KTH Mathematics

# Cover times, sign-dependent random walks, and maxima

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# Abstract

This thesis consists of four independent papers. The first three papers are closely connected while the fourth paper treats a slightly different problem.

In the first paper, results for cover times of simple random walks are generalized to hold for random walks with independent and identical exponentially distributed holding times. Transform techniques are used to derive the distributions. The transform for the density function of the cover time is appropriately scaled and a limit distribution is obtained. The distributions and the convergence to a limit distribution are graphically illustrated.

In the second paper the model is generalized to a sign-dependent random walk. A sign-dependent random walk is a simple random walk where the one-step transition probabilities can be different on the positive and negative half-line. The transform for the probability (density) function of the first passage time is derived, both for a sign-dependent random walk with constant holding times (probability generating function) and with independent and identical exponentially distributed holding times (Laplace transform). Some extensions of the first passage time are studied. The transform of the density function for the first passage time is scaled and a limit distribution is obtained. The Laplace transforms are numerically inverted and the distributions are illustrated by graphs.

The first passage time is used in the third paper to obtain transition probabilities for a sign-dependent random walk (also by transforms). After suitable scaling, weak convergence for a sign-dependent random walk to a certain diffusion is shown. In some special cases the transform for the limit distribution is possible to invert analytically. In the other cases numerical inversion is used.

The fourth paper concerns the number of maxima points in a three-dimensional cube where points are randomly placed. The main result is an exact explicit formula for the variance of the number of maxima. Previous formulas for the variance only contain the leading term in an asymptotic formula which give misleading estimates.


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# Contents

# Chapter 1

# Introduction

This thesis consists of two parts. The first part, Chapters 2–4, concerns mainly cover times, sign-dependent random walks and limits connected with them. Chapter 5 is the second part of the thesis, where the number of maxima in a three-dimensional cube is studied. A shorter published version of Chapter 5 is Carlsund (2003).

For a sign-dependent random walk $\{S_n\}_{n\geq 0}$, $S_n = X_1 + X_2 + \cdots + X_n$, the one-step transitions are given by

$$X_{n+1} = \begin{cases} 1 & \text{w.p.} & p_+ & \text{for } S_n > 0 \\ -1 & \text{w.p.} & q_+ & \text{for } S_n > 0 \\ & & & \\ 1 & \text{w.p.} & p_0 & \text{for } S_n = 0 \\ -1 & \text{w.p.} & q_0 & \text{for } S_n = 0 \\ & & & \\ 1 & \text{w.p.} & p_- & \text{for } S_n < 0 \\ -1 & \text{w.p.} & q_- & \text{for } S_n < 0. \end{cases}$$

This means that the sign-dependent random walk behaves like a simple random walk with parameter $p_+$ on the positive half-line and like an other simple random walk with parameter $p_-$ on the negative half-line. Between these two regions the origin is a reflecting filter. The parameter $p_0$ is chosen according to the desired grade of transference through the filter. The larger $p_0$ is, the less is the probability that the sign-dependent random walk passes downward the filter.

This model allows two different behaviors, one above zero and one below zero, and furthermore has a reflecting filter between these two regions. Thus it is a generalization of well-known models: a reflecting random walk ($p_0 = 1$), a random walk symmetric around zero ($p_0 = 0.5$, $p_+ = q_-$, $p_- = q_+$) and a random walk skew at zero ($p_+ = p_-$, $q_+ = q_-$ and $p_0$ is arbitrary). These discrete models have continuous analogous processes. Furthermore, the continuous processes are special cases of a continuous analogue to the sign-dependent random walk and has a similar behavior. It moves like a Brownian motion with certain parameters at the positive half-line and as an other Brownian motion on the negative half-line. The origin is a reflecting filter. The continuous analogues to the special cases above are discussed in Borodin and Salminen (2002): Brownian motion on $[0, \infty)$ with drift reflected at 0 (page 129–130), Brownian motion with alternating drift (page 128–129) and Brownian motion skew at zero (page 128).

A sign-dependent random walk has (to our knowledge) not been systematically studied in the literature before. The first subject of the first part of the thesis is to derive an expression for the probabilities $P(S_n = k)$. The first step to calculate these probabilities has been to derive a transform for the first passage time defined as follows. The starting point of the random walk is between an upper and a lower barrier and the first passage time is the first time the random walk visits any of these barriers, either the upper or the lower. An explicit transform for the first passage time has been obtained for deriving a transform of the probabilities $P(S_n = k)$.

The transforms are complicated. In a few special cases the transforms are analytically invertible. However, it is possible to get explicit expressions with a mathematical computer program, as Maple. After suitable scaling, the transforms are also used to derive transforms of limit distributions. Moreover, with coupling we show that the scaled sequence of the sign-dependent random walk is tight, implying weak convergence to a limit Markov process. The transform for the transition density of this limit process is also very complicated. However, it is invertible either analytically (in some cases) or numerically. We use a method proposed in a paper by Abate and Whitt (1995), which they refer to as Euler's method.

The second subject in the first part of the thesis is cover times. Let $\{W_t\}_{t \geq 0}$ be a stochastic process and let $m$ be a predetermined range. Then the cover time of the range $m$ is the first time the process has visited a range of length $m$. Denote the cover time by $T$, then

$$T = \inf \left\{ t \geq 0 : \sup_{0 \leq s \leq t} (W_s) - \inf_{0 \leq s \leq t} (W_s) \right\}.$$

To get a more intuitive idea of a cover time, let the stochastic process be a simple random walk. Let $m + 1$ nodes be placed on the circumference on a circle and let the random walk jump on the nodes, see Figure 1.1. The first time the random

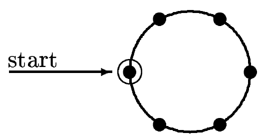


**Figure 1.1.** An example where $m = 5$ and start point is marked.

walk has visited all nodes on the circumference is the cover time.

The distributions of the cover time for simple random walks and Brownian Motions have been investigated in Chong, Cowan and Holst (2000). We generalize the results for the cover times to hold for a simple random walk with independent identical exponentially distributed holding times. We also derive a transform for the probability (density) function of the cover time for a sign-dependent random walk, both with constant holding times and independent exponentially distributed holding times. Furthermore, we have have studied transforms of the limit distributions. Some special cases are studied in detail with numerical results.

The last part of the thesis studies the number of maxima in a three-dimensional cube. Below follows an explanation of the definition of a maximum point. In the two-dimensional case we have the unit square. In this square we place a number of points uniformly. Pick one of the points. If none of the other points in the unit square has both a larger $x$-coordinate and a larger $y$-coordinate, then the chosen point is a maximum point. This means that there are no other point in the upper right rectangle from the point. Figure 1.2(a) is an example with three points and the total number of maxima is 2. In Figure 1.2(b) are totally 17 points with three maxima. The principle is the same in three dimensions. Here we have

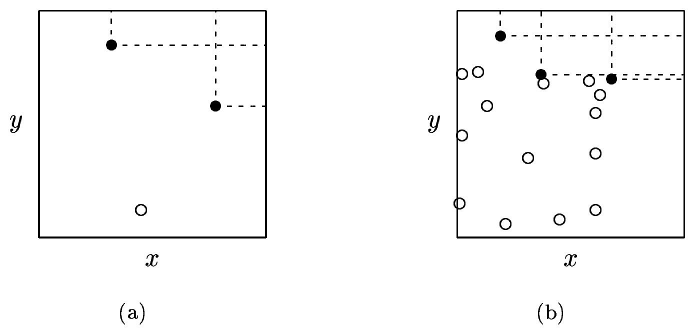


**Figure 1.2.** Two examples of maxima in two dimensions. The solid points are maxima and the circles are not.

a unit cube where we place a number of uniformly distributed points. A point is called a maximum point when none of the other points in the unit cube has larger $x$-coordinate, $y$-coordinate and $z$-coordinate.

In the two-dimensional case explicit expressions for the variance and the expected number of maxima have been known for a long time. However, the variance in the three-dimensional case was only known up to the leading term in an asymptotic expression. We have obtained an explicit formula for the variance of the number of maxima in three dimensions. This formula shows, due to slow convergence, that the leading term in the asymptotic expression is a misleading estimate of the variance. The distributional convergence of the number of maxima in $d$ dimensions has previously been shown to be normal (personal communication with

L. Devroye). We have used simulations of the distribution of the number of maxima and compared with both normal distributions and discrete distributions to illustrate the convergence.

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