponding to a graph H is denoted by $\pi(H)$. Clearly, if $\pi(H), \pi(H') \in \Pi_G$, then their join (in Π_n) belongs to Π_G. Since $\hat{0} \in \Pi_G$, we conclude that Π_G is a lattice, although it is easy to construct an example showing that Π_G is not a sublattice of Π_n.

By a *minimal cyclic set* of G, we mean an inclusion-minimal subset $S \subseteq [n]$ with the property that some directed G-cycle has S as vertex set. Clearly, such sets correspond to atoms of Π_G. We let $\widehat{G}$ denote the hypergraph on $[n]$ whose edges are precisely the minimal cyclic sets of G.

Directed graphs whose minimal cyclic sets all have cardinality two will be important to us. We call such graphs 2-*dense*. Thus, G is 2-dense iff every cycle

contains two vertices that themselves form a cycle in G, i.e. iff $\widehat{G}$ is an ordinary graph.

Recall that to any (undirected) graph $H = ([n], E)$, one associates the *graphical arrangement* $\mathcal{A}_H$. This is a hyperplane arrangement in $\mathbb{R}^n$ containing $|E|$ different hyperplanes, each given by a coordinate equation $x_i = x_j$ for $\{i, j\} \in E$. Its *intersection lattice*, $L(\mathcal{A}_H)$, is the lattice of all possible intersections of collections of such hyperplanes, ordered by reverse inclusion.

Theorem 3.4.1. *Suppose that G is 2-dense. If $\widehat{G}$ is connected, the order complexes $\Delta(\overline{\Pi_G})$ and $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$ are homotopy equivalent. If $\widehat{G}$ is disconnected, then $\Delta(\overline{\Pi_G})$ is contractible.*

Proof. Suppose that G is 2-dense and denote the edge set of $\widehat{G}$ by $E(\widehat{G})$. Let Σ denote the simplicial complex on the vertex set $E(\widehat{G})$ whose simplices are given by the disconnected subgraphs of $\widehat{G}$. By Lemma 3.2.3, we have $\Delta(\overline{\Pi_G}) \simeq \Sigma$.

If $\widehat{G}$ is disconnected, then Σ is just a simplex, and therefore contractible.

Now suppose that $\widehat{G}$ is connected. Taking transitive closure and then intersecting with $\widehat{G}$ yields a closure operator on $P(\Sigma)$. Its image is isomorphic to the poset of all partitions of $[n]$ that arise as sets of connected components in nonempty disconnected subgraphs of $\widehat{G}$. Clearly, this poset is isomorphic to $\overline{L(\mathcal{A}_{\widehat{G}})}$. By Lemma 3.2.2, the theorem follows. □

Remark. Requiring G to be 2-dense is not necessary in the above theorem. If G is not 2-dense, then $\mathcal{A}_{\widehat{G}}$ should be interpreted as the hypergraph subspace arrangement given by $\widehat{G}$. This generalization will not, however, be useful to us later in this paper. For more on hypergraph arrangements and subspace arrangements in general, we refer to the survey [9].

Björner's and Welker's proof of [18, Lemma 3.1] goes through to prove the more general statement below. We state it here to be able to point out where the 2-density assumption is being used. Below, $P \oplus Q$ denotes ordinal sum of posets.

Lemma 3.4.2 (See Lemma 3.1 in [18]). *If G is 2-dense, then Δ_G^{NSC} and $\Delta(\mathrm{Pos}_G \setminus \{\hat{0}\} \oplus \overline{\Pi_G})$ are homotopy equivalent.*

Proof. For convenience, let $Q = \mathrm{Pos}_G \setminus \{\hat{0}\} \oplus \overline{\Pi_G}$. Consider the natural order-preserving surjection $\varphi : P(\Delta_G^{NSC}) \to Q$ given by

$$\varphi(H) = \begin{cases} \mathrm{Tr}(H) \in \mathrm{Pos}_G \setminus \{\hat{0}\} & \text{if } H \text{ is acyclic,} \\ \pi(H) \in \overline{\Pi_G} & \text{otherwise.} \end{cases}$$

In order to use Lemma 3.2.1, we study the inverse images of φ.

To begin with, we pick $p \in \mathrm{Pos}_G \setminus \{\hat{0}\}$. Clearly, $\varphi^{-1}(Q_{\leq p})$ has a unique maximal element, namely the intersection of G and (the comparability graph of) p. This element is a cone point, and $\Delta(\varphi^{-1}(Q_{\leq p}))$ is contractible.

Now choose $\tau \in \overline{\Pi_G}$. Since G is 2-dense, any non-singleton block of τ contains a directed G-cycle of length two. Without loss of generality, suppose that 1 and 2 form such a cycle. Let $\Delta_2 = \Delta(\varphi^{-1}(Q_{\leq\tau}))$ and let $\Delta_1 \subseteq \Delta_2$ be the subcomplex comprising the graphs that contain no directed path from 1 to 2 except possibly the edge $(1,2)$. Now observe that adding the edge $(2,1)$ to $H \in \Delta_1$ affects the partition into strongly connected components at worst by merging the part which contains 1 with that which contains 2. This shows that $H \mapsto H \pm (2,1)$ maps Δ_1 into itself. Similarly, $H \mapsto H \pm (1,2)$ maps $\Delta_2 \setminus \Delta_1$ into itself. Thus, by Lemma 3.2.5, $\Delta(\varphi^{-1}(Q_{\leq\tau}))$ is contractible, and we are done. □

Using $\chi_{\widehat{G}}(t)$ to denote the chromatic polynomial of $\widehat{G}$, we are now in position to state the main theorem. Note that the case of G being not strongly connected is uninteresting since Δ_G^{NSC} is just a simplex in this case.

Theorem 3.4.3. *If G is 2-dense and strongly connected, then Δ_G^{NSC} is homotopy equivalent to a wedge of $(2n-4)$-dimensional spheres. The number of spheres is $|\chi'_{\widehat{G}}(0)|$.*

Proof. By Lemma 3.4.2 and the definition of ordinal sums, $\Delta_G^{NSC} \simeq \Delta(\mathrm{Pos}_G \setminus \{\hat{0}\}) * \Delta(\overline{\Pi_G})$.

If $\widehat{G}$ is disconnected, Δ_G^{NSC} is contractible by Theorem 3.4.1. In this case, the linear term of $\chi_{\widehat{G}}(t)$, and thus its absolute value $|\chi'_{\widehat{G}}(0)|$, vanishes as desired. We may therefore assume that $\widehat{G}$ is connected.

It is well-known, see e.g. Rota [65], that the characteristic polynomial of $L(\mathcal{A}_{\widehat{G}})$ and the chromatic polynomial of $\widehat{G}$ coincide, i.e.

$$\chi_{\widehat{G}}(t) = \sum_{x \in L(\mathcal{A}_{\widehat{G}})} \mu(\hat{0}, x) t^{\dim(x)},$$

where μ is the Möbius function of $L(\mathcal{A}_{\widehat{G}})$. Moreover, by a theorem of Björner [6], $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$ has the homotopy type of a wedge of $|\mu(\hat{0}, \hat{1})|$ spheres of dimension $\mathrm{codim}(\hat{1}) - 2$. Since the top element has dimension one in our case, we conclude that $\Delta(\overline{L(\mathcal{A}_{\widehat{G}})})$, and therefore $\Delta(\overline{\Pi_G})$, has the homotopy type of a wedge of $(n-3)$-dimensional spheres and that the number of spheres is the absolute value of the linear term of $\chi_{\widehat{G}}(t)$.

Theorem 3.3.3 shows that $\Delta(\mathrm{Pos}_G \setminus \{\hat{0}\}) \simeq S^{n-2}$, so, by Lemma 3.2.4, we are done. □

Remark. The number of spheres above, i.e. the absolute value of the linear term of the chromatic polynomial of $\widehat{G}$, has a nice interpretation due to Greene and Zaslavsky [40]. It is the number of acyclic orientations of $\widehat{G}$ having a unique fixed sink. See also [37].

Corollary 3.4.4 (Theorem 1.2 in [18]). *The complex of all not strongly connected directed graphs on* $[n]$ *is homotopy equivalent to a wedge of* $(n-1)!$ *spheres of dimension* $2n-4$.

Proof. If G is the complete directed graph, then $\widehat{G}$ is the complete undirected graph. The linear term in its chromatic polynomial is $(-1)^{n-1}(n-1)!$. □

Part II

Bruhat intervals

Paper 4.

Bruhat intervals of length 4 in Weyl groups

4.1 Introduction

The Bruhat ordering of Coxeter groups appears in a wide variety of mathematical contexts. As an example (perhaps the most prominent), suppose that G is a simply connected semisimple algebraic group over an algebraically closed field with Borel subgroup B. Then the incidences between the closed Schubert cells in the Bruhat decomposition of the flag variety G/B are governed by the Bruhat order on the corresponding Weyl group.

Answering a question of Björner [7], Dyer [28] showed that (up to isomorphism) only finitely many posets of each length n occur as intervals in the Bruhat order on finite Coxeter groups. It is natural to ask *which* posets. Work of Björner and Wachs [15] yields as a byproduct the solution for $n = 2, 3$.

In this paper we solve the classification problem when $n = 4$ for Weyl groups (i.e. finite Coxeter groups whose Coxeter graphs contain only labels from $\{2, 3, 4, 6\}$) and for the smaller classes of simply-laced Weyl groups (with labels from $\{2, 3\}$) and symmetric groups.

Our strategy is as follows. Relying on the results in [28], we need only inspect a small number (four) of particular Weyl groups. Using a computer, it is possible to single out all occurring isomorphism types in these groups by brute force; we report on such a search at the end of the paper. Most of the paper, however, is occupied by another approach, which we now describe. To each interval a so-called *crown index* is associated. Basically, it records some of the valencies in the Hasse diagram. Thus, it is much easier to compute than the entire isomorphism type of the interval. Aided by computer, we list all crown indices that occur in the four groups. Using the interpretation of Bruhat intervals as regular cell decompositions

of spheres given in [8] and the bearing of the crown index on this decomposition, we determine which isomorphism types occur within each crown index class.

Why not just be satisfied with the first method and leave all work to the computer? There are several reasons. First of all, there is always some uncertainty associated with computer-based proofs. For instance, are there bugs? Although our methods both rely on extensive computer calculations, they are independent of each other. Their agreement vastly diminishes the probability of error. Another reason is that the second method provides combinatorial insight. Structural statements such as Lemmata 4.3.3 and 4.3.4, Proposition 4.3.5 and the proof of Theorem 4.3.6 explain some of what is going on from a theoretical viewpoint. Also, the crown index approach may conceivably be possible to generalize to larger Bruhat intervals.

The rest of the paper is organized in the following way. We begin with a brief review of some important properties of the Bruhat order and the aforementioned results of Björner and Wachs and Dyer. Then, in Section 4.3, we perform the actual classification. The crown indices are defined and calculated in the first subsection. In the second subsection, the connection between Bruhat intervals of length 4, their interpretation as cell complexes and their crown index is stated to pave the way for the main theorem in the third subsection. Finally, in Section 4.4, we present a brute force computer calculation in agreement with the previous results.

Acknowledgement. The author would like to thank his advisor Anders Björner for the numerous suggestions that have aided and improved this work. An anonymous referee has also been very helpful.

4.2 Foundations

We will assume familiarity with basic Coxeter group theory. See e.g. Humphreys [50] for anything not explained here. We will also use basic notions from combinatorial topology regarding regular CW complexes and their face posets. We refer the reader to Björner [10] for details.

Throughout the paper, (W, S) will be a Coxeter system. Denote its set of reflections by T and let $\ell : W \to \mathbb{N}$ be the length function.

Definition 4.2.1. *The* Bruhat order *on W, denoted $Br(W)$, is the poset $(W, \leq)$, where $\leq$ is defined by:*

$$w \leq w' \text{ iff there exist } t_1, \ldots, t_i \in T \text{ such that } w' = wt_1 \ldots t_i$$
$$\text{and } \ell(wt_1 \ldots t_{j-1}) < \ell(wt_1 \ldots t_j) \text{ for all } j \leq i.$$

Although not obvious from the definition, it is a fact that $Br(W)$ is a ranked poset with ℓ as rank function. Important to us is that the operation of taking direct products carries over from Coxeter groups to Bruhat order, i.e. $Br(W \times W') \cong Br(W) \times Br(W')$. This reduces the interval classification problem to one concerning irreducible groups only. Another important property is that if W is finite, then

$Br(W)$ is self-dual. See e.g. Björner and Brenti [12, Chapter 2] for these and other basic facts about Bruhat order.

The following theorem of Björner will be vital to us. Earlier, a slightly weaker version of it was shown by Björner and Wachs [15]. Recall that a CW-complex is *regular* if every attaching map can be extended to a homeomorphism on the entire closed ball.

Theorem 4.2.2 (see [8]). *Pick $x, y \in W$ with $x < y$. The open interval $(x, y) \subseteq Br(W)$ is (isomorphic to) the face poset of a regular CW-decomposition of the $(\ell(y) - \ell(x) - 2)$-dimensional sphere.* □

From now on we use the term *iB-interval* instead of the phrase "interval of length i which appears in the Bruhat order of some Coxeter group".

Corollary 4.2.3 (See Figure 4.3). *We have:*

- *Any 2B-interval is isomorphic to the poset with one maximal element, one minimal element and two atoms.*
- *A 3B-interval is isomorphic to the face poset of a k-gon for some $k = 2, 3, \ldots$.*

Proof. Clearly, the proper parts (the posets with maximal and minimal elements removed) of the said posets are the only posets that are homeomorphic to S^0 and S^1, respectively. □

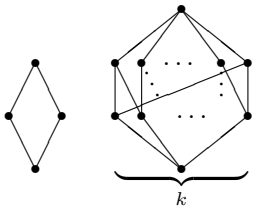


Figure 4.3. The only 2B-interval (left) and the face poset of a k-gon, i.e. a k-crown (right).

Adopting the notation of Björner and Brenti [12], we use the term *k-crown* for the face poset of a k-gon. Arbitrarily large k-crowns can appear in Bruhat orders of Coxeter groups (see e.g. [12, Section 2.7]). However, as we shall see, when we restrict ourselves to the finite case, the situation is much simpler.

A subgroup W' of W is a *reflection subgroup* if it is generated by a subset of T. Deodhar and Dyer independently proved that reflection subgroups are themselves Coxeter groups:

Theorem 4.2.4 (Deodhar [23], Dyer [27]). *Let W' be a reflection subgroup of W. Define $\chi(W') := \{t \in T \cap W' \mid \ell(t't) > \ell(t)\ \forall t' \in T \cap W' \setminus \{t\}\}$. Then $(W', \chi(W'))$ is a Coxeter system.* □

The following result of Dyer provides a strategy for classifying Bruhat intervals in the finite case.

Theorem 4.2.5 (Dyer [28]). *Suppose that W is finite. Pick $x, y \in W$ with $x < y$ and $\ell(y) - \ell(x) = n$. Then there exists a reflection subgroup W' of W with $|\chi(W')| \leq n$ such that the interval $[x, y] \subseteq Br(W)$ is isomorphic to some interval in $Br(W')$.* □

Hence, all nB-intervals that appear in finite Coxeter groups can be found among the finite Coxeter groups of rank $\leq n$. It is not *a priori* clear, though, that the nB-intervals that appear in Weyl groups all can be found among the *Weyl* groups of rank $\leq n$. That this fact and the corresponding statements for simply-laced groups and Young groups are true, follows from the next lemma. Here, we use the term *Young group* to mean a direct product of symmetric groups. The reason for this terminology is that such products are nothing but the parabolic subgroups of A_n usually referred to as Young subgroups. In particular, irreducible Young groups are symmetric groups.

Lemma 4.2.6. *The classes of Weyl groups, simply-laced Weyl groups and Young groups, respectively, are closed under taking reflection subgroups.*

Proof. It suffices to check that if W is an irreducible group from one of the three mentioned classes, then all reflection subgroups belong to this class. This is since the reflections in a direct product of Coxeter groups is the union of the reflections in the factors.

First suppose W is a Weyl group. By Theorem 4.2.4, it is sufficient to check that the angle between any pair of reflecting hyperplanes in the standard geometric realization of W belongs to $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$. Similarly, if W is a simply-laced Weyl group it suffices to check that all angles belong to $\{\frac{\pi}{2}, \frac{\pi}{3}\}$. With aid from e.g. the explicit list of root systems in [50, Section 2.10], the calculations are straightforward.

Now suppose that W is irreducible of type A. Then W is (isomorphic to) the symmetric group S_n (for some n), and its reflections are the transpositions. Let H be a subset of the transpositions. Construct a graph G_H on the vertex set $[n]$ with $\{i, j\}$ an edge iff $(i, j) \in H$. Clearly, $\langle H \rangle$ is the direct product of the symmetric groups on the vertex sets of each connected component in G_H. Hence, $\langle H \rangle$ is a Young group. □

Jantzen [52] has shown that 2-, 3- and 4-crowns are the only 3B-intervals that appear in Weyl groups. Except for dihedral groups, the only irreducible non-crystallographic Coxeter group of rank ≤ 3 is H_3. With aid from a table, e.g. Goresky [38], of Kazhdan-Lusztig polynomials one can check that if $u, v \in Br(H_3)$ with $\ell(u) = \ell(v) - 3$, then the KL-polynomial $P_{u,v}(x)$ is always one of 1, $1 + x$ and

$1+2x$. Hence, using the fact that if $[u,v] \subseteq Br(W)$ is a k-crown, then $P_{u,v}(x)$ is 1 if $k=2$ and $1+(k-3)x$ otherwise, we conclude that the only new crown appearing in H_3 is the 5-crown. Also, note that the product of three 1B-intervals, as well as the product of a 2B- and a 1B-interval, is a 3-crown. Hence no other k-crowns dwell among the reducible groups.

4.3 Classifying the intervals of length 4

4.3.1 Crown indices

Consider a 4B-interval $I = [x,y] \subseteq Br(W)$. Let a and c be an atom and a coatom of I, respectively. The subintervals $[a,y]$ and $[x,c]$ are of length 3. Hence they form crowns. Define $cr_I(a) = k$ if $[a,y]$ is a k-crown, and define $cr_I(c)$ similarly.

Definition 4.3.1. *To a 4B-interval $I = [x,y] \subseteq Br(W)$, we associate the* crown index $cr(I)$. *It is the ordered pair $cr(I) := (cr^{\downarrow}(I), cr^{\uparrow}(I))$ of the multisets*
$cr^{\downarrow}(I) := \{cr_I(a) \mid a \text{ is an atom of } I\}$ *and*
$cr^{\uparrow}(I) := \{cr_I(c) \mid c \text{ is a coatom of } I\}$.

Consider, for example, the Bruhat order $Br(I_2(4))$ of the dihedral group on 8 elements. It is of length 4 and its Hasse diagram is depicted in Figure 4.4. Picture 1 of Figure 4.7 shows its cell complex representation. The crown index is $cr(Br(I_2(4))) = (\{2,2\},\{2,2\})$. Hereafter, we will omit some commas and brackets and e.g. write $cr(Br(I_2(4))) = (22,22)$.

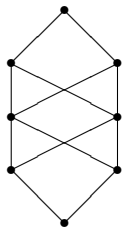


Figure 4.4. The Hasse diagram of $Br(I_2(4))$.

Obviously, if two 4B-intervals are isomorphic, then they have equal crown indices. Hence, a natural way to approach the classification problem is to compute all possible crown indices of 4B-intervals. Restricting ourselves to Weyl groups, it suffices to check all crown indices that appear in A_4, B_4, D_4 and F_4. This is since, except for G_2 (which, like all dihedral groups, only admits the crown index $(22,22)$), the irreducible groups of lower ranks are all parabolic subgroups of these four groups. Furthermore, one only has to check the "lower halves" of their Bruhat orders, since they are self-dual and, obviously, if $cr(I) = (A,B)$ and $I^{\star}$ is dual to I,

then $cr(I^\star) = (B, A)$. Thus, we need only consider one of the intervals (the "lower" one) in each orbit under the duality involution.

We have carried out the calculations in Maple 5 R3 with J. R. Stembridge's Maple package **Coxeter** v2.1 [71] using a program along the lines shown in Figure 4.5. The results are shown in Table 4.1.

```
S ← ∅

for each [u, v] ⊆ Br(W) with
ℓ(v) − ℓ(u) = 4 and ℓ(u) ≤ ℓ(Br(W))/2 − 2 do

    if cr([u, v]) is not in S then

        S ← S ∪ {cr([u, v]), cr(dual([u, v]))}

remove isomorphic copies from S

output S
```

Figure 4.5. Program computing all crown indices of 4B-intervals in $Br(W)$. Note that we only scan the lower half of the poset. The upper half is obtained by duality.

It is well-known, see e.g. [12, Exercise 5.10], that the Kazhdan-Lusztig polynomial of a 4B-interval $I = [u, v] \subseteq Br(W)$ is $P_{u,v}(x) = 1 + (\alpha(u, v) + \frac{1}{2}\beta(u, v) - 4)x$, where $\alpha(u, v)$ is the number of coatoms in I and $\beta(u, v)$ is the number of 2-crowns in I. Hence, the Kazhdan-Lusztig polynomial of I can easily be computed directly from the crown index, see Table 4.1.

Remark. In particular, the computations leading to Table 4.1 again verify the fact that only 2-, 3- and 4-crowns appear as 3B-intervals in Weyl groups.

4.3.2 Cell decompositions of the 2-sphere

Consider a 4B-interval I. Its lower intervals of length 3 are, as we have seen, the face posets of k-gons. Hence, according to Theorem 4.2.2, I (or rather I with its extremal elements removed; we will neglect this distinction) is the face poset of a regular cell decomposition of S^2 whose maximal cells are k-gons. From now on, we will not distinguish between a 4B-interval and the regular CW complex it determines.

A lot of information about the cell structure of I is provided by $cr(I)$. Its second part determines the number of k-gons involved, for each k. The first part determines the valencies of the vertices in the graph (1-skeleton) of the complex. For example, if $cr(I) = (2244, 3333)$, then I is a decomposition of S^2 into four 3-gons. Its 1-skeleton contains four vertices, two of valency 2 and two of valency

#	crown index	K-L	A_4	B_4	D_4	F_4
1.	$(22, 22)$	1	-	$\surd$	-	$\surd$
2.	$(233, 233)$	1	$\surd$	$\surd$	$\surd$	$\surd$
3.	$(334, 2233)$	$1 + x$	$\surd$	$\surd$	$\surd$	$\surd$
4.	$(2233, 334)$	1	$\surd$	$\surd$	$\surd$	$\surd$
5.	$(2334, 2334)$	$1 + x$	-	$\surd$	-	$\surd$
6.	$(3333, 2244)$	$1 + x$	-	$\surd$	-	$\surd$
7.	$(2244, 3333)$	$1 + x$	-	$\surd$	-	$\surd$
8.	$(3333, 3333)$	1	$\surd$	$\surd$	$\surd$	$\surd$
9.	$(33334, 33334)$	$1 + x$	$\surd$	$\surd$	$\surd$	$\surd$
10.	$(333333, 33444)$	$1 + x$	$\surd$	$\surd$	$\surd$	$\surd$
11.	$(33444, 333333)$	$1 + 2x$	$\surd$	$\surd$	$\surd$	$\surd$
12.	$(333344, 333344)$	$1 + 2x$	-	$\surd$	-	$\surd$
13.	$(334444, 3333334)$	$1 + 3x$	-	$\surd$	-	$\surd$
14.	$(3333334, 334444)$	$1 + 2x$	-	$\surd$	-	$\surd$
15.	$(3333444, 3333444)$	$1 + 3x$	-	$\surd$	$\surd$	$\surd$
16.	$(33333333, 444444)$	$1 + 2x$	-	$\surd$	$\surd$	$\surd$
17.	$(444444, 33333333)$	$1 + 4x$	-	$\surd$	$\surd$	$\surd$
18.	$(3344444, 33333344)$	$1 + 4x$	-	-	-	$\surd$
19.	$(33333344, 3344444)$	$1 + 3x$	-	-	-	$\surd$
20.	$(33334444, 33334444)$	$1 + 4x$	-	-	-	$\surd$

Table 4.1. All crown indices appearing in A_4, B_4, D_4 and F_4, and the corresponding Kazhdan-Lusztig (K-L) polynomials.

4. Consequently, there are 6 edges (1-cells). As will be seen, there is exactly one such decomposition, so that the crown index actually characterizes an isomorphism class in this case.

Recall that a graph is *2-connected* if it is connected and cannot be made disconnected by deleting a vertex together with all its incident edges. Any embedding of a planar loop-free 2-connected graph in the plane determines a unique cell decomposition of S^2; its 2-cells are the regions that the graph cuts out in the one-point compactification of the plane. We include a proof of the following straightforward lemma for the sake of completeness.

Lemma 4.3.2. *The cell decomposition of S^2 determined by an embedding of a planar loop-free 2-connected graph in the plane is regular.*

Proof. Let G be such a graph, and embed it in the plane. Since G is loop-free, the 1-skeleton of the cell decomposition is regular. Suppose the cell decomposition is not regular. Then there is an open 2-cell c whose boundary, the subgraph ∂c, is not homeomorphic to S^1. Hence, ∂c contains a vertex v of valency greater than 2. Take an open disc D such that $vert(G) \cap D = \{v\}$. Now $D \cap c$ is disconnected. Since c is arcwise connected, we can pick a closed simple loop $\gamma \subset c \cup \{v\}$ passing

through v. By the Jordan Curve Theorem, it cuts the plane into two regions. Both regions contain part of G, i.e. deleting v makes G disconnected, contradicting 2-connectivity. □

Similarly, given a regular cell decomposition of S^2, an embedding of its 1-skeleton (which, by regularity, is loop-free and 2-connected) in the plane is constructed as follows. Imagine that the 2-cells are transparent while the 1-skeleton is not. Put your eye outside the sphere close to the interior of any 2-cell, look into the sphere and draw what you see. Note that the 2-cell which corresponds to the unbounded region in the plane (the one you put your eye next to) can be chosen at will. We will repeatedly make use of this fact.

4.3.3 The classification

The following two lemmata substantially reduce the number of cell decompositions of S^2 that we need to consider.

Lemma 4.3.3. *Let I be any 4B-interval. If $cr(I)$ contains only the numbers 3 and 4, then the 1-skeleton of I has no multiple edges.*

Proof. We will prove the lemma by contradiction, so assume that there are multiple edges. Since $cr(I)$ does not contain 2, we can construct a planar embedding of the 1-skeleton of I (in the way described above) such that no edge-pair closes off a 2-gon. Pick vertices v_1 and v_2 which are endpoints of a pair of multiple edges e_1 and e_2.

Let C denote the (open) inner region bounded by e_1 and e_2, and let D be the outer one. There is at least one edge going from each v_i into C by 2-connectivity and the fact that C contains at least one vertex. The same holds for D. Hence, since the maximal valency is 4, exactly one edge goes from each v_i into each of C and D. Call these edges e_i^C and e_i^D, respectively. Denote the other endpoint of e_i^D by w_i, $i = 1, 2$. (Possibly, $w_1 = w_2$.) The situation is depicted in Figure 4.6.

Note that, since no additional edges emanate from v_1 or v_2, the path e_1^D, e_1, e_2^D forms part of the boundary of a 2-cell in I. The same is true for the path e_1^D, e_2, e_2^D. Since only 3-gons and 4-gons are allowed, at most one more edge is involved in each boundary; exactly one if $w_1 \neq w_2$, none otherwise.

If there is a single edge, f, between w_1 and w_2, or if $w_1 = w_2$, then the cycles $e_1^D, e_1, e_2^D(, f)$ and $e_1^D, e_2, e_2^D(, f)$ both form the boundaries of 2-cells. Moreover, these 2-cells are the only ones that contain e_1^D and e_2^D (and f). Thus, w_1 and w_2 are contained in no more edges, so they have valency 2, a contradiction.

We conclude that there must be two edges, f_1 and f_2, between w_1 and w_2. Now, by symmetry, the other endpoints of the e_i^C, call them u_i, must also be the endpoints of a pair of multiple edges. Moreover, all paths between the u_i and the w_i pass through some v_i. Recycling our arguments with w_i replacing v_i, f_i replacing e_i and so on, we produce an infinite path $\ldots, u_1, v_1, w_1, \ldots$ through distinct vertices, contradicting the well-known fact that I is finite. □

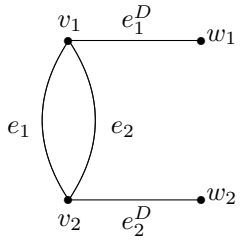


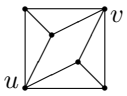
Figure 4.6. Illustration for Lemma 4.3.3.

Lemma 4.3.4. *If I is a 4B-interval in a Weyl group and $cr(I)$ contains only the numbers 3 and 4, then any nonempty intersection of a pair of 2-cells in I either consists of a single vertex or of an edge.*

Proof. Suppose that F and G are two 2-cells in I whose intersection contains two vertices u and v, but not an edge between them. Note that the dual interval $I^\star$ also contains 2-cells with this property. We have checked (see Case 20 in the proof of Theorem 4.3.6 below) that no interval with crown index $(33334444, 33334444)$ has such a pair of 2-cells. Hence, a glance at Table 4.1 shows that we may assume that I has at most 7 vertices.

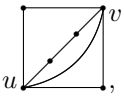
First, suppose that F and G are 4-gons. We divide the proof into several cases.

Case 1. There is no edge connecting u and v in $F \cup G$. When embedding the 1-skeleton of I in the plane, we can use one of F and G to close off the unbounded region. Noting that $I \setminus (F \cup G)$ contains at most one vertex, one easily sees that the following picture:



is the only way to avoid vertices of valency 2. (Here, the central 4-gon and the unbounded region correspond to F and G.) However, computer calculations show that every 4B-interval in B_4 and F_4 with crown index $(333344, 333344)$ has the property that the two vertices of valency 4 are connected by an edge. Thus, the above picture does not represent such an interval. (Note that this crown index does not occur in A_4 or D_4.)

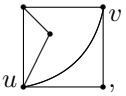
Case 2. There is an edge e connecting u and v in $F \cup G$. The intersection $F \cap G$ contains only the isolated points u and v. Suppose that $e \subseteq F$. Using G as the unbounded region, we obtain the partial picture:



,

where e is the curved edge. Now, something should be added to the picture inside the lower right 3-gon in order to increase the valency of the lower right vertex, but doing so would cause the graph not to be 2-connected (deleting the lower right vertex would make it disconnected).

Case 3. There is an edge e connecting u and v in $F \cup G$. The intersection $F \cap G$ contains an edge and an isolated point. Again, we assume that $e \subseteq F$ and use G as the unbounded cell, thereby obtaining the following partial picture:



where, again, e is the curved edge. Clearly, adding at most two vertices and some edges to the picture (in the complement of $F \cup G$), we cannot produce a simple graph in which all valencies are 3 or 4.

These are all possible configurations of two 4-gons that do not require multiple edges or vertices of valency 2, so we are done.

If F is a 3-gon and G is a 4-gon, then we may argue as above, the only relevant case being Case 2. If, finally, F and G are 3-gons, then we are done by Lemma 4.3.3. □

The preceding lemma will be of use to us only in the case when both F and G are 4-gons. We bothered to state the full version of it since, via a lemma of Björner, Edelman and Ziegler [13], it implies the following proposition, which we feel is interesting in its own right.

Proposition 4.3.5. *A 4B-interval in a Weyl group is a lattice iff it contains no 2-crown.*

Proof. Let $I = [u, v]$ be a 4B-interval in a Weyl group whose crown index does not contain 2. By [13, Lemma 2.1], a bounded poset of finite rank is a lattice iff the following property holds: if x and y have a common cover, then the meet $x \wedge y$ exists.

Now, if $x, y \in I$ have v as common cover, then their meet exists by Lemma 4.3.4. If they have some other common cover, z, then their meet exists since $[u, z]$ is a lattice. □

Remark. After the publication of [49], we have found out [29] that Dyer has shown (unpublished) that an arbitrary iB-interval is a lattice iff it contains no 2-crown. This result of course makes Lemma 4.3.4 and Proposition 4.3.5 superfluous.

Combining combinatorial arguments and computer calculations, we now verify the promised classification. Below, the numbering of graphs refers to Figure 4.7.

Theorem 4.3.6. *The 4B-intervals that occur in Weyl groups are those depicted in Figure 4.7. Those that occur already in simply-laced Weyl groups are 2, 3, 4, 8, 9,*

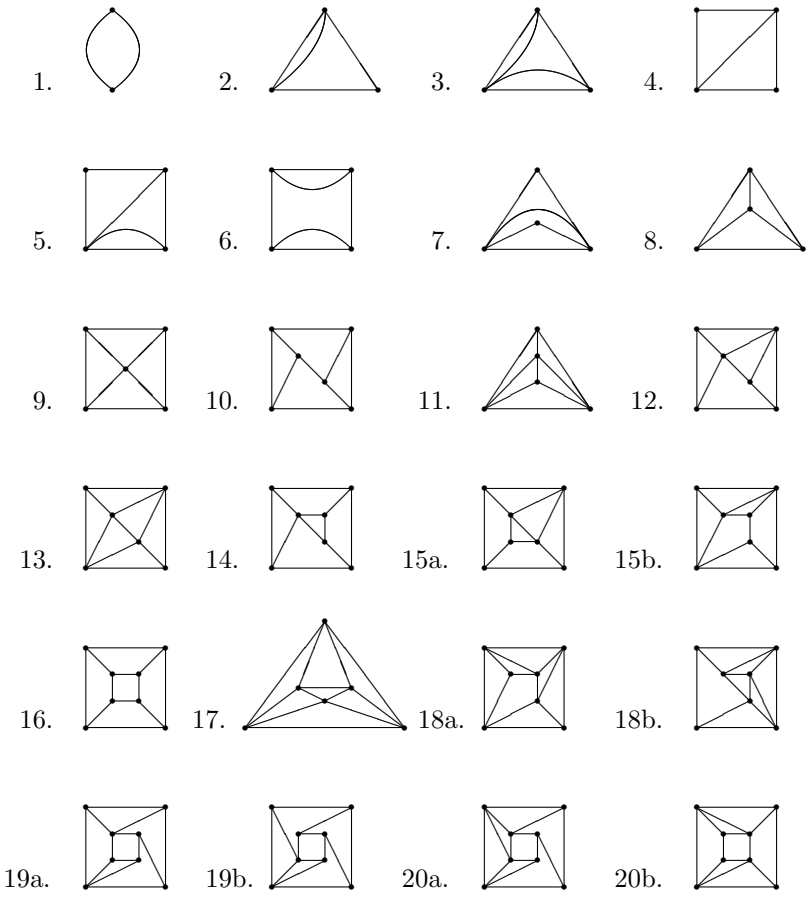


Figure 4.7. All 4B-intervals that appear in Weyl groups. (The pictures should be viewed as the cell decompositions of S^2 that they determine.)

10, 11, 15a, 16 and 17. Those that occur in symmetric groups are 2, 3, 4, 8, 9, 10 and 11.

Proof. To begin with, we check the intervals that *a priori* do not necessarily occur in irreducible groups of rank ≤ 4. The product of a 2-, 3- or 4-crown and a 1B-interval is isomorphic to 2, 8 and 9, respectively, whereas the product of two 2B-intervals is isomorphic to 8. Hence, all 4B-intervals not isomorphic to 2, 8 or 9 must occur in irreducible groups of rank ≤ 4. As mentioned before, all 4B-intervals in groups

of rank < 4 also appear in groups of rank 4. It remains to check the 4B-intervals in A_4, B_4, D_4 and F_4.

For each crown index in Table 4.1, we classify the 4B-intervals that give rise to it. We omit the cases 4, 7, 11, 14, 17 and 19, since they are obtained by duality from 3, 6, 10, 13, 16 and 18, respectively.

1. $cr(I) = (22, 22)$:
Clearly, 1 is the only decomposition of S^2 into two 2-gons.

2. $cr(I) = (233, 233)$:
Choosing a 3-cycle to close off the unbounded region, 2 is the only possible way to embed in the plane a graph with the prescribed valencies.

3. $cr(I) = (334, 2233)$:
Again, choosing a 3-cycle to close off the unbounded region, the only possible loop-free graph is 3.

5. $cr(I) = (2334, 2334)$:
Pick a 4-cycle v_1, v_2, v_3, v_4, v_1 to close off the unbounded region. There is one vertex, say v_1, with valency 4. The two remaining edges emanating from it end at different vertices (otherwise we would have another valency 4 vertex). If these were v_2 and v_4, the crown index would be $(2334, 2244)$. Hence, v_3 has valency 3, making 5 the only possible picture.

6. $cr(I) = (3333, 2244)$:
Let a 4-cycle close off the unbounded region. Each vertex is contained in one more edge. No diagonal edges can occur; this would force edges to cross. Hence, 6 is the remaining possibility.

In the remaining cases, we will implicitly use that there can be no multiple edges as ensured by Lemma 4.3.3.

8. $cr(I) = (3333, 3333)$:
The underlying graph must be the complete graph K_4. The only way to embed it in the plane is 8.

9. $cr(I) = (33334, 33334)$:
Choose a 4-cycle to close off the unbounded region. No diagonal edges in the 4-cycle are possible since they would force the "isolated" vertex to have valency 2. Hence the remaining four edges must all have one endpoint in the middle vertex. The result is 9.

10. $cr(I) = (333333, 33444)$:

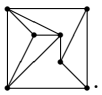
Close off the unbounded region with a 4-cycle. Since all vertices have valency 3 and there are 5 remaining edges, the middle vertices must be connected by an edge. The remaining two edges from each middle vertex must end in adjacent edges on the outer 4-cycle; otherwise a vertex in the outer 4-cycle would be forced to have valency 2. The only possibility left is 10.

12. $cr(I) = (333344, 333344)$:
Pick a 4-cycle to close off the unbounded region. There is one other 4-gon and two more vertices. Hence, the intersection of the two 4-gons must be an edge, by Lemma 4.3.4. Now, 12 is the only possibility.

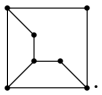
13. $cr(I) = (334444, 3333334)$:
As usual, let the unbounded region be a 4-gon. Vertices of valency 4 on the outer 4-cycle necessarily have edges to both inner vertices. Therefore, two vertices that are adjacent on the outer 4-cycle cannot both have valency 4; this would mean that one of the inner vertices must have valency 2 or (if the inner vertices are connected by an edge) that one of the remaining vertices on the outer 4-cycle must have valency 2. Hence, the four vertices of valency 4 are two diagonal vertices on the outer 4-cycle and both inner vertices, forcing the situation to be as in 13.

15. $cr(I) = (3333444, 3333444)$:
There are seven vertices, so each pair of 4-gons has nonempty intersection. By Lemma 4.3.4, either each such pair intersects in an edge, or there is a pair of 4-gons that intersect in a single vertex.

First suppose that every 4-gon pair intersects in an edge. Let the unbounded region be a 4-gon. Then the two edges that intersect the other 4-gons are are either disjoint or not. The former case implies this appearance:

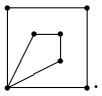


In the latter case, we have the following partial picture:

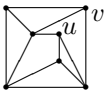


The only way to fill in three more edges without destroying one of the 4-gons or getting forbidden valencies is 15a.

Now suppose that there is a pair of 4-gons intersecting in a single vertex. Using one of them as the unbounded region, we get the partial picture:



The two vertices u and v that are not adjacent to the bottom left vertex in this picture are either joined by an edge or not. In the latter case, they are both forced to have valency 3 (to avoid 2-valencies), making this:



the only possibility. In the former case, exactly one of u and v has valency 4, not both (again, this would force 2-valencies). We may assume that it is v; otherwise we just interchange the roles of the two 4-gons. Now, 15b is the only possible picture.

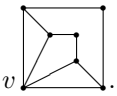
Thus, we have obtained four different candidates for 4B-intervals with the given crown index. Note that each one is self-dual. Computer calculations in B_4, D_4 and F_4 show that the induced subgraph on the four vertices of valency 3 is always either connected or of the following form:

,

which implies that 15a and 15b are precisely the occurring intervals. Moreover, 15b (the one corresponding to the graph just depicted) only occurs in the nonsimply-laced group F_4, whereas 15a appears in all three groups.

16. $cr(I) = (33333333, 444444)$:
Consider the intersection of two 4-gons. If it contained an isolated vertex, then this vertex would have had valency greater than 3. Hence, by Lemma 4.3.4 it either consists of one edge or is empty. We conclude that each 4-gon intersects four of the others, one in each edge. The interval must be isomorphic to the hollow cube, picture 16.

18. $cr(I) = (3344444, 33333344)$:
Since there are only seven vertices, the two 4-gons have nonempty intersection. By Lemma 4.3.4 this intersection is either a vertex or an edge. First suppose it is a vertex, v. There are no other edges emanating from v, and all other 2-cells are 3-gons. Hence, the four neighbours of v must form two pairs, each connected by an edge. Picking one of the 4-gons as the unbounded cell, we have the partial picture:



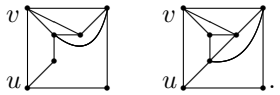
.

It is easily seen that 18a is the only way to complete this partial picture with three more edges which is consistent with the crown index.

Now suppose that the 4-gons intersect in an edge whose vertices we call u and v. Let w be the vertex which is not contained in any of the 4-gons. Note that any edge containing u or v must either be contained in one of the 4-gons, or it must contain w; otherwise a vertex on one of the 4-gons would be "isolated" and forced

to have valency 2. We claim that w cannot be adjacent to u or v, so that both u and v must have valency 3.

To prove the claim, suppose, for contradiction, that u and w are joined by an edge. Since w is not contained in a 4-gon, the subcomplex $star(w)$ must be the barycentric subdivision of either a 3-gon or a 4-gon. Moreover, for valency reasons, all neighbours of u except v are also neighbours of w, hence belong to $star(w)$. The valency of w determines which of the following two partial pictures we get:



(In the right hand picture we have made the choice to use as the unbounded cell the 4-gon which only contains two of w's neighbours. In the left hand picture, the situation is symmetric.) Clearly, neither of the pictures can be completed to one consistent with the crown index. We have a contradiction as desired.

Since u and v are the only vertices of valency 3, w has valency 4. Hence, $star(w)$ (which contains all vertices except u and v) is the barycentric subdivision of a 4-gon. We have arrived at picture 18b.

Computer calculations in F_4 show that there are intervals with this crown index both with and without the property that the vertices of valency 3 are connected by an edge. Hence, both 18a and 18b appear in F_4.

20. $cr(I) = (33334444, 33334444)$:
There are an abundance of cell decompositions of S^2 that are consistent with this crown index. However, only eight intervals with this crown index appear in F_4 (and we need only check the "bottom four"; the others are obtained by duality). By actually computing the Hasse diagrams of these four, we found that the only occurring isomorphism classes are 20a and 20b (note that they are self-dual). □

As [20, Conjecture 7.14], Brenti conjectures that no symmetric group contains a 5B-interval isomorphic to the 4-cube. Now, Theorem 4.3.6 shows that the following stronger statement holds:

Corollary 4.3.7. *No 4B-interval in any symmetric group is isomorphic to the 3-cube.* □

4.4 A completely computerized approach

Version 2.2 (the latest at the time of this writing) of J. R. Stembridge's Maple package **posets** [73] includes fast algorithms for isomorphism testing of posets. Using it, it is possible to go through every 4B-interval in $Br(W)$ for $W \in \{A_4, B_4, D_4, F_4\}$ and check whether or not it is isomorphic to one of the previous intervals.

Using Maple 7 with **Coxeter** v2.3 [72] and **posets** v2.2, we executed the program sketched in Fig. 4.8 on a regular PC with 1 GHz CPU frequency. After a

```
for each W in {A_4, D_4, B_4, F_4} do

    S ← ∅

    for each [u, v] ⊆ Br(W) with
    ℓ(v) − ℓ(u) = 4 and ℓ(u) ≤ ℓ(Br(W))/2 − 2 do

        if [u, v] is not isomorphic to any I in S then

            S ← S ∪ {[u, v], dual([u, v])}

    remove isomorphic copies from S

    output S
```

Figure 4.8. Sketch of the program used to compute all 4B-intervals in A_4, B_4, D_4 and F_4. As usual, we only go through the lower half of each Bruhat order, obtaining the upper by duality.

few days of running time, we obtained the following output. The program found 7, 10, 17 and 24 different intervals in A_4, D_4, B_4 and F_4, respectively, in complete agreement with Theorem 4.3.6.

Although beyond the scope of this article, we note that it could well be possible to perform an exhaustive search along the lines just described also for 5B-intervals. In this case, the relevant groups of rank 5 are A_5, B_5 and D_5. One would also have to check F_4, since it is not contained in the others as a parabolic subgroup.

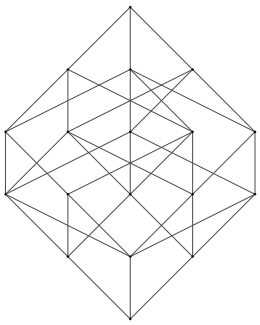
Complement.

Bruhat intervals of length 5 in symmetric groups

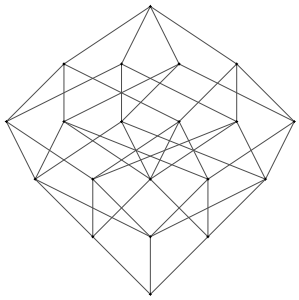
4.5 More computer calculations

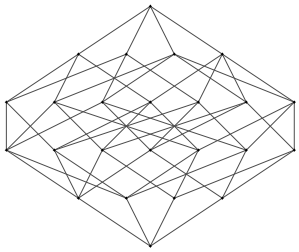
The all-computerized approach described in Section 4.4 has its limitations a bit beyond length 4. We applied the program in Figure 4.8 (with the obvious modifications) to the group A_5, in order to compute all its intervals of length 5. This resulted in the intervals whose Hasse diagrams are listed below, and their dual posets. Furthermore, this list includes all direct products of the 2B-interval and 2-, 3- and 4-crowns as well as the direct products of the 1B-interval and the seven 4B-intervals that appear in symmetric groups. Thus, we may draw the following conclusion:

Theorem 4.5.1. *The 5B-intervals that appear in symmetric groups are precisely the posets listed below and their duals. In particular, there are 25 such intervals.*

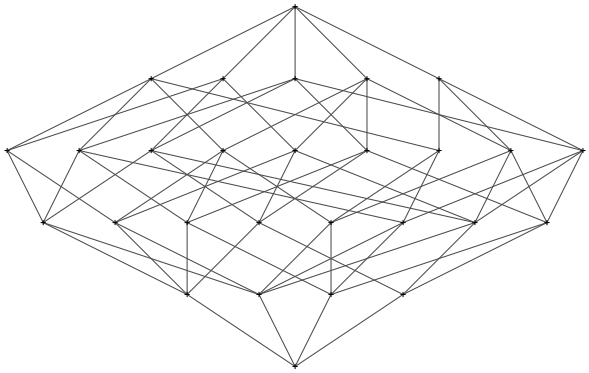


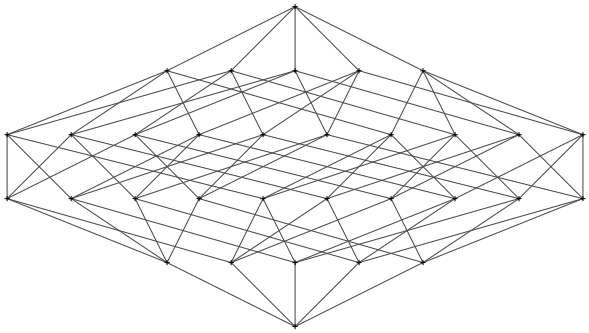
(self-dual)



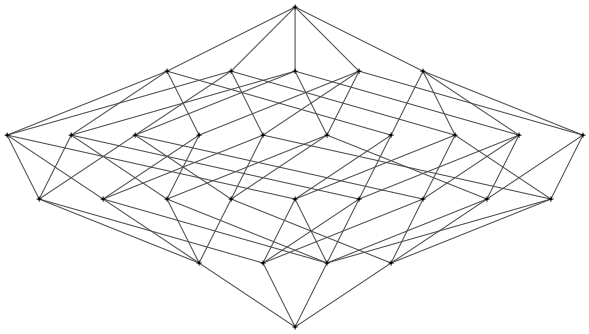


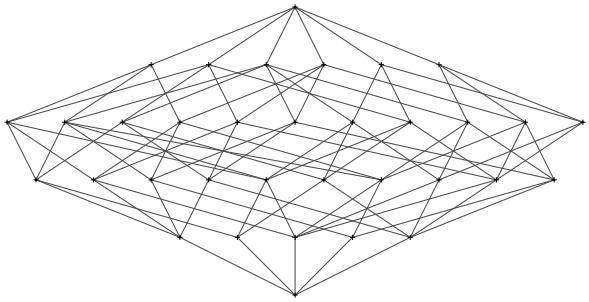
(self-dual)

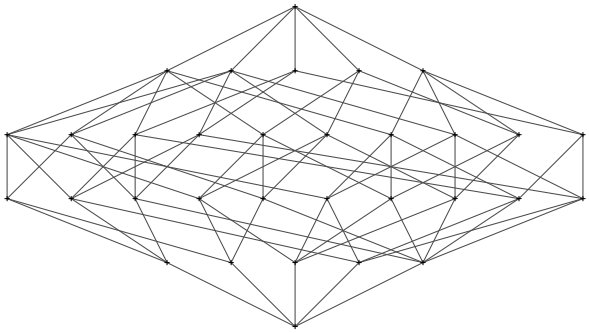


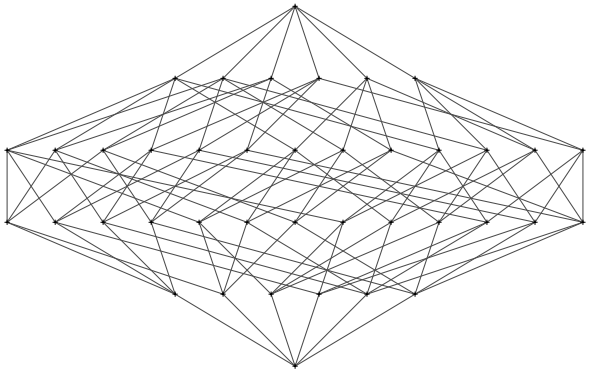


(self-dual)

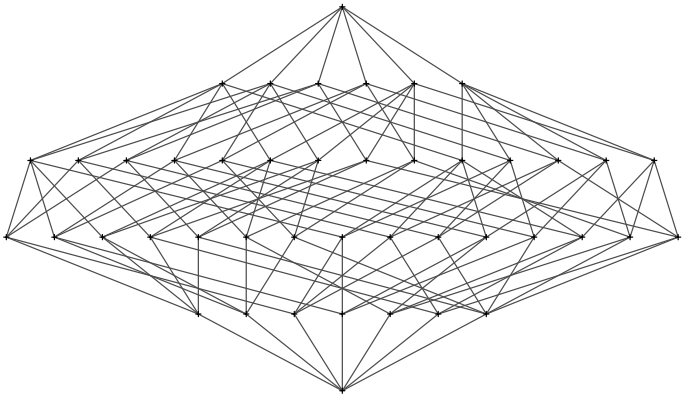


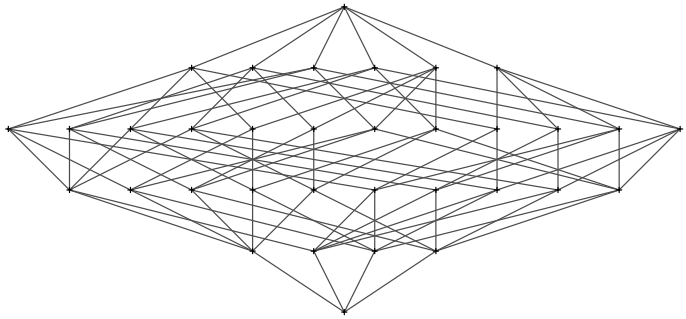


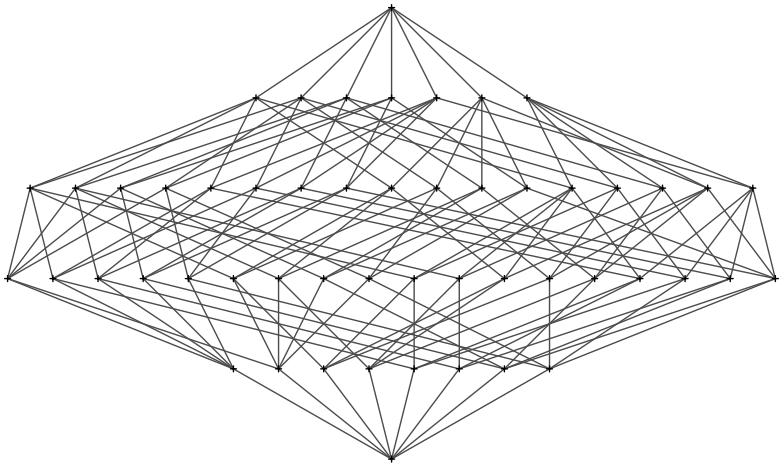


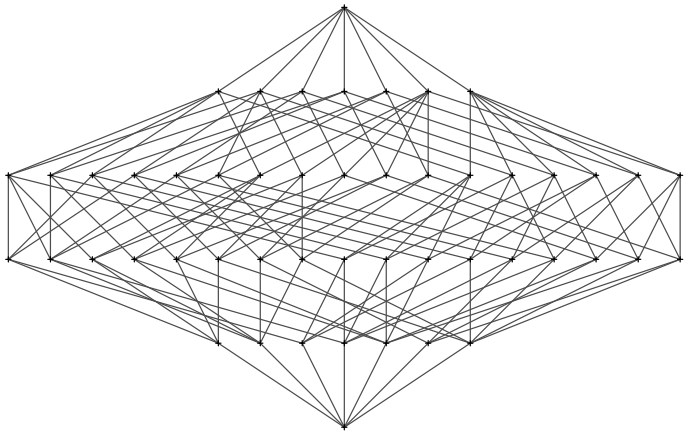


(self-dual)

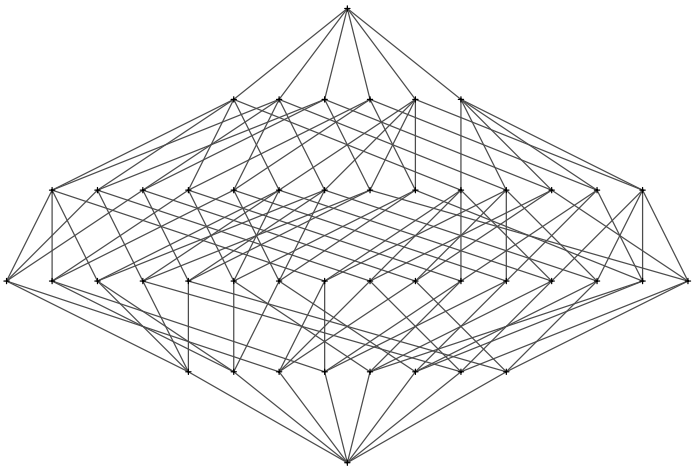


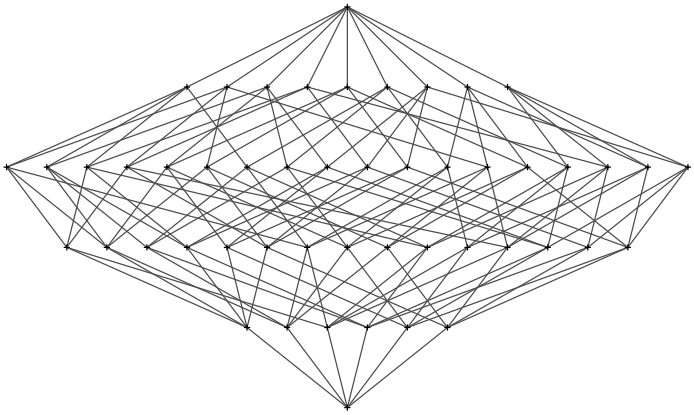






(self-dual)





Part III

Expected reflection distances

Paper 5.

Estimating the expected reversal distance after t random reversals

5.1 Introduction

Over the last decade, the computational biology community has been looking at the problem of estimating evolutionary distances between taxa from their gene order. The probably most common, and by far most studied, evolutionary operation in this context is the *reversal*: a segment (that is a sequence of consecutive genes) of the genome is taken out and inserted at the same place, but in reversed order. In 1995, Hannenhalli and Pevzner [44] presented a formula for the minimal number of reversals needed to transform one sequence of distinct genes into a given permutation of them.

For distant genomes, the true evolutionary distance is in general much longer than the shortest distance. In order to find a better estimate of the true distance, we may instead look at the expected distance. Such attempts have been made by Wang [76] and Eriksen [30], providing bounds for, and an approximation of, the expected reversal distance given the number of *breakpoints* between two genomes π and τ. (There is a breakpoint between two genes in π if they are adjacent in π but not in τ.)

The reversal distance "contains more information" than the breakpoint distance, so if we could find the expected reversal distance, we would probably obtain a biologically more relevant formula. For the same reasons, we would also expect this problem to be harder. The inverse problem seems to be of more reasonable difficulty:

Problem 5.1.1. *Compute the expected reversal distance after t random reversals, taken independently from a uniform distribution.*

In this paper (Section 5.3), we find an analogy between certain cycles used by Hannenhalli and Pevzner, and the ordinary cycles in the symmetric group. We reach the conclusion that one can obtain a good estimate to the expected reversal distance by solving the following analogous problem:

Problem 5.1.2. *Compute the expected transposition distance in the symmetric group after t random transpositions.*

In Sections 5.4 and 5.5 we find the solution to Problem 5.1.2 to be the following formula:

$$\mathbb{E}_{\mathrm{trp}}(n,t) = n - \sum_{k=1}^{n} \frac{1}{k} + \sum_{p=1}^{n-1} \sum_{q=1}^{\min(p,n-p)} a_{pq} \left(\frac{\binom{p}{2} + \binom{q-1}{2} - \binom{n-p-q+2}{2}}{\binom{n}{2}} \right)^t, \quad (5.1)$$

where

$$a_{pq} = (-1)^{n-p-q+1} \frac{(p-q+1)^2}{(n-q+1)^2(n-p)} \binom{n-p-1}{q-1} \binom{n}{p}.$$

Finally, in Section 5.6 we show how the inverse of (5.1) can be used as an estimate for the expected evolutionary reversal distance and investigate numerically how well this formula behaves compared to previous methods when it comes to predicting the true evolutionary distance.

It should be noted that "expected reversal distances" have been studied as early as in 1996 [4]. That paper, however, dealt with the expected reversal distance of a linear, unsigned genome taken from the uniform distribution.

Remark. In [32], we use analogous, but somewhat more involved, methods to solve Problem 5.1.2 for the complex reflection groups $G(r,1,n)$. The symmetric group is then the special case $r = 1$. For $r > 1$, there are no immediate applications to computational biology.

Acknowledgement. The authors are most grateful to Richard Stanley for pointing out a way to prove Theorem 5.5.1. We also thank Kimmo Eriksson for his careful reading of this article.

5.2 Preliminaries

Let S_n be the symmetric group on n elements and let $d_{\mathrm{trp}}(\pi)$ be the minimal transposition distance from a permutation π to the identity permutation, i.e. the minimal i such that π is a product of i transpositions. It is well-known since Cayley that $d_{\mathrm{trp}}(\pi) = n - c_{\mathrm{trp}}(\pi)$, where $c_{\mathrm{trp}}(\pi)$ is the number of cycles in π.

A *genome with n genes* is a signed, circular permutation on n elements. Two genomes are equivalent if you can obtain one from the other by reading it backwards and changing all signs. Disregarding the signs yields an *unsigned genome*. We will denote the set of all genomes with n genes by G_n. The identity genome is denoted by $id =$`1 2 ... n`. In this paper, we take the liberty of writing a genome $\pi \in G_n$ in a linear fashion. It is then understood that the leftmost gene should be attached to the rightmost gene.

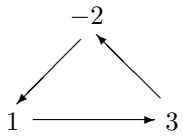


Figure 5.9. An example genome.

Example 5.2.1. *The genome in Figure 5.9 can be written as, for instance,* `1 3 -2`*, as* `3 -2 1` *and as* `-3 -1 2` *(reading in the opposite direction). Usually, we let* `1` *be the first element in the linear order.*

In this paper, we will consider an evolutionary event called *reversal* (or *inversion*). A reversal between π_i and π_j, where $i \neq j$, is an operation that takes the segment $\pi_{i+1}\pi_{i+2}\dots\pi_j$ out of the genome and inserts it at the same place backwards, changing the signs of all elements in the segment. This is depicted in Figure 5.10.

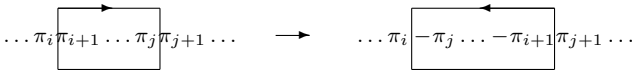


Figure 5.10. A reversal.

5.2.1 The cycle graph

The cycle graph of a genome π was introduced by Hannenhalli and Pevzner in 1995 [44] to find the *minimal reversal distance* $d_{\text{rev}}(\pi)$ between π and id. One should note that, since we can always rename the genes in two genomes such that one of them becomes the identity, this gives the minimal reversal distance between any pair of genomes.

Two genes a and b in a genome π are said to be *consecutive* if b follows directly after a or $-a$ follows directly after $-b$ in π. Observe that a and b is an ordered pair, so if a and b are consecutive in π, then b and a are in general not. There is a *breakpoint* between a and b in π (relative to id) if a and b are consecutive in id but not in π. We denote the number of breakpoints by $b(\pi)$.

Let U_{2n} denote the set of unsigned genomes with $2n$ genes. Following [44], we define the *genome transformation map* $\text{gtm} : G_n \longrightarrow U_{2n}$ as follows: each gene a in $\pi \in G_n$ is mapped to the pair of genes $(2a-1, 2a)$ if $a > 0$, and mapped to $(-2a, -2a-1)$ if $a < 0$. In the pair of genes obtained from a, we will denote the left element by a_L and the right by a_R. We then take these pairs in the same order as the corresponding genes appear in π. For instance, the genome $\pi =$ `1 -5 3 2 -4` is mapped to the unsigned genome $\text{gtm}(\pi) =$ `1 2 10 9 5 6 3 4 8 7`. Note that the number of breakpoints relative to the identity is preserved by this transformation, that is $b(\pi) = b(\text{gtm}(\pi))$.

The *cycle graph* $G(\pi)$ of $\pi \in G_n$ has the genes in $\text{gtm}(\pi)$ as vertices. There is a black edge between a_R and b_L if a and b are consecutive in π and there is a red edge between an odd and an even gene if these are consecutive in the identity on $2n$ genes. An example of a cycle graph can be viewed in Figure 5.11.

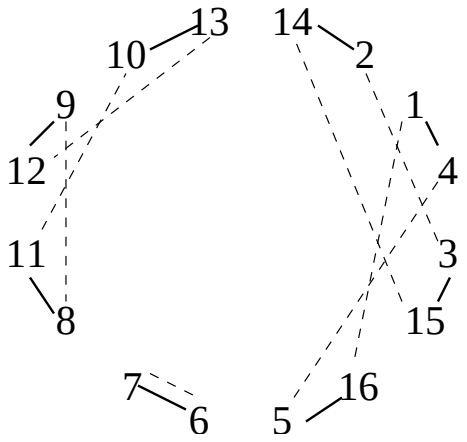


Figure 5.11. The cycle graph of $\pi = \ 1 \ -7 \ -5 \ -6 \ -4 \ -3 \ -8 \ 2$.

It is fairly easy to see that each vertex in $G(\pi)$ has valency two, and that no vertex has two edges of the same colour. Hence, the edges form alternating cycles. We will call the number of black edges in such a cycle the *length* of the cycle.

From now on, we assume the cycle graph of π to be drawn with its vertices on a circle, counterclockwise in the order given by $\text{gtm}(\pi)$. A cycle is *oriented* if it has length 1 or if, when we traverse it, we do not traverse all the black edges in the same direction (clockwise or counterclockwise). Otherwise, the cycle is *unoriented*.

We now present an equivalence relation on the cycles. It is the transitive closure of the relation R, which we now define. An *interval* on a genome is a segment of consecutive genes. We say that two cycles are related in R if, when we take one interval containing all the vertices of the first cycle and another interval containing all the vertices of the second cycle, the intervals are always intersecting. The

equivalence classes are called *components*. A component is *oriented* if it contains at least one oriented cycle and *unoriented* otherwise.

If there is an interval that contains (the vertices of) an unoriented component τ, but no other unoriented components, then τ is known as a *hurdle*. If there is an interval which contains exactly two unoriented components and possibly some oriented ones, and exactly one of these unoriented components is a hurdle, then this hurdle is a *super hurdle*. Finally, if a cycle graph contains an odd number of hurdles, all of which are super hurdles, then this graph is known as a *fortress*.

For $\pi \in G_n$, we define $c_{rev}(\pi)$ to be the number of cycles in $G(\pi)$. Similarly, $h(\pi)$ is its number of hurdles. Finally, $f(\pi)$ is one if $G(\pi)$ is a fortress, zero otherwise. Using these functions, we can formulate the theorem of Hannenhalli and Pevzner.

Theorem 5.2.2 ([44]). *The minimal reversal distance is given by*

$$d_{\mathrm{rev}}(\pi) = n - c_{\mathrm{rev}}(\pi) + h(\pi) + f(\pi).$$

It follows from Caprara [21] that genomes containing hurdles are very rare. For instance, for genomes of length 8, less than one percent of these contain hurdles, and for genomes of length 100, only one in 10^5 contains a hurdle. Thus, there is little harm in using the approximation $d_{\mathrm{rev}}(\pi) \approx n - c_{\mathrm{rev}}(\pi)$, observing that this formula is very similar to the one governing transposition distance in S_n.

5.3 The analogy

We shall now explore the analogy between unsigned transpositions and signed reversals. If we apply a transposition $\tau = (a\ b)$ to a permutation $\pi \in S_n$, one of the following things will happen.

- If a and b belong to different cycles in π, the number of cycles will decrease by one.
- If a and b belong the same cycle in π, the number of cycles will increase by one.

Thus, applying a transposition to π will change the transposition distance by one.

On the other hand, if we apply the reversal $a \dots b$ to $\pi \in G_n$, one of the following things will happen.

- If a_L and b_R belong to different cycles in $\mathrm{gtm}(\pi)$, the number of cycles will decrease by one.
- If a_L and b_R belong to the same cycle and the black edges connected to a_L and b_R are traversed in different directions when we traverse this cycle, the number of cycles will increase by one.

- If a_L and b_R belong to the same cycle and the black edges connected to a_L and b_R are traversed in the same direction when we traverse this cycle, the number of cycles will remain unchanged.

Applying a reversal to a genome π will thus change $d_{\text{rev}}(\pi)$ by one, unless the reversal cuts two equally directed black edges in the same cycle, while not creating or destroying a hurdle or altering the value of $f(\pi)$.

From this analysis, we find that if we apply a random transposition to a permutation π and the corresponding reversal to a genome σ with the same cycle structure, then the distances to the identities will in most cases change by an equal amount. This approximation holds particularly well for permutations and genomes close to the identity. It seems reasonable that the expected distances after t operations will be approximately equal, at least for $t \leq n$, say.

We must not carry this analogy too far; there are major dissimilarities between S_n and G_n. Still, as we will see in the paper, the similarities described above are sufficient to draw conclusions on the behaviour of genomes from the behaviour of permutations, when subject to reversals and transpositions, respectively.

5.4 The Markov chain approach

We wish to compute $\mathbb{E}_{\text{trp}}(n,t)$, the expected transposition distance in S_n given that t random transpositions have been applied to the identity permutation. One possible approach to calculating $\mathbb{E}_{\text{trp}}(n,t)$ would be to let each one of the $n!$ permutations in S_n correspond to a state in a Markov chain, where at each step we apply a transposition, chosen randomly from a uniform distribution. A more economical approach, however, is obtained from the observation that all permutations in some conjugacy class are equally probable. We thus let the conjugacy classes, each one corresponding to an integer partition of n, constitute the states in our Markov chain.

We adopt the convention of sorting the integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ in reverse lexicographic order.

Calculating the transition matrix is not too hard. Say that we wish to compute the probability that we go between states λ and μ. Such a transition is possible if λ, say, has two parts a and b which sum up to one part c of μ, all other parts in λ being equal to the other parts in μ. Then the probability that we go from λ to μ, given that λ has p parts equal to a and q parts equal to b, is $paqb/\binom{n}{2}$ if $a \neq b$ and $\binom{p}{2}a^2/\binom{n}{2}$ otherwise. The probability that we go from μ to λ, given that μ has r parts equal to c, is $cr/\binom{n}{2}$ if $a \neq b$ and $cr/2\binom{n}{2}$ otherwise. In order to obtain integer matrices, we multiply the transition matrices by $\binom{n}{2}$.

Example 5.4.1. *For $n = 4$, the transition matrix multiplied by $\binom{n}{2}$ is given by*

$$M_4 = \begin{pmatrix} 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 1 & 4 & 0 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 4 & 0 \end{pmatrix}$$

What we wish to calculate is $\mathbb{E}_{\mathrm{trp}}(n,t) = e_1 M_n^t w_n^T / \binom{n}{2}^t$, where $e_1 = (1,0,\ldots,0)$ and $w_n = (n - \ell(\lambda))_{\lambda \vdash n}$, where $\ell(\lambda)$ is the number of parts in λ. In other words, w_n contains the transposition distances from the corresponding conjugacy classes to the identity class. In order to do this, we can diagonalize $M_n = V_n D_n V_n^{-1}$. It follows from Ito [51] that each irreducible character χ^λ contributes to the spectrum an eigenvalue $\binom{n}{2}\chi^\lambda(2,1^{n-2})/\chi^\lambda(1^n)$, Hence, the eigenvalue of M_n indexed by λ is the sum $c(\lambda)$ of the *contents* of the diagram of λ, the content of row i, column j being $j - i$. In other words,

$$c(\lambda) = \sum_{i=1}^{\ell(\lambda)} \left(\binom{\lambda_i}{2} - (i-1)\lambda_i \right)$$

(this was known already by Frobenius). Moreover, the eigenvectors are given by the irreducible characters indexed by the corresponding partitions. Since the irreducible characters are orthonormal in the usual inner product

$$\langle \chi^\lambda, \chi^\nu \rangle = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)\chi^\nu(\mu)}{z_\mu},$$

we obtain the inverse of V_n from V_n^T by dividing each column by the appropriate z_λ.

Example 5.4.2. *For $n = 4$, we have the eigenvalues 6, 2, 0, -2 and -6. The matrix V_4 is given by*

$$V_4 = \begin{pmatrix} 1 & 3 & 2 & 3 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 2 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

and its inverse by

$$V_4^{-1} = \frac{1}{4!} \begin{pmatrix} 1 & -6 & 3 & 8 & -6 \\ 3 & -6 & -3 & 0 & 6 \\ 2 & 0 & 6 & -8 & 0 \\ 3 & 6 & -3 & 0 & -6 \\ 1 & 6 & 3 & 8 & 6 \end{pmatrix}$$

With this information, we find that

$$e_1 M_n^t w_n^T = \sum_{\lambda \vdash n} \chi^\lambda(1^n) \left(\frac{c(\lambda)}{\binom{n}{2}} \right)^t \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu) w_\mu}{z_\mu}$$

Theorem 5.4.3. *If $\lambda_3 \geq 2$, then*

$$\sum_{\mu \vdash n} \frac{\chi^\lambda(\mu) w_\mu}{z_\mu} = 0.$$

Otherwise, let $\lambda = (p, q, 1^{n-p-q})$. Then, for $p, q \geq 1$,

$$\sum_{\mu \vdash n} \frac{\chi^\lambda(\mu) w_\mu}{z_\mu} = (-1)^{n-p-q+1} \frac{p-q+1}{(n-q+1)(n-p)},$$

and, for $p = n, q = 0$,

$$\sum_{\mu \vdash n} \frac{\chi^{(n)}(\mu) w_\mu}{z_\mu} = n - \sum_{k=1}^{n} \frac{1}{k}.$$

We postpone the proof of this theorem to Subsection 5.5.1. Using it, we can give a closed formula for the expected transposition distance after t random transpositions in S_n.

Corollary 5.4.4. *The expected transposition distance after t random transpositions in S_n is given by*

$$n - \sum_{k=1}^{n} \frac{1}{k} + \sum_{p=1}^{n-1} \sum_{q=1}^{\min(p,n-p)} a_{pq} \left(\frac{\binom{p}{2} + \binom{q-1}{2} - \binom{n-p-q+2}{2}}{\binom{n}{2}} \right)^t,$$

where

$$a_{pq} = (-1)^{n-p-q+1} \frac{(p-q+1)^2}{(n-q+1)^2(n-p)} \binom{n-p-1}{q-1} \binom{n}{p}.$$

Proof. The character $\chi^\lambda(1^n)$ is given (see [60] or [70]) by the hook-length formula

$$\chi^\lambda(1^n) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

For $\lambda = (p, q, 1^{n-p-q})$, this yields

$$\begin{aligned} \chi^\lambda(1^n) &= \frac{n!(p-q+1)}{(q-1)!(n-p-q)!(n-p)(n-q+1)p!} \\ &= \frac{(p-q+1)}{(n-q+1)} \binom{n-p-1}{q-1} \binom{n}{p}. \end{aligned}$$

Since the sum of the contents of such a partition is

$$\binom{p}{2} + \binom{q-1}{2} - \binom{n-p-q+2}{2},$$

the corollary follows. □

Of interest is the behaviour of the expected distance as t grows (keeping n fixed). Depending on the parity of t, one of two limits is approached. It is not surprising that what we obtain for even (odd) t is exactly the expected distance of a randomly chosen even (odd) permutation of $[n]$ from a uniform distribution. We leave the verification of this statement to the reader.

Corollary 5.4.5. *We have*

$$\lim_{t\to\infty} \mathbb{E}_{\mathrm{trp}}(n, 2t) = n - \sum_{k=1}^{n} \frac{1}{k} + (-1)^{n-1} \frac{1}{n(n-1)}$$

and

$$\lim_{t\to\infty} \mathbb{E}_{\mathrm{trp}}(n, 2t+1) = n - \sum_{k=1}^{n} \frac{1}{k} + (-1)^{n} \frac{1}{n(n-1)}.$$

Proof. As t grows, all terms but one in the double sum of Corollary 5.4.4 tend to zero, the exception being given by $p = q = 1$. This term is $(-1)^{t+n-1}\frac{1}{n(n-1)}$. Substituting $2t$ and $2t+1$, respectively, for t yields the result. □

5.5 Decomposing the length function

Recall that the *length*, $\ell(\lambda)$, of a partition λ is its number of parts. In this section we will use elements of symmetric functions theory in order to prove our main technical tool: a decomposition formula for ℓ. To this end, we briefly review the material we need. For terminology not explained here, we refer the reader e.g. to Macdonald [60] or Stanley [70, Chapter 7].

Let R^n be the vector space (over $\mathbb{Q}$, say) of *class functions*, i.e. functions $f : P_n \longrightarrow \mathbb{Q}$, where P_n is the set of integer partitions of n. The irreducible S_n-characters, $\{\chi^\lambda\}_{\lambda\vdash n}$ form an orthonormal basis of R^n with respect to the inner product defined by $\langle f, g\rangle = \frac{1}{n!}\sum_{\pi\in S_n} f(\mathrm{type}(\pi))g(\mathrm{type}(\pi))$. As a vector space, R^n is isomorphic to the space Λ^n of symmetric functions of degree n via the *characteristic map*, $\mathrm{ch}^n : R^n \longrightarrow \Lambda^n$, defined by $f \mapsto \sum_{\lambda\vdash n} \frac{p_\lambda}{z_\lambda} f(\lambda)$. Here, $\frac{n!}{z_\lambda}$ is the number of permutations of cycle type λ in S_n and $\{p_\lambda\}_{\lambda\vdash n}$ is the Λ^n-basis of *power sums*. We will use one more basis of Λ^n. The *Schur function* s_λ is the image of χ^λ under ch^n, hence the Schur functions form a basis.

If λ and μ are two partitions such that the Ferrers diagram of λ is contained in that of μ, then μ/λ denotes the part of the μ-diagram not contained in λ. We call μ/λ a *border strip* if it is connected (meaning that we can walk from any square to any other without crossing corners) and contains no 2×2 subsquare. The *height*, $ht(\mu/\lambda)$, of the border strip μ/λ is one less than the number of rows in its diagram.

Richard Stanley pointed out to us the usefulness of the following two equations to proving Theorem 5.5.1 below. Letting the first t y-variables be equal to one and

the rest be zero in [70, 7.20], then differentiating with respect to t, putting $t = 1$ and considering only terms of degree n yields

$$\sum_{\lambda \vdash n} \frac{\ell(\lambda)}{z_\lambda} p_\lambda = \sum_{k=1}^{n} \frac{1}{k} s_{(n-k)} p_{(k)}. \tag{5.2}$$

The following is a special case of [70, 7.72]. The sum is over all $\lambda \vdash n$ such that $\lambda/(n-k)$ is a border strip.

$$s_{(n-k)} p_{(k)} = \sum_{\lambda} (-1)^{ht(\lambda/(n-k))} s_\lambda. \tag{5.3}$$

We are now ready to prove the theorem.

Theorem 5.5.1. *Let $\lambda \vdash n$. We have $\ell(\lambda) = \sum_{\mu \vdash n} c_\mu \chi^\mu(\lambda)$, where*

$$c_\mu = \begin{cases} \sum_{k=1}^{n} \frac{1}{k} & \text{if } \mu = (n), \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)} & \text{if } \mu = (p, q, 1^{n-p-q}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Living in R^n, ℓ can be written uniquely as a linear combination of the $\{\chi^\mu\}_{\mu \vdash n}$. Hence, $\ell = \sum c_\mu \chi^\mu$ for some coefficients c_μ. Passing to Λ^n yields

$$\sum_{\mu \vdash n} c_\mu s_\mu = \sum_{\mu \vdash n} \frac{p_\mu}{z_\mu} \ell(\mu).$$

Using (5.2), we get

$$\sum_{\mu \vdash n} c_\mu s_\mu = \sum_{k=1}^{n} \frac{1}{k} s_{(n-k)} p_{(k)},$$

which, with the aid of (5.3), turns into

$$\sum_{\mu \vdash n} c_\mu s_\mu = \sum_{k=1}^{n} \sum_{\mu} \frac{1}{k} (-1)^{ht(\mu/(n-k))} s_\mu,$$

so that

$$c_\mu = \sum \frac{1}{k} (-1)^{ht(\mu/(n-k))},$$

where the sum now is over all k such that $\mu/(n-k)$ is a border strip. This immediately shows that $c_\mu = 0$ unless $\mu = (n)$ or $\mu = (p, q, 1^{n-p-q})$ for some $p \geq q \geq 1$. Now, $(n)/(n-k)$ is always a border strip of height zero, so $c_{(n)} = \sum_{k=1}^{n} \frac{1}{k}$. Finally, $(p, q, 1^{n-p-q})/(n-k)$ is a border strip if and only if $q = n-k+1$ or $p = n-k$. Thus,

$$\begin{aligned} c_{(p,q,1^{n-p-q})} &= (-1)^{n-p-q+1} \frac{1}{n-q+1} + (-1)^{n-p-q} \frac{1}{n-p} \\ &= (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)}. \end{aligned}$$

□

5.5.1 Proof of Theorem 5.4.3

Now we show that Theorem 5.4.3 is a consequence of Theorem 5.5.1. Note that $\mu \mapsto w_\mu$ is a class function and that for fixed $\lambda \vdash n$ we have

$$\sum_{\mu \vdash n} \frac{\chi^\lambda(\mu) w_\mu}{z_\mu} = \langle \chi^\lambda, w_\bullet \rangle.$$

Hence, $\sum_{\mu\vdash n} \frac{\chi^\lambda(\mu)w_\mu}{z_\mu}$ is the coefficient of χ^λ when the class function $w_\bullet$ is written as a linear combination of the irreducible S_n-characters. Now, $w_\mu = n - \ell(\mu)$. Hence, with c_μ as in Theorem 5.5.1, the coefficient of the trivial character $\chi^{(n)}$ is $n - c_{(n)}$, whereas the coefficient of χ^μ, $\mu \neq (n)$, is $-c_\mu$. This concludes the proof of Theorem 5.4.3.

5.5.2 Computing the variance

The methods used above apply not only to computing the expected transposition distance given n and t, but also to computing the variance. The formulae in this case are messier and we confine ourselves to briefly sketching the computations.

Since variance and expectation are related according to $V(X) = E(X^2) - E(X)^2$, what we need to compute is the expected value of the square of the transposition distance. Applying our Markov chain machinery, this amounts to computing the coefficients when the class function $\mu \mapsto (n - \ell(\mu))^2 = n^2 - 2n\ell(\mu) + \ell(\mu)^2$ is written as a linear combination of the irreducible S_n-characters. Passing to the space of symmetric functions, what we need to compute are the coefficients d_μ in the expansion $\sum_{\mu\vdash n} \frac{p_\mu}{z_\mu} \ell(\mu)^2 = \sum_{\mu\vdash n} d_\mu s_\mu$.

Again, we need two equations. The first is obtained in the same way as (5.2) except that we differentiate twice instead of once with respect to t.

$$\sum_{\lambda\vdash n} \frac{p_\lambda}{z_\lambda} \ell(\lambda)(\ell(\lambda) - 1) = \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{1}{jk} s_{(n-j-k)} p_{(j)} p_{(k)} \tag{5.4}$$

The other equation we need is a special case of [70, Theorem 7.17.3]. The first sum is over all $\lambda \vdash n$ such that $\lambda/(n-j-k)$ is a border strip, and the second sum is over all *border strip tableaux* of shape $\lambda/(n-j-k)$ and *type* $(\max(j,k), \min(j,k))$ (see [70] for definitions).

$$s_{(n-j-k)} p_{(j)} p_{(k)} = \sum_\lambda \sum_T (-1)^{ht(T)} s_\lambda. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$d_\mu = c_\mu + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{1}{jk} \sum_T (-1)^{ht(T)},$$

where the second sum is over all border strip tableaux of shape $\mu/(n-j-k)$ and type $(\max(j,k), \min(j,k))$. In particular, $d_\mu = 0$ unless $\mu_4 \leq 2$.

5.6 Experimental results

We have deduced a closed formula for the expected transposition distance after t transpositions. We shall now use it as an approximation for the expected reversal distance after t reversals. By taking the inverse, we obtain an estimate for the expected number of reversals applied in creating a genome with some given reversal distance.

5.6.1 Predicting the true reversal distance

By computing the inverse of $\mathbb{E}_{\text{trp}}$ numerically, we may use it in experiments. We have performed 10000 simulations of evolutionary processes, in which genomes of length 400 have had between 200 and 600 reversals applied to them. We have then used three methods to estimate this evolutionary distance from the resulting genome:

Expected transposition distance. This is of course the method presented in this paper.

Expected reversal distance given breakpoint distance. The method presented in Eriksen [30]. It is fairly accurate, but considers breakpoints only.

Minimal reversal distance. The by now classical method of Hannenhalli and Pevzner. This is exact, but really measures something different from what we want to measure.

Figure 5.12 shows that the estimated evolutionary distance depends approximately linearly on the true evolutionary distance if we use any of the first two estimation methods, but not the third. We also see that we should probably not use any of these methods for more distant genomes than those in our experiments, since the results are getting unreliable at the right end of the diagram. This is only natural, since the distribution of genomes after t reversals will approach the uniform distribution as t grows.

Turning to Table 5.2, we have gathered the mean absolute error and standard deviation obtained using these three methods. As expected, the minimal reversal distance estimates the true evolutionary distance quite poorly. The other two methods are better and quite at par with each other. Looking at the absolute error, the expected transposition distance seems a better choice than the expected reversal distance given breakpoints. Looking at standard deviations, the situation is the opposite. Also note that their arithmetical mean is a slight improvement over both these estimates taken separately. It is an interesting question whether a more sophisticated use of these two methods will give further improvements.

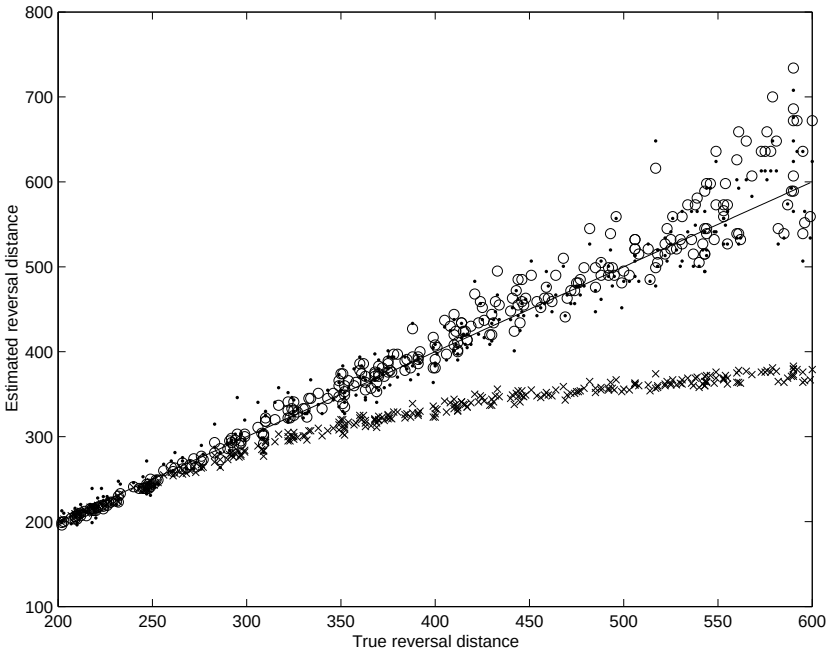


Figure 5.12. Results from using three different methods to obtain the evolutionary reversal distance in 300 of our simulations. The circles come from $\mathbb{E}_{\text{trp}}$, the dots from $\mathbb{E}_{\text{rev}}$and the crosses from the minimal reversal distance. The two former methods keep their linearity throughout this range, whereas the minimal reversal distance estimate is far from linear.

	$\mathbb{E}_{\text{trp}}$	$\mathbb{E}_{\text{rev}}$	$\frac{\mathbb{E}_{\text{trp}}+\mathbb{E}_{\text{rev}}}{2}$	d_{rev}
Mean abs	16.2	18.0	15.8	83.5
St. d.	25.8	24.2	23.0	108.4

Table 5.2. The mean absolute error and the standard deviation obtained from using four methods to estimate the evolutionary reversal distance in simulations. The genomes had length 400, the evolutionary reversal distances were between 200 and 600 and the number of simulations was 500.

5.7 Conclusions

In computational biology, one has to find the fine balance between models that are relevant and models that facilitate computation. For gene order rearrangements, the "reversals only" model has met both criteria as far as regarding minimal dis-

tances, but computation has proved harder for expected distances. With this in mind, it seems natural to look for models with similar behaviour to the reversals model, but with properties better suited for computation.

One such model is ordinary permutations with transpositions. The symmetric group has been well studied over the years and its computational accessibility is undisputed. The interesting question is whether it is suited as a model in the biological context.

In this paper we have seen that as far as expected distances go, we get results that compare well to the best results obtained through other methods. This should encourage us to look for further areas where we could benefit from this model.

One related problem is the *reversal median problem*: determine a genome G (a *median*) which minimizes the sum of the reversal distances from G to three given genomes. This problem is NP-hard, but attempts to use the reversal median have met with some success [5, 61]. The use of transpositions is new to this area, and we hope that it can be useful in the future, for instance by studying the transposition median.

Paper 6.

Expected reflection distance in $G(r, 1, n)$ after t random reflections

6.1 Introduction

Consider the graph G'_{1n} with the symmetric group S_n as vertex set and an edge $\{\pi, \tau\}$ if and only if $\pi = \tau t$ for some transposition t. In [31], the authors computed the expected distance (in the graph-theoretic sense) from the identity after a random walk on G'_{1n} with a fixed number of steps. The motivation was that a random walk on G'_{1n} is a good approximation of a random walk on a graph originating from computational biology: its vertices are the genomes with n genes and its edges correspond to evolutionary events called reversals. Thus, a random walk on the latter graph is thought to simulate evolution. Solving the inverse problem, to compute the expected number of steps given a fixed distance, would then provide a measure for how closely related two taxa are.

In this paper we generalize the mathematical results of [31]. More precisely, we solve the problem described above with S_n replaced by the complex reflection group $G(r, 1, n)$ and the transpositions replaced by the set of reflections.

Our approach is analogous to that in [31]. We view the random walk as a Markov process with a certain transition matrix. This yields an expression for the expected distance which involves two unknown parts, namely the eigenvalues of the said matrix and the inner product of certain (virtual) $G(r, 1, n)$-characters. The eigenvalues are computed using the Murnaghan-Nakayama type formula for $G(r, 1, n)$ given by Ariki and Koike [1]. The inner product is computed using elements of the "symmetric functions" theory of $G(r, 1, n)$-representations, thereby generalizing the corresponding result in [31] to $G(r, 1, n)$.

The paper is organised as follows. In Section 6.2 we review some material about the groups $G(r,1,n)$ and define the appropriate graphs. Thereafter, in Section 6.3, we give a brief sketch of the symmetric functions-like theory that governs $G(r,1,n)$-representations. We then describe the Markov chain approach and state the main theorem in Section 6.4. We do not prove it until in Section 6.7, though, since the proof relies on computation of the eigenvalues and inner product described above; these computations take place in Sections 6.5 and 6.6, respectively.

6.2 The group $G(r,1,n)$ and its reflection graph

Choose positive integers r and n. Let ζ be a primitive rth root of unity in $\mathbb{C}$. We will view $G(r,1,n)$ as the group of permutations π of the set $\{\zeta^i j \mid i \in [r],\, j \in [n]\}$ such that $\pi(\zeta^i j) = \zeta^i \pi(j)$ for all $i \in [r]$, $j \in [n]$. The special cases $r = 1$ and $r = 2$ yield the symmetric group S_n and the hyperoctahedral group B_n, respectively. Both are real reflection groups. In general, $G(r,1,n)$ is a complex reflection group, namely the symmetry group of the regular complex polytope known as the generalized cross-polytope β_n^r (see [66]). Also note that $G(r,1,n)$ is isomorphic to the wreath product $\mathbb{Z}_r \wr S_n$.

An *r-partition* λ of n, written $\lambda \vdash_r n$, is an r-tuple of integer partitions $\lambda = (\lambda^1, \ldots, \lambda^r)$ such that $n = \sum |\lambda^i|$. Consider $\pi \in G(r,1,n)$. It gives rise to an r-partition $\mathrm{type}(\pi) = (\lambda^1, \ldots, \lambda^r) \vdash_r n$ as follows. Write down the disjoint cycle decomposition of π and consider only the absolute values of the entries. This causes some cycles to coincide; those that do belong to the same equivalence class called a *class cycle*. Each class cycle c corresponds to a part in λ^i, i being determined by the requirement that $\pi^k(j) = \zeta^{i-1} j$ for the smallest $k > 0$ such that $|\pi^k(j)| = |j|$, where j is any entry in (a representative of) c. The size of the part is the number of entries in c divided by r. It is straightforward to verify that π and τ are conjugate if and only if $\mathrm{type}(\pi) = \mathrm{type}(\tau)$. Thus, the r-partitions of n index the conjugacy classes (and, hence, the complex irreducible characters) of $G(r,1,n)$.

Example 6.2.1. *With $r = n = 4$ and $\zeta = i = \sqrt{-1}$, the element*

$$(1\ \ -2)(i\ \ -2i)(-1\ 2)(-i\ 2i)(3\ \ -4\ \ -3\ 4)(3i\ \ -4i\ \ -3i\ 4i) \in G(r,1,n)$$

contains two class cycles and has type $(\square, \emptyset, \square, \emptyset)$.

The element π is a *reflection* if λ^1 has exactly $n-1$ parts. We let $T = T(n,r)$ denote the set of reflections. Note that $T(n,1)$ is just the set of transpositions in S_n.

Although an arbitrary $t \in T$ is not in general conjugate to t^{-1}, we still have $t^{-1} \in T$. Hence, there is no ambiguity in the definition we now give. We let G'_{rn} be the graph with the elements of $G(r,1,n)$ as vertices and an edge $\{x, y\}$ if and only if $x = yt$ for some $t \in T$. We call G'_{rn} the *reflection graph* of $G(r,1,n)$.

It is well-known that the reflection distance $w_{rn}(\pi)$, i.e. the graph-theoretic distance between the identity and π in G'_{rn}, is given by n minus the number of class cycles of π that contain exactly r cycles. In other words,

$$w_{rn}(\pi) = n - \ell(\lambda^1). \tag{6.6}$$

Remark. In case $r \in \{1, 2\}$, the reflection graph is just the undirected version of the Bruhat graph on the Coxeter group $G(r, 1, n)$ defined by Dyer [28].

6.3 Irreducible characters and "symmetric functions"

In this section, we briefly review some of the theory of $G(r, 1, n)$-representations, which in many ways resembles the theory of symmetric functions (the $r = 1$ case). We refer to Macdonald [60, Chapter I, Appendix B] for more details. Some knowledge of "ordinary" symmetric functions will be assumed, see e.g. [60] or Stanley [70, Chapter 7].

The irreducible characters of $G(r, 1, n)$ are indexed by the r-partitions of n; we write χ^λ for the character indexed by $\lambda \vdash_r n$. They form an orthonormal basis of the $\mathbb{C}$-vector space $R^n(r)$ of class functions on $G(r, 1, n)$ with respect to the inner product

$$\langle f, g \rangle = \sum_{\lambda \vdash_r n} \frac{f(\lambda)\overline{g(\lambda)}}{Z_\lambda},$$

where

$$Z_\lambda = z_{\lambda^1} \dots z_{\lambda^r} r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)}.$$

For $i \in [r]$, let $x_i = (x_{i1}, x_{i2}, \dots)$. Given $\lambda \vdash_r n$, we define

$$P_\lambda = \prod_{i=1}^{r} p_{\lambda^i}(x_i) \in \mathbb{C}[[x_1, \dots, x_r]],$$

where the p_μ are the ordinary power sum functions. Let $\Lambda^n(r)$ denote the $\mathbb{C}$-span of $\{P_\lambda\}_{\lambda \vdash_r n}$. It turns out that the *characteristic map* $\mathrm{ch}^n : R^n(r) \longrightarrow \Lambda^n(r)$ given by $f \mapsto \sum_{\lambda \vdash_r n} \frac{P_\lambda}{Z_\lambda} f(\lambda)$ is a vector space isomorphism.

Multiplication of power series turns $\Lambda(r) = \bigoplus_{n \geq 0} \Lambda^n(r)$ into a graded algebra. The same holds for $R(r) = \bigoplus_{n \geq 0} R^n(r)$ (under a suitably defined multiplication whose nature needs not concern us here). Taking the characteristic map on each component then yields an isomorphism of graded algebras $\mathrm{ch} : R(r) \longrightarrow \Lambda(r)$ which we also call the characteristic map.

Now, consider another set of variables: for $i \in [r]$, put $\tilde{x}_i = (\tilde{x}_{i1}, \tilde{x}_{i2}, \dots)$. The connection between the x_i and the $\tilde{x}_i$ is governed by the transformation rules

$$p_m(\tilde{x}_i) = \sum_{j\in[r]} \frac{1}{r} T_{i,j} p_m(x_j)$$

and

$$p_m(x_i) = \sum_{j\in[r]} \overline{T_{j,i}} p_m(\tilde{x}_j),$$

where T is the character table of $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$. In particular, adopting the convention that the trivial $\mathbb{Z}_r$-character corresponds to the first row in T and the (conjugacy class consisting of the) identity element corresponds to the first column, we have $T_{i1} = T_{1i} = 1$ for all i. Hence,

$$p_m(\tilde{x}_1) = \sum_{j\in[r]} \frac{1}{r} p_m(x_j) \tag{6.7}$$

and

$$p_m(x_1) = \sum_{j\in[r]} p_m(\tilde{x}_j). \tag{6.8}$$

The main reason to care about this second set of variables is the following. For $\lambda \vdash_r n$, define

$$\widetilde{S}_\lambda = \prod_{i\in[r]} s_{\lambda^i}(\tilde{x}_i) \in \mathbb{C}[[\tilde{x}_1, \tilde{x}_2, \dots]],$$

where the s_μ are the ordinary Schur functions. Then $\widetilde{S}_\lambda$ is the image of χ^λ under the characteristic map.

6.4 The Markov chain

We wish to view the walk on G'_{rn} as a Markov process. We can then use the properties of the transition matrix to compute the expected reflection distance. Our approach is analogous to the approach in [31].

Associated with the Cayley graph G'_{rn} is its adjacency matrix M'_{rn} with rows and columns indexed by the vertices in G'_{rn} and with entries indicating the number of edges (one or zero) between the corresponding vertices. The probability that a random walk on G'_{rn} starting in the identity ends up in π depends only on the type of π. Hence, to reduce the size of the problem, we may group the permutations into conjugacy classes, each indexed by its type. We then get the corresponding (multi-) graph G_{rn} with adjacency matrix $M_{rn} = (m_{ij})$, the number m_{ij} denoting the number of G'_{rn}-edges from some permutation of type i to the set of permutations of type j.

Example 6.4.1. *The group $G(2,1,2)$ has 8 elements of 5 different types. If the latter are ordered according to*

$$(\ydiagram{1},\emptyset),(\ydiagram{2},\emptyset),(\ydiagram{1,1},\ydiagram{1,1}),(\emptyset,\ydiagram{1}),(\emptyset,\ydiagram{2}),$$

we get

$$M_{22}=\begin{pmatrix}0&2&2&0&0\\1&0&0&1&2\\1&0&0&1&2\\0&2&2&0&0\\0&2&2&0&0\end{pmatrix}$$

We view this adjacency matrix as a transition matrix in a Markov chain (after normalizing by the common row sum $|T|$). It is easy to see that the expected reflection distance after t reflections taken from a uniform distribution is given by (see for instance [31]) $e_1M_{rn}^t w_{rn}^T/|T|^t$, where w_{rn} is a row vector containing the reflection distances from the different types to the identity. The row vector e_1 has 1 in the first position and zeroes everywhere else.

In order to compute $e_1M_{rn}^t w_{rn}^T$, we wish to diagonalize M_{rn}. It follows from Ito [51] that its eigenvalues are given by

$$\operatorname{eig}(M_{rn},\lambda)=\sum_i \frac{n_i\chi^\lambda(i)}{\chi^\lambda(1^n,\emptyset,\dots,\emptyset)}, \tag{6.9}$$

for $\lambda\vdash_r n$. Here, n_i is the number of elements of type i in $G(r,1,n)$, and the sum is taken over all reflection types i. For $r=1$, the eigenvalues equal the sum $c(\lambda)$ of the *contents* of the diagram of λ, the content of row i, column j being $j-i$. In other words,

$$c(\lambda)=\frac{\binom{n}{2}\chi^\lambda(2,1^{n-2})}{\chi^\lambda(1^n)}=\sum_{i=1}^{\ell(\lambda)}\left(\binom{\lambda_i}{2}-(i-1)\lambda_i\right) \tag{6.10}$$

(see [31, 51]). We will compute the other eigenvalues in Section 6.5.

The eigenvector corresponding to $\operatorname{eig}(M_{rn},\lambda)$ is given by the values of χ^λ on the various conjugacy classes, see [51]. Hence, viewing the character table C as a matrix, we can diagonalize: $M_{rn}=C^TD(C^T)^{-1}$, where D is the diagonal matrix with the eigenvalues on the diagonal. Using the orthogonality of irreducible characters, we compute $(C^T)^{-1}$; it is obtained from C by dividing each column by the corresponding Z_μ. We obtain

$$e_1M_{rn}^t w_{rn}^T=\sum_{\lambda\vdash_r n}\chi^\lambda((1^n),\emptyset,\dots,\emptyset)(\operatorname{eig}(M_{rn},\lambda))^t\sum_{\mu\vdash_r n}\frac{\chi^\lambda(\mu)w_{rn}(\mu)}{Z_\mu}.$$

In Section 6.6, we decompose $w_{rn}(\mu)$ into a linear combination of the irreducible $G(r,1,n)$-characters, thus obtaining an expression for the second sum.

Combining all parts, we obtain the main theorem, thus extending the corresponding result for $r=1$ in [31].

Theorem 6.4.2. *Assume $r,n \in \mathbb{N}$ and $rn > 1$. Then the expected reflection distance after t random reflections in $G(r,1,n)$ is given by*

$$n - \frac{1}{r}\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{r}\sum_{p=1}^{n-1}\sum_{q=1}^{\min(p,n-p)} a_{pq}\left(\frac{r\left(\binom{p}{2}+\binom{q-1}{2}-\binom{n-p-q+2}{2}+n\right)-n}{r\binom{n+1}{2}-n}\right)^t + \frac{r-1}{r}\sum_{p=0}^{n-1}\sum_{q=1}^{n-p} b_{pq}\left(\frac{r\left(\binom{p}{2}+\binom{q}{2}-\binom{n-p-q+1}{2}+p\right)-n}{r\binom{n+1}{2}-n}\right)^t,$$

where

$$a_{pq} = (-1)^{n-p-q+1}\frac{(p-q+1)^2}{(n-q+1)^2(n-p)}\binom{n}{p}\binom{n-p-1}{q-1}$$

and

$$b_{pq} = \frac{(-1)^{n-p-q+1}}{n-p}\binom{n}{p}\binom{n-p-1}{q-1}.$$

The proof of this theorem is postponed until Section 6.7, since it uses the material derived in the following two sections.

Example 6.4.3. *Returning to the case $n = r = 2$, M_{22} diagonalizes as*

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 1 & 2 & 2 & 1 & 2 \\ 1 & -2 & 2 & 1 & -2 \\ 2 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & 1 & -2 \\ 1 & -2 & -2 & 1 & 2 \end{pmatrix}.$$

We recognize the leftmost matrix in this expression; it is the transpose of the character table of $G(2,1,2)$.

Plugging $n = r = 2$ into Theorem 6.4.2, we get (for $t > 0$)

$$2 - \frac{3}{4} + \frac{1}{2}\frac{(-1)}{2}0^t + \frac{1}{2}\left(\frac{1}{2}\cdot(-1)^t - \frac{1}{2}\cdot 0^t - 2\cdot 0^t\right) = \frac{5}{4} + \frac{(-1)^t}{4}.$$

The asymptotics are now fairly easy to deal with. For $r = 1, 2$ we have a dependence on the parity of t reflecting the bipartite nature of G_{rn}. For larger r, G_{rn} is no longer bipartite, and this behaviour disappears.

Corollary 6.4.4. *As t goes to infinity, the expected reflection distance in $G(r,1,n)$ approaches*

$$n - \frac{1}{r}\sum_{k=1}^{n}\frac{1}{k} + \delta,$$

where

$$\delta = \begin{cases} \frac{(-1)^{n-1}}{n(n-1)} & \text{if } r = 1 \text{ and } t \text{ is even,} \\ \frac{(-1)^{n}}{n(n-1)} & \text{if } r = 1 \text{ and } t \text{ is odd,} \\ \frac{(-1)^{n}}{2n} & \text{if } r = 2 \text{ and } t \text{ is even,} \\ \frac{(-1)^{n+1}}{2n} & \text{if } r = 2 \text{ and } t \text{ is odd,} \\ 0 & \text{if } r \geq 3. \end{cases}$$

Proof. The case $r = 1$ was carried out in [31], so suppose $r \geq 2$. Consider the expressions inside the large brackets preceded by a_{pq} and b_{pq} in Theorem 6.4.2; call them B_1 and B_2, respectively. It is easily checked that $|B_1| < 1$ for all p, q. It is equally simple to verify that $|B_2| < 1$ unless $p = 0$, $q = 1$, $r = 2$ in which case we have $B_2 = -1$. Noting that $b_{01} = (-1)^n/n$ proves the corollary.

□

6.5 The eigenvalues of the adjacency matrices

To compute the eigenvalues of M_{rn}, we use the following $G(r, 1, n)$-version of the Murnaghan-Nakayama formula which can be found in Ariki and Koike [1].

Theorem 6.5.1 ([1]). *For fixed i and j, let μ^i_j be the jth part of μ^i in $\mu = (\mu^1, \ldots, \mu^r)$, and let ζ be a primitive rth root of unity in $\mathbb{C}$. Then*

$$\chi^\lambda(\mu) = \sum_{p=1}^{r} \sum_{|\Gamma| = \mu^i_j} (-1)^{\mathrm{ht}(\Gamma)} \zeta^{(i-1)(p-1)} \chi^{(\lambda^1, \ldots, \lambda^p - \Gamma, \ldots, \lambda^r)}(\mu - \mu^i_j),$$

where the second sum runs over all rim hooks Γ of size μ^i_j in λ^p and $\mathrm{ht}(\Gamma)$ is one less than the number of rows in Γ.

From (6.9), it follows that the eigenvalue corresponding to $\lambda \vdash_r n$ is

$$\frac{r\binom{n}{2}\chi^\lambda((2, 1^{n-2}), \emptyset, \ldots, \emptyset)}{\chi^\lambda(1^n, \emptyset, \ldots, \emptyset)} + \sum_{i=2}^{r} \frac{n\chi^\lambda(1^{n-1}, \emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset)}{\chi^\lambda(1^n, \emptyset, \ldots, \emptyset)}, \tag{6.11}$$

where, in the second sum, the ith argument of χ^λ is 1.

We thus need to compute some entries in the character table of $G(r, 1, n)$.

Lemma 6.5.2. *For any $\lambda = (\lambda^1, \ldots, \lambda^r) \vdash_r n$, we have*

$$\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = \binom{n}{|\lambda^1|, \ldots, |\lambda^r|} \prod_{k=1}^r \chi^{\lambda^k}(1^{|\lambda^k|}),$$
$$\chi^\lambda((2,1^{n-2}), \emptyset, \ldots, \emptyset) =$$
$$\sum_{p=1}^r \binom{n-2}{|\lambda^1|, \ldots, |\lambda^p|-2, \ldots, |\lambda^r|} \chi^{\lambda^p}(2,1^{|\lambda^p|-2}) \prod_{k \neq p} \chi^{\lambda^k}(1^{|\lambda^k|})$$

and

$$\chi^\lambda(1^{n-1}, \emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset) =$$
$$\sum_{p=1}^r \binom{n-1}{|\lambda^1|, \ldots, |\lambda^p|-1, \ldots, |\lambda^r|} \zeta^{(i-1)(p-1)} \prod_{k=1}^r \chi^{\lambda^k}(1^{|\lambda^k|}).$$

(In the last equation, the ith argument of χ^λ is 1.)

Proof. For $\mu = (1^n, \emptyset, \ldots, \emptyset)$, the Murnaghan-Nakayama rule becomes

$$\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = \sum_{p=1}^r \sum_{\square} \chi^{(\lambda^1, \ldots, \lambda^p - \square, \ldots, \lambda^r)}(1^{n-1}, \emptyset, \ldots, \emptyset),$$

where the inner sum runs over all outer squares of λ^p. Thus, the character equals the number of ways to remove one outer square at a time from the Ferrers' diagrams of λ. This number is $\binom{n}{|\lambda^1|, \ldots, |\lambda^r|} \prod_{k=1}^r \chi^{\lambda^k}(1^{|\lambda^k|})$, since $\chi^{\lambda^k}(1^{|\lambda^k|})$ is the number of ways to remove one outer square at a time from λ^k.

The two other equations follow similarly. When $\mu = ((2,1^{n-2}), \emptyset, \ldots, \emptyset)$, we first remove a rim hook of size 2 from some λ^p. For $\mu = (1^{n-1}, \emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset)$, we start by removing the square corresponding to μ^j.

□

We are now ready to compute the eigenvalues.

Theorem 6.5.3. *Let $\lambda = (\lambda^1, \ldots, \lambda^r) \vdash_r n$. The eigenvalue of M_{rn} corresponding to λ is given by*

$$\mathrm{eig}(M_{rn}, \lambda) = r \sum_{p=1}^r c(\lambda^p) + r|\lambda^1| - n.$$

Proof. Combining equation (6.11) and Lemma 6.5.2, the eigenvalue becomes

$$r\sum_{p=1}^{r}\binom{|\lambda^p|}{2}\frac{\chi^{\lambda^p}(2,1^{|\lambda^p|-2})}{\chi^{\lambda^p}(1^{|\lambda^p|})}+\sum_{p=1}^{r}|\lambda^p|\sum_{j=2}^{r}\zeta^{(j-1)(p-1)}.$$

But

$$\binom{|\lambda^p|}{2}\frac{\chi^{\lambda^p}(2,1^{|\lambda^p|-2})}{\chi^{\lambda^p}(1^{|\lambda^p|})}=\mathrm{eig}(M_{1|\lambda^p|},\lambda^p)=c(\lambda^p),$$

and

$$\sum_{j=2}^{r}\zeta^{(j-1)(p-1)}=\sum_{j=1}^{r}\zeta^{(j-1)(p-1)}-1=\begin{cases}r-1 & \text{if } p=1,\\ -1 & \text{otherwise.}\end{cases}$$

□

6.6 Decomposing the distance function

Recall from (6.6) the distance function w_{rn} in the reflection graph of $G(r,1,n)$. Being a class function, it can be written as a linear combination of the irreducible $G(r,1,n)$-characters. In this section, we will make this decomposition explicit using the material reviewed in Section 6.3. In [31], the symmetric group case ($r=1$) was settled using a similar approach. However, the fact that $x_i \neq \tilde{x}_i$ for larger r calls for greater care.

Before stating the main theorem we need some preliminary results.

Proposition 6.6.1. *The complete symmetric functions satisfy*

$$\prod_{i=1}^{r}\sum_{n\geq 0}h_n(x_i)=\left(\sum_{n\geq 0}h_n(\tilde{x}_1)\right)^r.$$

Proof. Throughout the proof, lower case Greek letters with or without superscripts, such as μ and μ^i, will denote ordinary integer partitions.

First, we manipulate the left hand side a little to obtain

$$\prod_{i=1}^{r}\sum_{n\geq 0}h_n(x_i)=\prod_{i=1}^{r}\sum_{\mu}\frac{p_\mu(x_i)}{z_\mu}=\sum_{(\mu^1,\ldots,\mu^r)}\frac{p_{\mu^1}(x_1)\ldots p_{\mu^r}(x_r)}{z_{\mu^1}\ldots z_{\mu^r}}.$$

Turning to the right hand side, using (6.7), we get

$$\left(\sum_{n\geq 0} h_n(\tilde{x}_1)\right)^r = \left(\sum_{\mu} \frac{p_\mu(\tilde{x}_1)}{z_\mu}\right)^r = \left(\sum_{\mu} \frac{p_{\mu_1}(\tilde{x}_1)\dots p_{\mu_{\ell(\mu)}}(\tilde{x}_1)}{z_\mu}\right)^r$$
$$= \left(\sum_{\mu} \frac{1}{z_\mu} \prod_{i=1}^{\ell(\mu)} \frac{p_{\mu_i}(x_1)+\dots+p_{\mu_i}(x_r)}{r}\right)^r$$
$$= \sum_{(\mu^1,\dots,\mu^r)} \frac{1}{z_{\mu^1}\dots z_{\mu^r} r^{\ell(\mu^1)+\dots+\ell(\mu^r)}} \prod_{j=1}^{r}\prod_{i=1}^{\ell(\mu^j)} (p_{\mu_i^j}(x_1)+\dots+p_{\mu_i^j}(x_r)).$$

For appropriate coefficients $K_{\mu^1,\dots,\mu^r}$, this expression can be written as

$$\sum_{(\mu^1,\dots,\mu^r)} K_{\mu^1,\dots,\mu^r} p_{\mu^1}(x_1)\dots p_{\mu^r}(x_r).$$

Fix $\lambda^1,\dots,\lambda^r$. We must show that $K_{\lambda^1,\dots,\lambda^r} = (z_{\lambda^1}\dots z_{\lambda^r})^{-1}$.

Consider the last expression for the right hand side above. For a term indexed by $(\mu^1,\dots,\mu^r)$, let f_i^j be the number of parts that equal i in μ^j. Similarly, let e_i^j be the number of parts that equal i in λ^j and put $N_i = \sum_j e_i^j$. Clearly, the only terms that contribute to $K_{\lambda^1,\dots,\lambda^r}$ are those for which $\sum_j f_i^j = N_i$ for all i. Below, the sums are over all such $\mu^1,\dots,\mu^r$ (so that, in particular, $\ell(\lambda^1)+\dots+\ell(\lambda^r) = \ell(\mu^1)+\dots+\ell(\mu^r)$). We get

$$K_{\lambda^1,\dots,\lambda^r} = \frac{1}{r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)}} \sum_{(\mu^1,\dots,\mu^r)} \frac{1}{z_{\mu^1}\dots z_{\mu^r}} \prod_{i\geq 1} \binom{N_i}{e_i^1,\dots,e_i^r}$$
$$= \frac{1}{r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)}} \sum_{(\mu^1,\dots,\mu^r)} \frac{\prod_{k=1}^r (\prod_j e_j^k!)}{z_{\lambda^1}\dots z_{\lambda^r} \prod_{k=1}^r (\prod_j f_j^k!)} \prod_{i\geq 1} \binom{N_i}{e_i^1,\dots,e_i^r}$$
$$= \frac{1}{r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)} z_{\lambda^1}\dots z_{\lambda^r}} \sum_{(\mu^1,\dots,\mu^r)} \prod_{i\geq 1} \binom{N_i}{f_i^1,\dots,f_i^r}.$$

To simplify this sum, consider the following situation: we have r boxes of distinguishable balls, the ith box containing N_i balls, and we wish to paint the balls using (at most) r colours. Of course, colouring the balls one by one, this can be done in $r^{\sum N_i} = r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)}$ ways. Another way to colour the balls is this: first choose $(\mu^1,\dots,\mu^r) \vdash_r n$ with $\sum_j f_i^j = N_i$ for all i. In box i, there will be f_i^j balls with colour j; this box can be coloured in $\binom{N_i}{f_i^1,\dots,f_i^r}$ ways. Thus,

$$K_{\lambda^1,\dots,\lambda^r} = \frac{1}{r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)} z_{\lambda^1}\dots z_{\lambda^r}} r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)} = \frac{1}{z_{\lambda^1}\dots z_{\lambda^r}},$$

as desired. □

Let $L \in R(r)$ be the function $\lambda \mapsto \ell(\lambda^1)$. Sometimes, by abuse of notation, we will let L denote its restriction to $R^n(r)$.

Lemma 6.6.2. *We have*

$$\mathrm{ch}(L) = \frac{1}{r}\sum_{n\geq 1}\frac{1}{n}p_n(x_1)\prod_{i=1}^{r}\left(\sum_{m\geq 0}h_m(x_i)\right)^{\frac{1}{r}}.$$

Proof. Again, in this proof symbols such as μ and μ^i will denote ordinary integer partitions.

Letting the first t y-variables equal one and the rest be zero in [70, 7.20] yields

$$\sum_{\mu}\frac{p_\mu}{z_\mu}t^{\ell(\mu)} = \exp\left(\sum_{n\geq 1}\frac{t}{n}p_n\right).$$

Hence, letting t_i be independent indeterminates,

$$\prod_{i=1}^{r}\sum_{\mu}\frac{p_\mu(x_i)}{z_\mu}t_i^{\ell(\mu)} = \exp\left(\sum_{i=1}^{r}\sum_{n\geq 1}\frac{t_i}{n}p_n(x_i)\right).$$

Differentiating with respect to t_1, we obtain

$$\sum_{\mu}\frac{p_\mu(x_1)}{z_\mu}\ell(\mu)t_1^{\ell(\mu)-1}\prod_{i=2}^{r}\sum_{\nu}\frac{p_\nu(x_i)}{z_\nu}t_i^{\ell(\nu)} = \sum_{n\geq 1}\frac{p_n(x_1)}{n}\exp\left(\sum_{i=1}^{r}\sum_{m\geq 1}\frac{t_i}{m}p_m(x_i)\right),$$

which, after putting all $t_i = \frac{1}{r}$, becomes

$$r\sum_{(\mu^1,\ldots,\mu^r)}\frac{p_{\mu^1}(x_1)\ldots p_{\mu^r}(x_r)}{z_{\mu^1}\ldots z_{\mu^r}r^{\ell(\mu^1)+\cdots+\ell(\mu^r)}}\ell(\mu^1) = \sum_{n\geq 1}\frac{p_n(x_1)}{n}\exp\left(\frac{1}{r}\sum_{i=1}^{r}\sum_{m\geq 1}\frac{p_m(x_i)}{m}\right).$$

Now, the left hand side is in fact $r\mathrm{ch}(L)$. The fact that $\exp\left(\sum_{m\geq 1}\frac{p_m}{m}\right) = \sum_{m\geq 0}h_m$ concludes the proof.

□

We are now in position to state and prove the main result of this section.

Theorem 6.6.3. *For all $\lambda \vdash_r n$, $L(\lambda) = \sum_{\mu\vdash_r n} c_\mu\chi^\mu(\lambda)$, where*

$$c_\mu = \begin{cases} \sum_{k=1}^{n}\frac{1}{rk} & \text{if } \mu^1 = (n),\\ (-1)^{n-p-q}\frac{p-q+1}{r(n-q+1)(n-p)} & \text{if } \mu^1 = (p,q,1^{n-p-q}),\\ \frac{(-1)^{n-p-q}}{r(n-p)} & \text{if } \mu^1 = (p) \text{ and } \mu^i = (q,1^{n-p-q}) \text{ for some } i,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Passing to $\Lambda^n(r)$, we want to compute the coefficients c_μ in the expression $\mathrm{ch}^n(L) = \sum_{\mu \vdash_r n} c_\mu \widetilde{S}_\mu$.

Combining Lemma 6.6.2 and Proposition 6.6.1 yields

$$\mathrm{ch}(L) = \frac{1}{r} \sum_{n \geq 1} \frac{1}{n} p_n(x_1) \sum_{m \geq 0} h_m(\tilde{x}_1),$$

which, with the aid of (6.8), becomes

$$\mathrm{ch}(L) = \frac{1}{r} \sum_{n \geq 1} \frac{1}{n} \sum_{i=1}^{r} p_n(\tilde{x}_i) \sum_{m \geq 0} h_m(\tilde{x}_1).$$

Now, define coefficients c_μ^i by writing

$$\sum_{m \geq 0} h_m(\tilde{x}_1) \sum_{n \geq 1} \frac{1}{n} p_n(\tilde{x}_i) = \sum_{n} \sum_{\mu \vdash_r n} c_\mu^i \widetilde{S}_\mu,$$

so that $r c_\mu = \sum_{i=1}^{r} c_\mu^i$. For the rest of this proof, we let $\mu \vdash_r n$ be fixed.

First, we consider the case $i = 1$. Using [70, 7.72], it is not difficult to show that

$$c_\mu^1 = \begin{cases} \sum_{k=1}^{n} \frac{1}{k} & \text{if } \mu^1 = (n), \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)} & \text{if } \mu^1 = (p, q, 1^{n-p-q}), \\ 0 & \text{otherwise;} \end{cases}$$

see the proof of [31, Theorem 3] for the details.

Now, pick $i > 1$. It is well-known that

$$p_n = \sum_{q=1}^{n} (-1)^{n-q} s_{(q, 1^{n-q})}.$$

Hence, recalling that $s_m = h_m$, we obtain

$$c_\mu^i = \begin{cases} \frac{(-1)^{n-p-q}}{n-p} & \text{if } \mu^1 = (p) \text{ and } \mu^i = (q, 1^{n-p-q}), \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. □

What we really need is the decomposition of the distance function w_{rn}, not L. It is now easily obtained.

Corollary 6.6.4. *For all $\lambda \vdash_r n$, $w_{rn}(\lambda) = \sum_{\mu \vdash_r n} d_\mu \chi^\mu(\lambda)$, where*

$$d_\mu = \begin{cases} n - \sum_{k=1}^{n} \frac{1}{rk} & \text{if } \mu^1 = (n), \\ -c_\mu & \text{otherwise.} \end{cases}$$

Here, c_μ is as in Theorem 6.6.3.

Proof. We know that $w_{rn} = n\chi_{\mathrm{triv}} - L$, where χ_{triv} is the trivial character. Since the trivial character is indexed by $(n, \emptyset, \ldots, \emptyset)$, the result follows. □

6.7 Proof of Theorem 6.4.2

We now turn to the proof of our main theorem. We have already shown that the expected reflection distance after t random reflections is given by

$$\sum_{\lambda \vdash_r n} \chi^\lambda(1^n, \emptyset, \ldots, \emptyset) \left(\frac{\mathrm{eig}(M_{rn}, \lambda)}{|T|}\right)^t \sum_{\mu \vdash_r n} \frac{\chi^\lambda(\mu) w_{rn}(\mu)}{Z_\mu}$$
$$= \sum_{\lambda \vdash_r n} \chi^\lambda(1^n, \emptyset, \ldots, \emptyset) \left(\frac{\mathrm{eig}(M_{rn}, \lambda)}{|T|}\right)^t \langle \chi^\lambda, w_{rn} \rangle.$$

If we decompose $w_{rn}(\mu) = \sum_{\lambda \vdash_r n} d_\lambda \chi^\lambda(\mu)$ and use that the number of reflections is $|T| = r\binom{n+1}{2} - n$, we obtain

$$\sum_{\lambda \vdash_r n} d_\lambda \chi^\lambda(1^n, \emptyset, \ldots, \emptyset) \left(\frac{\mathrm{eig}(M_{rn}, \lambda)}{r\binom{n+1}{2} - n}\right)^t.$$

The coefficients d_λ are zero for most λ, the exceptions being $\lambda^1 = (n)$, $\lambda^1 = (p, q, 1^{n-p-q})$ and $\lambda = (p, \emptyset, \ldots, \emptyset, (q, 1^{n-p-q}), \emptyset, \ldots, \emptyset)$.

If $\lambda^1 = (n)$, we have $d_\lambda = n - \sum_{k=1}^n \frac{1}{rk}$, $\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = 1$ (since χ^λ is the trivial character) and $\mathrm{eig}(M_{rn}, \lambda) = r\binom{n}{2} + (r-1)n = r\binom{n+1}{2} - n$, so we get

$$d_\lambda \chi^\lambda(1^n, \emptyset, \ldots, \emptyset) \left(\frac{\mathrm{eig}(M_{rn}, \lambda)}{r\binom{n+1}{2} - n}\right)^t = n - \sum_{k=1}^n \frac{1}{rk}.$$

Similarily, if $\lambda^1 = (p, q, 1^{n-p-q})$, we obtain $d_\lambda = (-1)^{n-p-q+1} \frac{p-q+1}{r(n-q+1)(n-p)}$,

$$\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = \frac{n!(p-q+1)}{(q-1)!(n-p-q)!(n-p)(n-q+1)p!}$$
$$= \frac{(p-q+1)}{(n-q+1)} \binom{n-p-1}{q-1} \binom{n}{p}$$

and $\mathrm{eig}(M_{rn}, \lambda) = rc(p, q, 1^{n-p-q}) + (r-1)n$.

Finally, if $\lambda^1 = (p)$ and $\lambda^i = (q, 1^{n-p-q})$, we have $d_\lambda = \frac{(-1)^{n-p-q+1}}{r(n-p)}$,

$$\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = \binom{n}{p} \chi^{(p)}(1^p) \chi^{(q, 1^{n-p-q})}(1^{n-p}) = \binom{n}{p} \binom{n-p-1}{q-1}$$

and

$$\mathrm{eig}(M_{rn}, \lambda) = r\left(\binom{p}{2} + \binom{q}{2} - \binom{n-p-q+1}{2} + p\right) - n.$$

Putting it all together yields the theorem.

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