A comparison of 5G candidate waveforms subject to phase noise impairment at mm-wave frequencies

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Acknowledgments

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Abstract

Frequencies above 6 GHz are being considered by mobile communication industry for the deployment of future 5G networks. Large channel bandwidths above 6 GHz are likely to be used to serve diverse use cases and meet extreme user requirements. However in the higher carrier frequencies, especially the millimeter-wave frequencies (above 30 GHz), there can be severe degradations in the transmitted and received signals due to Radio-Frequency (RF) impairments such as phase noise introduced by the local oscillators. 5G radio interface, operating higher carrier frequencies, has to be robust against phase noise. In this thesis, the effect of phase noise has been investigated for three different multi-carrier waveforms, namely Orthogonal Frequency Division Multiplexing (OFDM), Offset QAM Filter-Bank Multi-Carrier (OQAM-FBMC) and QAM Filter-Bank Multi-Carrier (QAM-FBMC). These waveforms have been considered as potential candidates for 5G radio interface. We develop analytical tools to evaluate the performance of these three waveforms subject to phase noise in terms of Signal-to-Interference Ratio (SIR). The derived tools can be used to obtain SIR under any phase noise model with a known phase noise spectral density. We have used a specific phase noise model (the mmMagic phase noise model) in our waveform evaluations and comparisons at three different carrier frequencies: 6 GHz, 28 GHz, and 82 GHz. The theoretical results are further verified using Monte Carlo evaluations for SIR and Symbol Error Rate (SER). The evaluation results reveal that OFDM is relatively more robust to phase noise than OQAM-FBMC and QAM-FBMC. It has been also observed that the choice of the overlapping factor in FBMC based waveforms can play a role in the performance. For the given phase noise model, OFDM has been observed to be robust to phase noise even at very high frequency (up to 82 GHz), which makes it a strong candidate for 5G radio interface.
Sammanfattning

Frekvenser över 6 GHz övervägs av mobilkommunikationsbranschen för utbyggnaden av framtida 5G nätverk. Men i frekvenser över 30 GHz kan det finnas allvarliga försämringar i signalerna på grund av fasbruset som införs genom lokala oscillatorer. Rapporten presenterar effekten av fasbruset undersökts för tre olika multi-carrier vågformer, nämligen Orthogonal Frequency Division Multiplexing (OFDM), Offset QAM Filter-Bank Multi-Carrier (OQAM-FBMC), och QAM Filter-Bank Multi-Carrier (QAM-FBMC). Vi utvecklar analysverktyg för att utvärdera och jämföra prestanda för dessa tre vågformer som omfattas av fasbrus vid 6 GHz, 28 GHz, och 82 GHz. De teoretiska resultaten verifieras ytterligare med hjälp av Monte Carlo-simuleringar. Utvärderingsresultaten visar att OFDM är relativt mer robust att fasbrus än OQAM-FBMC och QAM-FBMC.
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Chapter 1

Introduction

1.1 Motivation

Mobile communication systems operating in higher frequencies than those currently allocated to 4G networks are being considered by industry as a very promising approach to boost capacity in 5G networks significantly. Such a system can potentially use the much larger spectrum available in high frequencies. Moreover, in order to support user data rates of Gbps and above, contiguous bandwidths larger than 100 MHz (being the widest bandwidth currently defined for 4G) are required. Depending on the realization of the 5G system, bandwidths in the order of several GHz may be needed for efficient high capacity data delivery. Such wide contiguous blocks of bandwidth are not available below 6 GHz, where the spectrum is highly fragmented, but can be found in higher frequencies above 6 GHz, and in particular in the millimetre-wave (mm-wave) frequency bands. A millimetre-wave air-interface, operating in frequencies beyond 30 GHz, can serve extreme demands on capacity, throughput, latency, mobility, and reliability, by making use of the large available bandwidths. A mm-wave Radio Access Technology (RAT) is therefore envisioned to be an integral part of the 5G multi-RAT system.

The foundation of a successful mm-wave radio access technology is based on the waveform design. The currently used waveform in 4G systems is OFDM. However, the propagation characteristics for the frequencies in mm-wave bands are relatively unfavourable compared to the frequencies below 6 GHz. Furthermore, the signals transmitted and received at very high frequencies are subject to severe RF impairments (as phase noise). However, the small wavelength also brings benefits, allowing for a much larger number of antenna elements to be integrated into the devices. An efficient waveform design has to overcome the mm-wave specific (propagation and RF impairment related) challenges, while harnessing the benefits of large available channel bandwidth and massive number of antennas. Therefore, analyzing the effects of phase noise in the performance of different waveforms is a very timely and interesting topic related with the 5G standardization [1] [2].

1.2 Previous work

Not much work has been previously done related with the analysis of the effects of phase noise for the 5G candidate waveforms, mainly due to the fact that for the frequencies used by the current mobile networks (below 6 GHz) phase noise is not a significant degradation, but
also because some of the waveforms proposed for 5G have been presented recently and not much previous research related with them has been done. In [3] the effects of phase noise in continuous time OFDM were presented and the concept of weighting functions (what will be later on explained) was introduced. In [4] the effects of phase noise in OFDM were studied for different phase noise models and different receiver types (coherent and differential receiver). In [5] expressions for the power of the interferences produced by phase noise in OQAM-FBMC were presented, however without taking into account the effects of the intrinsic interferences that affect OQAM-FBMC.

1.3 Goals

The main goal of this master’s thesis is to analyse and compare the effects of phase noise in OFDM, QAM-FBMC and OQAM-FBMC for the mm-wave band. The mentioned waveforms are three of the most promising candidate Multi-Carrier Modulation (MCM) for 5G. The comparison should give a clear idea of which waveform offers better performance in the presence of phase noise for frequencies above 6 GHz. The performance is evaluated using as metrics the Signal-to-Interference Ratio (SIR) and the Symbol Error Rate (SER).

In order to carry out the comparison, we derive a method to evaluate the performance of the different MCM subject to phase noise. The most significant strength of the developed method is that it provides tools to analyse the effects of phase noise for any oscillator whose Power Spectral Density (PSD) is known and with any set of waveform parameters (as the signal bandwidth or the symbol length). So the method can be used to compare the performance of the three waveforms subject to phase noise for any case, which is a very helpful tool when deciding which is the best waveform for 5G in terms of robustness against phase noise.

1.4 Societal Aspects

Usually people think that the main difference between 4G and 5G is only related with transmission speed. However, 5G will not only offer higher transmission speed than 4G, it is supposed to offer other important properties, as higher energy efficiency and a big variety of services and cases (as machine-to-machine and vehicle-to-vehicle). These new properties will help to the development of different innovations that will have high impact in the society, as the drive-less car, the smart cities, the remote surgery and many other. One important aspect in order to achieve these innovations is the study of the communications in the mm-Wave band. Therefore, the study of the effects of phase noise in the communications in the mm-Wave band is a key aspect in order to achieve the previous mention innovations.

1.5 Tasks sequence

In this section we present the time distribution of the different stages in which the thesis has been divided:

- Stage 1: Bibliographic research and state of the art about 5G, MCM, OFDM, QAM-FBMC, OQAM-FBMC and phase noise, in order to get a solid background regarding the thesis topics.
• Stage 2: Study of the effects of phase noise in OFDM, deriving tools that allow us to evaluate the SIR theoretically. Perform simulations to check the accuracy of the theoretical results.

• Stage 3: Analysis of the effects of phase noise in QAM-FBMC, deriving tools that allow us to evaluate the SIR theoretically.

• Stage 4: Examination of the effects of phase noise in OQAM-FBMC, deriving tools that allow us to evaluate the SIR theoretically. Performance of simulations to check the accuracy of the theoretical results.

• Stage 5: Performance of simulations to study the effect of phase noise in the SER for the different waveforms.

• Stage 6: Analysis of the results and report writing.

• Stage 7: Preparation of the master’s thesis presentation, including the slides elaboration.

1.6 Outline

The rest of the report is organized as follows:

• Chapter 2 presents the theoretical background needed to follow the rest of the thesis. Concretely, concepts about MCM, OFDM, QAM-FBMC, OQAM-FBMC and phase noise are discussed.

• Chapter 3, 4 and 5 contain the derivations needed to study the effect of phase noise in OFDM, QAM-FBMC and OQAM-FBMC respectively.

• Chapter 6 presents the comparison of the results obtained for the three waveforms.

• Chapter 7 summarizes the thesis, presents the conclusions and gives some ideas of likely future work.

• Appendix A, B and C contain derivations related with Chapter 3, 4 and 5 respectively.
## 1.7 Acronyms and Nomenclature

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>CP</td>
<td>Cyclic Prefix</td>
</tr>
<tr>
<td>DTFT</td>
<td>Discrete Time Fourier Transform</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>FIR</td>
<td>Finite impulse response</td>
</tr>
<tr>
<td>ICI</td>
<td>Inter-Carrier Interference</td>
</tr>
<tr>
<td>IDTFT</td>
<td>Inverse Discrete Time Fourier Transform</td>
</tr>
<tr>
<td>IFFT</td>
<td>Inverse Fast Fourier Transform</td>
</tr>
<tr>
<td>ISI</td>
<td>Inter-Symbol Interference</td>
</tr>
<tr>
<td>LPF</td>
<td>Low Pass Filter</td>
</tr>
<tr>
<td>LTE</td>
<td>Long Term Evolution</td>
</tr>
<tr>
<td>MCM</td>
<td>Multi-Carrier Modulation</td>
</tr>
<tr>
<td>OFDM</td>
<td>Orthogonal Frequency Division Multiplexing</td>
</tr>
<tr>
<td>OQAM-FBMC</td>
<td>Offset Quadrature Amplitude Modulation - Filter Bank Multi-Carrier</td>
</tr>
<tr>
<td>PLL</td>
<td>Phase-Locked Loop</td>
</tr>
<tr>
<td>PN-CPE</td>
<td>Phase Noise - Common Phase Error</td>
</tr>
<tr>
<td>PN-ICI</td>
<td>Phase Noise - Inter-Carrier Interference</td>
</tr>
<tr>
<td>PN-ISI</td>
<td>Phase Noise - Inter-Symbol Interference</td>
</tr>
<tr>
<td>PSD</td>
<td>Power Spectral Density</td>
</tr>
<tr>
<td>QAM-FBMC</td>
<td>Quadrature Amplitude Modulation - Filter Bank Multi-Carrier</td>
</tr>
<tr>
<td>RAT</td>
<td>Radio Access Technology</td>
</tr>
<tr>
<td>SCM</td>
<td>Single-Carrier Modulation</td>
</tr>
<tr>
<td>SER</td>
<td>Symbol Error Rate</td>
</tr>
<tr>
<td>S-ICI</td>
<td>Self - Inter-Carrier Interference</td>
</tr>
<tr>
<td>SIR</td>
<td>Signal-to-Interference Ratio</td>
</tr>
<tr>
<td>S-ISI</td>
<td>Self - Inter-Symbol Interference</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>VCO</td>
<td>Voltage Controlled Oscillator</td>
</tr>
<tr>
<td>WSS</td>
<td>Wide-Sense Stationary Stochastic</td>
</tr>
</tbody>
</table>
### Notation

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$t$</td>
<td>Continuous time variable</td>
</tr>
<tr>
<td>$n$</td>
<td>Discrete time variable</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency variable</td>
</tr>
<tr>
<td>$\nu := \frac{f}{f_s}$</td>
<td>Normalized frequency variable</td>
</tr>
<tr>
<td>$f_c$</td>
<td>Carrier frequency</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Sampling frequency</td>
</tr>
<tr>
<td>$B_W$</td>
<td>Signal bandwidth</td>
</tr>
<tr>
<td>$T$</td>
<td>Symbol period</td>
</tr>
<tr>
<td>$\Delta f$</td>
<td>Sub-carrier spacing</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of sub-carriers</td>
</tr>
<tr>
<td>$N_{\text{act}}$</td>
<td>Number of active sub-carriers</td>
</tr>
<tr>
<td>$N_g$</td>
<td>Number of null bands in one of the sides of the spectrum</td>
</tr>
<tr>
<td>$I$</td>
<td>Set of indices of the active sub-carriers, $I = {N_g, ..., N - N_g - 1}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Overlapping factor</td>
</tr>
<tr>
<td>$i$</td>
<td>Index of modulated sub-carrier</td>
</tr>
<tr>
<td>$l$</td>
<td>Index of demodulated sub-carrier</td>
</tr>
<tr>
<td>$s$</td>
<td>Index of modulated symbol</td>
</tr>
<tr>
<td>$q$</td>
<td>Index of demodulated symbol</td>
</tr>
<tr>
<td>$d$</td>
<td>Symbol overlapping index</td>
</tr>
<tr>
<td>$\tilde{\cdot}$</td>
<td>Symbol used to identify the continuous time signals</td>
</tr>
</tbody>
</table>

### Operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\langle \cdot, \cdot \rangle$</td>
<td>Hermitian inner product operator</td>
</tr>
<tr>
<td>$*$</td>
<td>Convolution operator</td>
</tr>
<tr>
<td>$\circ$</td>
<td>Circular convolution operator</td>
</tr>
<tr>
<td>$\overline{X}$</td>
<td>Denotes complex conjugation of $X$</td>
</tr>
<tr>
<td>$\mathcal{F}{\cdot}$</td>
<td>Fourier transform operator</td>
</tr>
<tr>
<td>$\mathcal{F}^{-1}{\cdot}$</td>
<td>Inverse Fourier transform operator</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>Set of integers</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>Set of real numbers</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>Set of complex numbers</td>
</tr>
<tr>
<td>$j$</td>
<td>Complex variable, $j^2 = -1$</td>
</tr>
<tr>
<td>$\Re{\cdot}$</td>
<td>Real part operator</td>
</tr>
<tr>
<td>$\Im{\cdot}$</td>
<td>Imaginary part operator</td>
</tr>
<tr>
<td>$\mathbb{E}{\cdot}$</td>
<td>Expected value operator</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathcal{N}(\mu, \sigma^2)$</td>
<td>Gaussian distribution with mean $\mu$ and variance $\sigma^2$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>Dirac delta function</td>
</tr>
<tr>
<td>$\delta_{i,j}$</td>
<td>Kronecker delta function</td>
</tr>
</tbody>
</table>
Chapter 2

Background

In this chapter we provide a brief discussion of the essential theoretical background needed throughout the thesis. First the concept of Multi-Carrier Modulations (MCM) is presented, followed by a detailed explanation of the different properties for the three MCM studied in this thesis (OFDM, QAM-FBMC and OQAM-FBMC). Finally, the concept of phase noise and its effects on a modulated signal are presented.

2.1 Multi-Carrier Modulations

In classical Single-Carrier modulations (SCM), the information is transmitted over one single carrier frequency using high transmission rates, i.e., wide signal bandwidth and short symbol duration. Multi-Carrier Modulations (MCM) is a group of waveforms in which the information is split and transmitted simultaneously over different carrier frequencies (called sub-carriers) each one with low transmission rates (compared to SCM), i.e., narrow sub-carrier bandwidth and long symbol duration [6, pp. 27-30].

The main advantage of MCM over SCM which makes them preferred for high speed wireless transmissions is their robustness against frequency selective fading channels. Their robustness is explained because a selective fading channel can be divided into parallel frequency flat fading narrow subchannels in MCM (one for each sub-carrier). This effect can be seen in Figure 2.1. Thus, the channel equalization for MCM is an easy process performed in the frequency domain separately for each subchannel. Contrary, in SCM subject to a frequency selective fading channel the equalization is performed in the time domain using complex procedures.

![Figure 2.1: Effect of dividing a selective fading channel in narrow subchannels.](image-url)
In Figure 2.2 the general scheme for the modulator and demodulator of a MCM system is shown. In the scheme, \( X_{i,s} \) is any mapped symbol transmitted in sub-carrier \( i \) and symbol \( s \), \( \tilde{p}_i(t) \) is the filter used for the transmission in sub-carrier \( i \) (usually it is a band-pass filter centered in the sub-carrier frequency \( f_i \)) and \( \tilde{x}(t) \) is the base-band transmitted signal obtained from the sum of the signal for every sub-carrier, which expression is given by
\[
\tilde{x}(t) = \sum_{s \in \mathbb{Z}} \sum_{i \in I} X_{s,i} \tilde{p}_i(t - sT),
\]
where \( i \) is the index for the sub-carriers, \( s \) is the index for the different symbols, \( I \) is the set of indices of the active sub-carriers and \( T \) is the symbol duration \(^1\).

![Figure 2.2: Modulator and Demodulator scheme for a general MCM system.](image)

The demodulation in MCM is done using the matched filter demodulation. It is the optimal demodulation for maximizing the Signal-to-Noise Ratio (SNR) in presence of background noise when the filters for each sub-carrier are Linear Time-Invariant (LTI) \([7]\). The matched filter demodulation consists of the convolution of the received signal \((r(t))\) with the complex conjugated time-reversed version of the transmitter filter, followed by the sampling at rate of \(1/T\). Therefore, for MCM the expression for the demodulated signal in sub-carrier \( l \) and symbol \( q \) follows as
\[
R_{l,q} = \tilde{r}(t) * \tilde{p}_l(-t) = \sum_{t=qT} \langle \tilde{r}(t), \tilde{p}_l(t - qT) \rangle = \sum_{s \in \mathbb{Z}} \sum_{i \in I} X_{s,i} \langle \tilde{p}_i(t - sT), \tilde{p}_l(t - qT) \rangle \tag{2.2}
\]
(a) follows from interpreting the convolution as the correlation between the received signal and the filter for sub-carrier \( l \) and symbol \( q \); (b) follows from considering the ideal case (i.e., \( r(t) = x(t) \)) and from (2.1). In (2.2) \( * \) is the convolution operator, \( \langle \cdot, \cdot \rangle \) denotes the inner product and \( \tilde{p}_l(t) \) denotes complex conjugation of \( \tilde{p}_l(t) \).

In order to make a classification of the different types of MCM, the orthogonality condition is presented. We will assume that there is orthogonality when perfect reconstruction is achieved (i.e., \( R_{l,q} = X_{l,q} \)). From (2.2) the orthogonality condition can be derived \(^2\)
\[
\langle \tilde{p}_i(t - sT), \tilde{p}_l(t - qT) \rangle = \delta_{i,l} \delta_{s,q}, \tag{2.3}
\]
where \( \delta_{i,l} \) is the Kronecker delta with index \( i \) and \( l \). Intuitively, the previous expression means that there is orthogonality among the sub-carriers and symbols when other sub-carriers or symbols do not interfere with the current one.

\(^1\)We define the active sub-carriers as the ones carrying out a symbol. i.e., \( X_{i,s} \neq 0 \), while the null sub-carriers are the ones in which \( X_{i,s} = 0 \).

\(^2\)The orthogonality condition presented is for the case in which the filters have a total energy of 1, i.e., \( \int_{-\infty}^{\infty} |\tilde{p}_i(t)|^2 \, dt \overset{1}{=} 1 \) \( \forall i \in I \).
According to the orthogonality definition in (2.3), the MCM can be classified in two types:

- **Orthogonal MCM**, in which the orthogonality condition in (2.3) is fulfilled. They offer perfect reconstruction in the ideal case, i.e., $\forall l \in I$ and $\forall q \in \mathbb{Z}$ $R_{l,q} = X_{l,q}$ if $r[n] = x[n]$. The most common waveform of this group is OFDM.

- **Non-Orthogonal MCM**, in which the orthogonality condition in (2.3) is not fulfilled because of the effect of some degradations that we are going to call self-interferences. These degradations are intrinsic to the waveform and usually cannot be removed. The self-interferences can be split in two types:
  - **Self Inter-Carrier Interference (S-ICI)**, which are the interferences of other sub-carriers in the one that has been demodulated.
  - **Self Inter-Symbol Interference (S-ISI)**, which are the interferences of other symbols in the current demodulated one.

These degradations trigger that perfect reconstruction is not possible even in the ideal case, i.e., $\forall l \in I$ and $\forall q \in \mathbb{Z}$ $R_{l,q} \neq X_{l,q}$ even if $r[n] = x[n]$. In order to understand better the effects of the non-orthogonality, a modified orthogonality condition is defined

$$
\langle \tilde{p}_i(t - sT), \tilde{p}_l(t - qT) \rangle = \begin{cases} 
1 & \text{if } i = l \text{ and } s = q \\
\varepsilon_{i,l} & \text{if } i \neq l \text{ and } s = q \\
\beta_{i,l,s,q} & \text{else}
\end{cases}
$$

(2.4)

where $\varepsilon_{i,l}$ and $\beta_{i,l,s,q}$ determine how severe is the effect of the S-ICI and S-ISI respectively. Some non-orthogonal MCM fulfill $\forall i \in I$, $\forall l \in I$, $\forall s \in \mathbb{Z}$ and $\forall q \in \mathbb{Z}$ $|\varepsilon_{i,l}| < \lambda$ and $|\beta_{i,l,s,q}| < \lambda$ with $\lambda << 1$. When these conditions are fulfilled we define (2.4) as the relaxed orthogonality condition, and the MCM can be considered as a semi-orthogonal MCM, for example QAM-FBMC.

The time-frequency lattice representation is an instructive tool used in MCM to study how efficient is the utilization of the resources. It is a grid representation in which the distribution of the mapped symbols among the sub-carriers and the time resources is shown. In Figure 2.3 a general example of time-frequency lattice is plotted for MCM, where $\Delta f$ is the sub-carrier spacing and $T_{\text{symbol}}$ is the time between consecutive symbols.

The lattice representation is very useful because clearly shows the symbol’s density in terms of frequency and time resources. The density is given by $\frac{\Delta f}{T_{\text{symbol}}}$ and the best achievable density is 1 [8]. However, $\Delta f T_{\text{symbol}} > 1$ is a necessary condition for fulfilling the orthogonality condition in (2.3) [8]. The orthogonal MCM with $\Delta f T_{\text{symbol}} < 1$ (without S-ICI and S-ISI but with poorer use of the resources) and the non-orthogonal MCM with an efficient use of the resources but with S-ICI or S-ISI.

Next, three 5G candidate MCM with different features are presented (OFDM, QAM-FBMC and OQAM-FBMC).
2.1.1 OFDM

Orthogonal Frequency Division Multiplexing (OFDM) is the most popular and used MCM. The scheme for the modulator and the demodulator of an OFDM system is shown in Figure 2.4, where each sub-carrier carries a symbol $X_{i,s} \in \mathbb{C}$.

If one compares Figure 2.4 and Figure 2.2, it is obvious that in OFDM the filter for each sub-carrier is a frequency shift version of a base filter ($\tilde{p}(t)$), so it is defined as

$$\tilde{p}_i(t) := \tilde{p}(t) e^{j 2\pi f_i t},$$

where $\tilde{p}(t)$ is a rectangular filter with length $T$ and $f_i$ is the frequency for sub-carrier $i$. The
base filter is defined as
\[
\tilde{p}(t) := \begin{cases} 
\frac{1}{\sqrt{T}} & \text{if } 0 \leq t \leq T, \\
0 & \text{else}
\end{cases}
\]
(2.6)
in which the amplitude \(\frac{1}{\sqrt{T}}\) is used to achieve a total energy of 1 for the base filter, i.e.,
\[
\int_{-\infty}^{+\infty} |\tilde{p}(t)|^2 \, dt = 1.
\]

Next, the orthogonality for OFDM is discussed. One important fact that has to be taken into consideration is that in OFDM different symbols do not overlap each other. Therefore, no S-ISI effect appears and the orthogonality condition in (2.3) can be simplified by removing the condition related with the symbol index (which is always true for OFDM), obtaining
\[
\langle \tilde{p}_i(t), \tilde{p}_l(t) \rangle = \delta_{i,l}
\]
(2.7)
From (2.7), the orthogonality for OFDM is studied
\[
\langle \tilde{p}_i(t), \tilde{p}_l(t) \rangle = \int_0^T \tilde{p}_i(t) \overline{\tilde{p}_l(t)} \, dt \\
= \int_0^T \tilde{p}(t) e^{j2\pi f_i t} \overline{\tilde{p}(t) e^{-j2\pi f_l t}} \, dt \\
= \int_0^T e^{j2\pi (f_i - f_l) t} \, dt
\]
From the previous expression, the relation between sub-carriers needed to fulfill the orthogonality condition in (2.7) (and avoid S-ICI) can be deduced
\[
f_i - f_l = \frac{i - l}{T} = (i - l) \Delta f,
\]
in which \(\Delta f\) is the sub-carrier spacing, \(\Delta f := \frac{1}{T}\). Therefore, OFDM is an orthogonal MCM when the sub-carrier spacing is equal to the inverse of the symbol period. Finally, taking into account the orthogonality condition the expression for the filter in sub-carrier \(i\) is given by
\[
\tilde{p}_i(t) = \tilde{p}(t) e^{j2\pi \frac{i}{T} t}
\]
(2.8)
In Figure 2.5 the shape of the frequency response for three orthogonal sub-carriers is represented. The filter used for each sub-carrier has a rectangular impulse response of length \(T\). Therefore, its frequency response has a sinc shape with null values in increments of \(\Delta f = \frac{1}{T}\), avoiding S-ICI.

Another important characteristic of OFDM is Cyclic Prefix (CP). It is a repeated version of the last \(T_{CP}\) seconds of the OFDM symbol in its beginning. The main goal of the CP is to avoid the ISI produced by channel multi-paths and keeping the orthogonality among sub-carriers (avoiding ICI). ISI and ICI produced by the channel are avoided when the spread delay of the channel is shorter than the duration of the CP. If the spread delay is longer than the CP, ISI appears and the orthogonality among sub-carriers is lost.

In Figure 2.6 the lattice representation for OFDM has been plotted. The symbol density for OFDM is \(\frac{T}{\tau} \frac{1}{T_{CP}} < 1\), it reflects that OFDM does not offer the most efficient use of the resources.
One of the main advantages of OFDM is that the modulation and demodulation can be done in the digital domain in a very computationally efficient way using the Fast Fourier Transform (FFT). In Figure 2.7 the most common OFDM modulator and demodulator are presented. In the modulator, the N-point Inverse Fourier Transform (IFFT) splits the mapped symbols in N orthogonal sub-carriers with very few computation. In the demodulator the N-point FFT recovers the symbol for each of the sub-carriers.

Even though OFDM offers good performance and low complexity for the modulation and demodulation, there are two aspects that can be improved using other waveforms. These two
aspects are:

- **The Cyclic prefix.** Although the CP gives robustness against ISI and ICI produced by the channel, it occupies time resources where no information is transmitted.

- **The rectangular pulses.** Although the orthogonality condition is fulfilled, the spectral shape of the rectangular pulses (sinc shape) has high power side-lobes, what produces significant out-of-band emissions.

As has been showed previously, OFDM offers many advantages, that is why it has been used as waveform in many standards for wireless systems, for example IEEE 802.11g/n/ac and LTE. However, one of the challenges for 5G networks is to increase capacity. So new waveforms trying to improve the two previous cons have become very popular lately. Thus, two of these waveforms are studied next (QAM-FBMC and OQAM-FBMC).

![Figure 2.7: OFDM implementation using the FFT.](image)

2.1.2 QAM-FBMC

Quadrature Amplitude Modulation - Filter-Bank Multi-Carrier (QAM-FBMC) is a MCM that was first introduced in [9]. In Figure 2.8 the QAM-FBMC modulator and demodulator are shown, each sub-carrier carries a symbol $X_{i,s} \in \mathbb{C}$.

![Figure 2.8: Continuous time QAM-FBMC Modulator and Demodulator.](image)

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If one compares Figure 2.8 and Figure 2.2, it is clear that the filter for each sub-carrier is defined as
\[
\tilde{p}_i(t) := \begin{cases} 
\tilde{p}_{\text{even}}(t) e^{j2\pi f_i t} & \text{if } i \text{ is even} \\
\tilde{p}_{\text{odd}}(t) e^{j2\pi f_i t} & \text{if } i \text{ is odd,}
\end{cases}
\] (2.9)
where \(\tilde{p}_{\text{even}}(t)\) and \(\tilde{p}_{\text{odd}}(t)\) are the base filters for the even and odd sub-carriers respectively. Both filters are LTI filters which must be designed to be mutually orthogonal for the sub-carrier spacing \(\Delta f\). The length of the filters is \(L = K T\), where \(K\) is the overlapping factor, \(T = \frac{1}{\Delta f}\) and \(\Delta f\) is defined as the sub-carrier spacing (as in OFDM).

The length of one modulated QAM-FBMC symbol is \(K\) times longer than the duration of one OFDM symbol (for the same sub-carrier spacing). The motivation for using filters with longer impulse response is to obtain better frequency localization than in OFDM, with less out-of-band emissions. In Figure 2.9 the spectrum of QAM-FBMC is plotted together with the spectrum of OFDM for 1 and 300 sub-carriers. As can be seen in the graph, the filters used by QAM-FBMC have lower side-lobes than the one for OFDM and produce lower out-of band emissions.

Increasing the length of the filters also produces a not efficient use of the time resources. In order to get an efficient utilization of the time resources using long pulses, QAM-FBMC introduces overlapping between consecutive symbols. In Figure 2.8 the overlapping operation is carried out by the block Overlap & Sum. In Figure 2.10 the overlapping structure for QAM-FBMC is shown, where \(K\) is the overlapping factor.

In Figure 2.11 the lattice representation for QAM-FBMC has been plotted. The symbol density is 1, which means that QAM-FBMC offers a very efficient use of the resources. However, as the density is equal to 1, OQAM-FBMC is a Non-orthogonal MCM that suffers from S-ICI and S-ISI.

Next, the orthogonality condition is studied for QAM-FBMC. From (2.4)
\[
\langle \tilde{p}_i(t - sT), \tilde{p}_l(t - qT) \rangle = \begin{cases} 
1 & \text{if } i = l \text{ and } s = q \\
\varepsilon_{i,l} & \text{if } i \neq l \text{ and } s = q \\
\beta_{i,l,s,q} & \text{else}
\end{cases}
\] (2.10)
in which \( \exists i \in I, \exists l \in I, \exists s \in \mathbb{Z} \) and \( \exists s \in \mathbb{Z} \) \( \varepsilon_{i,l} \neq 0 \) and \( \beta_{i,l,s,q} \neq 0 \), so the orthogonality condition is not fulfilled. The cause of the non-orthogonality are the self-interferences, concretely:

- The **S-ICI** produced because the base filters are not ideally orthogonal, producing interference between different sub-carriers.
- The **S-ISI**, produced by the interference of the overlapping symbols in the current demodulated symbol.

The authors in [9] assure that \( \forall i \in I, \forall l \in I, \forall s \in \mathbb{Z} \) and \( \forall q \in \mathbb{Z} \) \( |\varepsilon_{i,l}| < \lambda \) and \( |\beta_{i,l,s,q}| < \lambda \) with \( \lambda << 1 \) for QAM-FBMC. So, we can conclude that QAM-FBMC fulfills the relaxed orthogonality condition and we can consider it as a semi-orthogonal MCM.

Regarding the effects of the channel multi-paths, QAM-FBMC does not use cyclic prefix to avoid their effects. The motivation for this choice is that even without the effects of the
channel, QAM-FBMC suffers S-ISI and S-ICI. Adding the CP would improve the robustness of the waveform against the ICI and ISI effect produced by the multipaths, but it would worsen the orthogonality conditions because of the overlap (producing more S-ICI and S-ISI). Therefore it is reasonable not to use CP. Moreover, not using CP leads to a more efficient use of the time resources (one of the weakness of OFDM).

In [9] a computationally efficient version of the modulator and demodulator in the digital domain can be found, it is based on using the FFT.

To sum up, QAM-FBMC improves the use of the time resources and achieves a lower level of out-of-band emissions than OFDM. However, it suffers from S-ICI and S-ISI, so perfect reconstruction is not achieved. Studying the robustness of QAM-FBMC against phase noise can be helpful to decide which waveform is more suitable for 5G.

2.1.3 OQAM-FBMC

Offset Quadrature Amplitude Modulation - Filter-Bank Multi-Carrier (OQAM-FBMC) is a MCM that was first introduced in [10]. In Figure 2.12 the OQAM-FBMC modulator and demodulator are shown, where each sub-carrier is modulated with a symbol \( X_{i,s} \in \mathbb{R} \).

\[
\tilde{p}(t) := \tilde{p}(t) e^{j2\pi f_0 t},
\]

where \( \tilde{p}(t) \) is the base filter for OQAM-FBMC defined in [11] for different overlapping factor \( K \). The filter has a length \( L = KT \), where \( T = \frac{1}{\Delta f} \).

As in the QAM-FBMC case, the motivation for using filters with longer impulse response is to obtain better frequency localization than in OFDM. In Figure 2.13 the comparison between the spectrum of the three studied waveforms for 1 and 300 sub-carriers is plotted. It can be seen how increasing the overlapping factor in OQAM-FBMC reduces the level of out-of-band emissions. OQAM-FBMC with \( K = 3 \) and \( K = 4 \) has lower out-of-band emissions than QAM-FBMC and OFDM.

In order to make an efficient use of the time resources, OQAM-FBMC introduces overlapping between consecutive symbols (as QAM-FBMC). In Figure 2.14 the overlapping structure for QAM-FBMC is shown, where \( K \) is the overlapping factor.
One important characteristic of the base filter used by OQAM-FBMC is that it does not introduce S-ICI if a phase difference of $\frac{\pi}{2}$ is kept between the symbols carried by adjacent sub-carriers [10]. Therefore, in order to avoid the S-ICI, the sub-carriers carry pure real and imaginary symbols alternatively. Moreover, in order to use the resources efficiently an Offset-mapping is used, so the symbol rate is doubled (with respect to OFDM and QAM-FBMC) and the symbols carried by each sub-carrier are shifted by a phase of $\frac{\pi}{2}$ each $\frac{T}{2}$.

The phase distribution is easily understood looking into the lattice representation in Figure 2.15. In order to generate such a series of symbols, real symbols ($X_{i,s} \in \mathbb{R}$) multiplied by a phase term $\theta_{i,s} = j^{(i+s)}$ are used. In the demodulation the phase term $\theta_l = j^{(-l)}$ and the $\mathbb{R} \{ \}$ and $\mathbb{I} \{ \}$ operators are used. Thus, the main difference between OQAM-FBMC respect to QAM-FBMC and OFDM is that in OQAM-FBMC each sub-carrier carries one $X_{i,s} \in \mathbb{R}$ while in the other $X_{i,s} \in \mathbb{C}$.

In Figure 2.15 the lattice representation for OQAM-FBMC has been plotted. In this case it is important to emphasise that the set of 1 real and 1 imaginary symbol for OQAM-FBMC is considered equal than 1 complex symbol for QAM-FBMC and OFDM, in order to make a fair comparison. Therefore, the symbol density is 1, which means that OQAM-FBMC offers a very efficient use of the resources but also shows that it is a non-orthogonal MCM that suffers from degradations due to the non-orthogonality.
Next, the orthogonality condition is studied for OQAM-FBMC. Due to the use of the Offset-mapping, the orthogonality condition in (2.4) has to be modified including the phase terms and the $\Re\{\}$ operator, giving

\[
\Re\left\{\langle \theta_{i,s} \tilde{p}_i (t - \frac{T}{2}), \theta_{l,q} \tilde{p}_l (t - \frac{T}{2}) \rangle\right\} = \begin{cases} 
1 & \text{if } i = l \text{ and } s = q \\
\varepsilon_{i,l} \beta_{i,l,s,q} & \text{if } i \neq l \text{ and } s = q \\
\beta_{i,l,s,q} & \text{else}
\end{cases},
\]

(2.12)

in which $\varepsilon_{i,l}$ and $\beta_{i,l,s,q}$ are related with the S-ICI and S-ISI degradations respectively. As has been studied previously, OQAM-FBMC is not affected by S-ICI if the phase difference between neighbour sub-carriers is $\frac{\pi}{2}$, so $\forall i \in I$ and $\forall l \in I$ $\varepsilon_{i,l} = 0$. However, due to the overlap between consecutive symbols it is affected by S-ISI effect, so $\exists i \in I$, $\exists l \in I$, $\exists s \in \mathbb{Z}$ and $\exists s \in \mathbb{Z}$ $\beta_{i,l,s,q} \neq 0$. Thus the orthogonality condition is not fulfilled because of the S-ISI effect.

The authors in [11] assure that $\forall i \in I$, $\forall l \in I$, $\forall s \in \mathbb{Z}$ and $\forall q \in \mathbb{Z}$ $|\beta_{i,l,s,q}| < \lambda$ with $\lambda << 1$. Therefore, we can conclude that OQAM-FBMC fulfills the relaxed orthogonality condition and we can consider it as a semi-orthogonal MCM.

Regarding the effects of the multipaths in the channel, OQAM-FBMC does not use cyclic prefix to avoid them. The motivation is the same than for the case of QAM-FBMC.

\[3\] The expression in (2.12) is only valid for even $l$. For odd $l$ the $\Im\{\}$ operator would replace the $\Re\{\}$ operator.
One of the weaknesses of OQAM-FBMC compared to (OFDM and QAM-FBMC) is its performance in complex channels. When the channel has complex coefficients the phase difference of $\frac{\pi}{2}$ between neighbour sub-carriers could be modified, producing the appearance of S-ICI.

In [10] a computationally efficient version of the modulator and demodulator in the digital domain can be found, it is based in the use of the FFT and poly-phase networks.

To sum up, OQAM-FBMC improves the use of the time resources respect to OFDM and achieves lower level of out-of-band emissions than OFDM (but higher than QAM-FBMC). Moreover, for real channels it is not affected by S-ICI, so it suffers less degradations than QAM-FBMC but more than OFDM. However, for complex channels it suffers from severe S-ICI.

2.2 Phase noise

Oscillators are important elements of transmitters and receivers in wireless systems. The main function of oscillators is to up-convert a base-band signal to a radio-frequency signal at the transmitter and down-convert a radio-frequency signal to a base-band signal at the receiver. Ideally, an oscillator generates a perfect sinusoidal signal with frequency $f_o$. In practical situations, the signal generated by oscillators is not perfect and has low random fluctuations in the phase, which are usually called phase noise. An oscillator with a central frequency $f_o$ and the effects of phase noise can be modelled as

$$V(t) = e^{j(2\pi f_o t + \tilde{\phi}(t))},$$

in which $\tilde{\phi}(t)$ is a stochastic process that modifies the phase of the ideal sinusoidal signal, called phase noise.

The main effect of phase noise in the oscillator output signal is an spreading on its Power Spectral Density (PSD), which ideally is a Dirac Delta in the central frequency $f_o$. This effect can be seen in Figure 2.16.

![PSD](image)

(a) Ideal oscillator

![PSD](image)

(b) Real oscillator

Figure 2.16: Spreading produced by phase noise in the PSD of the oscillator.

The specific properties of phase noise depend on each particular oscillator. There are several types of oscillator, but the most used ones for Wireless systems are the ones using a Phase-Locked Loop Synthesizer (PLL). In Figure 2.17 the main structure of an oscillator based on PLL is shown. In this model, the phase of the Voltage Controlled Oscillator (VCO) is compared with a reference signal produced by a Free-Running Oscillator (Reference oscillator), the difference between both phases is filtered using a low pass-filter (LPF) and applied as a control signal to the
VCO which produces the output sinusoidal signal. Using this structure, the phase deviations are lower than using a Free-Running Oscillator. Moreover, phase noise in this case can be modelled as a wide-sense stationary stochastic process with finite power and $\phi(t) \sim \mathcal{N}(0, \sigma^2)$ (zero-mean Gaussian random variable). Whereas in the case of Free-Running Oscillator, phase noise is modelled as a Wiener stochastic process with infinite power (which is a less suitable model for the analytical derivations) [12].

\[
\phi(t) \sim \mathcal{N}(0, \sigma^2)
\]

Figure 2.17: Model of an oscillator based on PLL.

Another important characteristic of phase noise is that its power increases with an increase in the oscillator frequency ($f_o$). For $f_o$ in the mm-wave band (above 30 GHz) it is more difficult to produce oscillators with good phase noise properties, so usually the phase noise power is high, producing a significant degradation in the demodulated signal. However, for the frequencies used for 2G, 3G and 4G (below 6 GHz) the power of phase noise is very low, so it is not a significant degradation. Therefore, this is the main motivation to study the effects of phase noise in the mm-wave band for the different waveforms.

In order to study the effects of phase noise we are going to use the phase noise model used in the mmMagic project 4, which is based on the oscillator model using the PLL. Let us denote the PSD of phase noise by $S_{\phi}(f)$. The PSD for mmMagic phase noise model can be seen in Figure 2.18 for different oscillator frequencies. In the figure, the $x$-axis corresponds to the frequency offset with respect to the oscillator frequency ($f_o$).

\[
S_{\phi}(f) [\text{dBW}]
\]

Figure 2.18: Power Spectral Density for different $f_o$ for the mmMagic project phase noise model.

4mmMAGIC project (Millimetre-Wave Based Mobile Radio Access Network for Fifth Generation Integrated Communications) is a European project aiming to develop novel radio access technologies for mobile communication in the frequency range 6-100 GHz [13]
2.2.1 Effect of phase noise in a generic signal

Let us consider $\tilde{x}(t)$ as the base-band modulated signal. In the transmitter, the base-band signal is up-converted to a radio-frequency signal using an oscillator with frequency $f_c$

$$\bar{x}_c(t) = \tilde{x}(t) e^{j(2\pi f_c t + \tilde{\phi}(t))},$$  \hspace{1cm} (2.13)

where $\phi_t(t)$ is the phase noise impulse response for the transmitter oscillator.

In the receiver, the received radio-frequency signal is given by

$$\tilde{r}_c(t) = \tilde{x}_c(t) * h(t) + w(t),$$

where $*$ denotes convolution operation, $h(t)$ is the impulse response of the radio-channel and $w(t)$ is AWGN noise.

Although in a real transmission the effects of the channel can not be disregarded, in all the following derivations an ideal flat channel with non additive noise will be assumed, i.e., $h(t) = \delta(t)$ and $w(t) = 0$ for any $t$. Taking this assumption, the previous expression can be modified as

$$\tilde{r}_c(t) = \tilde{x}_c(t)$$  \hspace{1cm} (2.14)

The motivation for this assumption is to study the effects only produced by phase noise.

Next, the received radio-frequency signal is down-converted in order to get base band signal

$$\tilde{r}(t) = \tilde{r}_c(t) e^{-(j2\pi f_c + \tilde{\phi}_r(t))} \overset{(a)}{=} \tilde{x}_c(t) e^{-(j2\pi f_c + \tilde{\phi}_r(t))} \overset{(b)}{=} \tilde{x}(t) e^{j(\tilde{\phi}_t(t) - \tilde{\phi}_r(t))},$$  \hspace{1cm} (2.15)

where $\tilde{\phi}_r(t)$ is the phase noise introduced by receiver’s oscillator; (a) follows from (2.14); (b) follows from (2.13).

A random variable with the effect of phase noise of both oscillators is included in (2.14), resulting in the following expression

$$\tilde{r}(t) = \tilde{x}(t) e^{j\tilde{\phi}(t)},$$  \hspace{1cm} (2.16)

in which $\tilde{\phi}(t) = \tilde{\phi}_t(t) - \tilde{\phi}_r(t)$.

As $\phi(t) \sim N(0, \sigma^2)$ with very small $\sigma^2$ and $\sin(x) \approx x$ if $|x| \ll 1$, we are going to use the following approximation

$$e^{j\tilde{\phi}(t)} \approx 1 + j\tilde{\phi}(t)$$  \hspace{1cm} (2.17)

This approximation has been widely used in the literature [3] [4] and simplifies significantly the analysis of the effect of phase noise in the different waveforms. Finally, the expression for the received base-band signal with the effects of phase noise is given by

$$\tilde{r}(t) \approx \tilde{x}(t) (1 + j\tilde{\phi}(t))$$  \hspace{1cm} (2.18)
Chapter 3

OFDM

Orthogonal Frequency Division Multiplexing (OFDM) is an orthogonal MCM widely used in wireless systems (for example IEEE 802.11g/n/ac and LTE) as discussed in Section 2.1.1. An important feature of a waveform for 5G is its robustness against phase noise (as was explained in Chapter 1). So, this chapter presents the derivations and evaluations carried out in order to study the effects of phase noise in OFDM. Concretely, the effects of phase noise are studied for the continuous and discrete time OFDM with the objective of showing the effects of sampling. So, for both cases, we start deriving the expressions for the modulated and demodulated signals. Next we study the effect of phase noise in the demodulated signal, deriving expressions for the power of interference terms produced by phase noise. In the following we derive the expressions for the Symbol-to-Interference Ratio (SIR). Once the expression for the SIR for both cases (continuous and discrete) are obtained, an analysis of the functions that determine the value of the SIR is carried out (we refer to these functions as weighting functions). Next, a comparison between the continuous and discrete time is done. Finally, the SIR and SER are evaluated for some concrete implementation parameters.

3.1 Continuous time Modulated and Demodulated signal

**Modulated signal** The continuous time base-band modulated signal for an OFDM symbol is given by

$$\tilde{x}(t) = \sum_{i \in I} X_i \tilde{p}(t) e^{j2\pi i T t} = \sum_{i \in I} X_i \tilde{p}_i(t),$$

(3.1)

where $X_i$ is any mapped symbol that modulates sub-carrier $i$, $I$ is the set of indices for the active sub-carriers, $T$ is the length of the OFDM symbol (without the contribution of the CP), $\tilde{p}(t)$ is the base filter defined in (2.6) and $\tilde{p}_i(t)$ is the filter for sub-carrier $i$ defined in (2.8).

**Demodulated signal** In Section 2.1 the demodulation using matched filter was presented, it is the most common demodulation procedure for MCM. OFDM uses the matched filter demodulation, so the demodulated symbol in sub-carrier $l$ follows from (2.2) and it is given by

$$\tilde{R}_l = \tilde{r}(t) \ast \tilde{g}_l(t) \bigg|_{t=0},$$

(3.2)

where $\tilde{g}_l(t)$ is the matched filter used in sub-carrier $l$. It is defined as

$$\tilde{g}_l(t) := \tilde{p}_l(-t)$$

(3.3)
3.2 Continuous time Demodulated signal subject to phase noise

Once the expressions for the modulated and demodulated signal are known, we are going to study the effects of phase noise in the demodulated signal.

The demodulated signal subject to phase noise is given by

$$\tilde{R}_l = \tilde{r}(t) \ast \tilde{g}(t) \mid_{t=0}$$

$$(a)$$ follows from the definition of the demodulated signal in (3.2) while $\ast$ is the convolution operator; $$(b)$$ follows from the effect of phase noise in the received signal in (2.18); $$(c)$$ follows from (3.1); $$(d)$$ follows from the distributivity property of the convolution; $$(e)$$ follows by defining $\tilde{X}_{i,l} := \tilde{p}_i(t) \ast \tilde{g}(t) \mid_{t=0}$ and $\tilde{\phi}_{i,l} := \left[ j\tilde{\phi}(t) p_i(t) \right] \ast \tilde{g}(t) \mid_{t=0}$.

The terms in (3.4) that only depend on the impulse response of the transmitter and receiver filter are

$$\tilde{\lambda}_{i,l} = \tilde{p}_i(t) \ast \tilde{g}(t) \mid_{t=0} \quad (a) \quad \langle \tilde{p}_i(t), \tilde{p}_l(t) \rangle \quad (b)$$

$$(c)$$ follows from the convolution definition in $t=0$ and from (3.3); $$(b)$$ follows from the orthogonality condition for OFDM in (2.7). This is a very important result, because it means that when the orthogonality condition is fulfilled there are no self-interferences in the demodulation.

The terms in (3.4) that do not only depend on the impulse response of the transmitter and receiver filter but also on the phase noise are

$$\tilde{\phi}_{i,l} = \left[ j\tilde{\phi}(t) \tilde{p}_i(t) \right] \ast \tilde{g}(t) \mid_{t=0} \quad (a) \quad \int_{-\infty}^{\infty} j\tilde{\phi}(\tau) \tilde{p}_i(\tau) \tilde{g}(\tau) d\tau \quad (b)$$

$$(c)$$ follows from the convolution definition in $t=0$; $$(b)$$ follows from (3.3). These terms are related with the interference terms produced by phase noise.

Taking into consideration the previous definitions in (3.5) and (3.6), we can rewrite (3.4) as

$$\tilde{R}_l = X_l + \tilde{N}_l^{CPE} + \tilde{N}_l^{ICI},$$

(3.7)
where $\tilde{N}_l^{CPE}$ and $\tilde{N}_l^{ICI}$ are two interference terms produced by phase noise. They are defined as

$$\tilde{N}_l^{CPE} := X_l \tilde{\varphi}_{l,l}$$  \hspace{1cm} (3.8)
$$\tilde{N}_l^{ICI} := \sum_{i \in I^{ICI}} X_i \tilde{\varphi}_{i,l}$$  \hspace{1cm} (3.9)

where $I^{ICI}$ is the set of indices for the interfering sub-carriers. It is defined as $I^{ICI} = I \setminus \{l\}$.

As shown in (3.7), phase noise introduces two interference terms in the demodulated signal. We are going to classify these interferences as:

- **Phase noise - Common Phase Error (PN-CPE).** It is identified with the term $\tilde{N}_l^{CPE}$ and it is a rotated version of the ideally transmitted symbol $X_l$. The rotation is given by the term $\tilde{\varphi}_{l,l}$, which is equal for every sub-carrier $l$. This fact makes the PN-CPE correction a very easy process using pilot symbols.

- **PN - Inter-Carrier Interference (PN-ICI).** The introduction of phase noise worsens the orthogonality between sub-carriers, so interference from the rest of the sub-carriers appears in the demodulated sub-carrier, producing the interference term $\tilde{N}_l^{ICI}$.

### 3.2.1 Interference power

In this section the power of the interference terms $\tilde{N}_l^{CPE}$ and $\tilde{N}_l^{ICI}$ produced by phase noise are derived.

The power of the interference terms is given by

$$P_{\tilde{N}^{CPE}} := \mathbb{E} \left[ |X_l \tilde{\varphi}_{l,l}|^2 \right] = \mathbb{E} \left[ |X_l|^2 \right] \mathbb{E} \left[ |\tilde{\varphi}_{l,l}|^2 \right] = E_X \left( \mathbb{E} \left[ |\tilde{\varphi}_{l,l}|^2 \right] \right)$$  \hspace{1cm} (3.10)

$$P_{\tilde{N}^{ICI}} := \mathbb{E} \left[ \left| \sum_{i \in I^{ICI}} X_i \tilde{\varphi}_{i,l} \right|^2 \right] = \sum_{i \in I^{ICI}} \mathbb{E} \left[ |X_i|^2 \right] \mathbb{E} \left[ |\tilde{\varphi}_{i,l}|^2 \right] = E_X \left( \sum_{i \in I^{ICI}} \mathbb{E} \left[ |\tilde{\varphi}_{i,l}|^2 \right] \right)$$  \hspace{1cm} (3.11)

In (3.10) and (3.11) it has been assumed that $\{X_i\} \forall i \in I$ is a sequence of zero-mean independent random variables and $\{\tilde{\varphi}_{i,l}\}$ is a sequence of random variables whose elements are independent from the elements of $\{X_i\} \forall i \in I$ and $\forall l \in I$. Moreover, $E_X$ is the power of the mapped symbol and it is defined as $E_X = \mathbb{E} \left[ |X_l|^2 \right] \forall l \in I$.

The expression for the power of the interference terms in (3.10) and (3.11) is dependent on the term $\mathbb{E} \left[ |\tilde{\varphi}_{i,l}|^2 \right]$. According to Appendix A.1, $\mathbb{E} \left[ |\tilde{\varphi}_{i,l}|^2 \right]$, can be expressed as

$$\mathbb{E} \left[ |\tilde{\varphi}_{i,l}|^2 \right] = \int_{-\infty}^{+\infty} S_\phi(f) \tilde{W}_{i,l}(f) \, df,$$  \hspace{1cm} (3.12)

where $S_\phi(f)$ is the PSD of phase noise and $\tilde{W}_{i,l}(f)$ is defined as

$$\tilde{W}_{i,l}(f) := \left| \text{sinc} \left( \frac{f - i - l T}{T} \right) \right|^2$$  \hspace{1cm} (3.13)
Applying the previous result in (3.12) to (3.10) and (3.11) the final expressions for the power of the interference terms is obtained as

\[ P_{\tilde{N}_{CPE}} = E_X \left( \int_{-\infty}^{\infty} \tilde{W}_{CPE}^l(f) S_\phi(f) df \right) \]  
\[ P_{\tilde{N}_{ICI}} = E_X \left( \int_{-\infty}^{\infty} \tilde{W}_{ICI}^l(f) S_\phi(f) df \right) \]  

(3.14)  
(3.15)

in which

\[ \tilde{W}_{CPE}^l(f) := \tilde{W}_{CPE}^l(f) \]  
\[ \tilde{W}_{ICI}^l(f) := \sum_{i \in I_{ICI}} \tilde{W}_{ICI}^i(f) \]  

(3.16)  
(3.17)

### 3.2.2 Signal-to-Interference Ratio due to phase noise

From (3.7) the expression for the SIR in the demodulated signal in sub-carrier \( l \) subject to phase noise is easily derived, giving

\[ \text{SIR}_l = \frac{E_X}{P_{\tilde{N}_{CPE}} + P_{\tilde{N}_{ICI}}} = \frac{1}{\int_{-\infty}^{\infty} \tilde{W}_{CPE}^l(f) + \tilde{W}_{ICI}^l(f) S_\phi(f) df} \]  

(3.18)

\( (a) \) follows from (3.14) and (3.15). Moreover, \( P_{\tilde{N}_{CPE}} \) and \( P_{\tilde{N}_{ICI}} \) are the power of the interference terms produced by the PN-CPE and PN-ICI respectively and \( E_X \) is the power of the transmitted symbol in the active sub-carrier \( l \), and it is defined as \( E_X := E \left| X_l \right|^2 \).

**Effect of removing PN-CPE**  The expression for the SIR without the effect of PN-CPE in sub-carrier \( l \) is easily derived from (3.18) by removing the CPE contribution, giving

\[ \text{SIR}_l = \frac{1}{\int_{-\infty}^{\infty} \tilde{W}_{ICI}^l(f) S_\phi(f) df} \]  

(3.19)

### 3.3 Discrete time Modulated and Demodulated signal

**Modulated signal**  The discrete time modulated signal for an OFDM symbol is given by

\[ x[n] = \sqrt{\frac{T}{N}} \tilde{x}(t) \bigg|_{t = \frac{n}{f_s}} = \sum_{i \in I} X_i p[n] e^{j2\pi \frac{n}{N}} = \sum_{i \in I} X_i p_i[n] \]  

(3.20)

where \( N \) is the total number of samples of one OFDM symbol (without the CP length), \( X_i \) is the modulated symbol in sub-carrier \( i \), \( I \) is the set with the index of active sub-carriers, \( f_s \) is the sampling frequency, \( p[n] \) is the discrete time base filter, \( p_i[n] \) is the discrete time filter for sub-carrier \( i \) and \( \sqrt{\frac{T}{N}} \) is a scaling term whose goal is to have the same total energy for \( x[n] \) as for \( \tilde{x}(t) \).

The base filter for the discrete time implementation can be considered as a sampled and scaled version of the filter in (2.6), defined as

\[ p[n] := \sqrt{\frac{T}{N}} \tilde{p}(t) \bigg|_{t = \frac{n}{f_s}} = \begin{cases} \frac{1}{\sqrt{N}} & \text{if } 0 \leq n \leq N - 1 \\ 0 & \text{else} \end{cases} \]  

(3.21)

The scaling is motivated by having a filter with a total energy equal to 1, i.e., \( \sum_{n=-\infty}^{+\infty} |p[n]|^2 = 1 \).
The filter for sub-carrier \( i \) for the discrete time implementation is a shifted version of the base filter in (3.21), given by

\[
p_i[n] := p[n] e^{j2\pi \frac{n}{N}} \quad (3.22)
\]

**Demodulated signal**  As in the continuous time case, the demodulation in the discrete time case is performed using the matching filter demodulation. Therefore, the demodulated signal in sub-carrier \( l \) follows from (2.2) and it is given by

\[
R_l = r[n] * g_l[n] \bigg|_{n=0},
\]

where \( r[n] \) is the sampled base-band received signal, \( * \) is the convolution operator and \( g_l[n] \) is the matched filter for sub-carrier \( l \), which is defined as

\[
g_l[n] := \overline{p_l[-n]} \quad (3.24)
\]

### 3.4 Discrete time Demodulated signal subject to phase noise

In this section we study the effect that has the sampling of the continuous signal and the following discrete time demodulation in the degradations produced by phase noise.

The sampled base-band received signal with a sample rate \( f_s \) subject to phase noise is given by

\[
r[n] = \sqrt{\frac{T}{N}} \tilde{r}(t) \bigg|_{t = \frac{n}{f_s}} \overset{(a)}{=} \sqrt{\frac{T}{N}} \tilde{x}(t) (1 + j\tilde{\phi}(t)) \bigg|_{t = \frac{n}{f_s}} \overset{(b)}{=} x[n] (1 + j\phi[n]) \quad (3.25)
\]

(a) follows from the definition of the received signal with the effects of phase noise in (2.18); (b) follows from (3.20) and from the sampled version of \( \tilde{\phi}(t) \), which is

\[
\phi[n] = \tilde{\phi}(t) \bigg|_{t = \frac{n}{f_s}} \quad (3.26)
\]

The demodulated signal subject to phase noise using a discrete time demodulator is given by

\[
R_l \overset{(a)}{=} r[n] * g_l[n] \bigg|_{n=0} \overset{(b)}{=} X_l + N_{lCPE}^l + N_{lICI}^l \quad (3.27)
\]

(a) follows from the definition of the demodulated signal for the discrete time implementation in (3.23); (b) follows from the same procedure as in (3.4) and (3.7) but replacing the continuous signals by the sampled ones. Therefore, the noise terms for the discrete case are defined as

\[
N_{lCPE}^l := X_l \varphi_{l,l} \quad (3.28)
\]

\[
N_{lICI}^l := \sum_{i \in I_{IC1}} X_i \varphi_{i,l} \quad (3.29)
\]
In the previous expressions (3.28) and (3.29), $\varphi_{i,l}$ is a random variable that contains the phase noise terms for the discrete case. Similarly to the case of $\tilde{\varphi}_{i,l}$ in (3.6), $\varphi_{i,l}$ is defined as

$$\varphi_{i,l} := (j\phi[n] p_i[n]) * g_l[n] = \sum_{s=-\infty}^{\infty} j\phi[s] p_i[s] p_l[s],$$  \hfill (3.30)$$

where we have the same derivation as in (3.6) replacing the continuous signals by its sampled version.

From the previous results in (3.28) and (3.29) we can conclude that the two different kind of interferences produced in the discrete time case are the same than in the continuous time case, i.e., the interference types are PN-CPE and PN-ICI. In the following sections we study the differences on the power of this interferences between the discrete and continuous time cases.

It is important to say that in the sampling no anti-aliasing filter has been used. The motivation for this decision is that for frequencies outside the sampling bandwidth neither the signal nor the phase noise have significant components, so the aliasing introduced by the sampling is negligible. Not considering the anti-aliasing filter gives a more suitable set of equations.

### 3.4.1 Interference power

In this section the power of the interference terms $N_{CPE}^i$ and $N_{ICI}^i$ is derived. The derivation follows the same steps than (3.10) and (3.11). Therefore, the power of the noise terms is given by

$$P_{N_{CPE}} := E_X \left( E \left[ |\varphi_{CPE}|^2 \right] \right)$$ \hfill (3.31)$$

$$P_{N_{ICI}} := E_X \left( \sum_{i \in I_{ICI}} E \left[ |\varphi_{ICI}|^2 \right] \right)$$ \hfill (3.32)$$

According to Appendix A.2, $E[\varphi_{i,l}]$ is given by

$$E[\varphi_{i,l}] = \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l}(\nu) \, d\nu,$$ \hfill (3.33)$$

in which $\nu$ is the normalized frequency ($\nu := \frac{f}{f_s}$), $S_{\phi}(\nu) := f_s \sum_{k=-\infty}^{\infty} S_{\tilde{\phi}}((\nu - k) f_s)$ is the PSD of the sampled phase noise and $W_{i,l}(\nu)$ is defined as

$$W_{i,l}(\nu) := \frac{1}{N^2} \left| \frac{\sin \left( \pi \left( \nu - \frac{i}{N} \right) \right)}{\sin \left( \pi \left( \nu - \frac{l}{N} \right) \right)} \right|^2$$ \hfill (3.34)$$

Applying the previous result in (3.33) to (3.31) and (3.32) the final expressions for the power of the interference terms is obtained

$$P_{N_{CPE}} = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,CPE}(\nu) \, d\nu \right)$$ \hfill (3.35)$$

$$P_{N_{ICI}} = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,ICI}(\nu) \, d\nu \right)$$ \hfill (3.36)$$
where

\[ W_CPE(l) : = W_{l,l}(\nu) \]  \hspace{1cm} (3.37)

\[ W_{ICI}(\nu) = \sum_{i \in I_C} W_{i,l}(\nu) \]  \hspace{1cm} (3.38)

### 3.4.2 Signal-to-Interference Ratio due to phase noise

From (3.27) the expression for the SIR in the demodulated signal in sub-carrier \( l \) subject to phase noise is easily derived, obtaining

\[
SIR_l = \frac{E_X}{P_{N_{CPE}} + P_{N_{ICI}}} = \frac{1}{\int_{-0.5}^{+0.5} [W_{CPE}(\nu) + W_{ICI}(\nu)] S_\phi(\nu) d\nu}
\]  \hspace{1cm} (3.39)

(a) follows from (3.35) and (3.36). \( P_{N_{CPE}} \) and \( P_{N_{ICI}} \) are the power of the interference terms produced by the PN-CPE and PN-ICI respectively and \( E_X \) is the power of the transmitted symbol in the active sub-carrier \( l \), and it is defined as \( E_X : = E \left[ |X_l|^2 \right] \).

**Effect of removing PN-CPE** The expression for the SIR without the effect of PN-CPE in sub-carrier \( l \) is easily derived from (3.39) by removing the PN-CPE contribution, giving

\[
SIR_l = \frac{E_X}{P_{N_{ICI}}} = \frac{1}{\int_{-0.5}^{+0.5} W_{ICI}(\nu) S_\phi(\nu) d\nu}
\]  \hspace{1cm} (3.40)

### 3.5 Weighting functions

In this section we present the tools usually used in MCM to compute the power of the interferences caused by phase noise. In general, in the literature these tools are known as weighting functions. Previously they were used in [3].

Weighting functions provide a graphical understanding of the level of the interferences produced by phase noise. They shape its PSD in order to get the total power of the interferences, i.e., they indicate for which frequencies the PSD of phase noise has less and more influence in the total power of the interference terms. This effect can be seen in Figure 3.1. Thus, if the shape of the weighting functions and the PSD of phase noise are known, it can be understood how the power of the interference terms changes as a function of different parameters of the waveforms (as \( \Delta f \)).

In Section 3.2.1 we derived the expressions for the power of the interferences produced by phase noise for continuous time OFDM. The final expressions were

\[
P_{N_{CPE}} = E_X \left( \int_{-\infty}^{\infty} \widehat{W}_{CPE}(f) S_\phi(f) df \right)
\]

\[
P_{N_{ICI}} = E_X \left( \int_{-\infty}^{\infty} \widehat{W}_{ICI}(f) S_\phi(f) df \right)
\]

In the previous expression \( S_\phi(f) \) is multiplied by some functions that change its shape. This functions are \( \widehat{W}_{CPE}(f) \) and \( \widehat{W}_{ICI}(f) \), which we are going to call weighting functions for the PN-CPE and PN-ICI effects respectively. They were previously defined in (3.16) and (3.17).
Note: From now on we introduce some notation to refer to the positions of the sub-carriers. When we talk about the DC sub-carrier we refer to the active sub-carrier in the middle of the spectrum \((l = \frac{N}{2})\) and when we talk about EDGE sub-carrier we refer to the first active sub-carrier of the spectrum \((l = \text{N}_g)\).

3.5.1 Shape analysis

Next, the shape of the weighting functions for the continuous time case and the way it influences the power of the interference terms is studied. The power of the interferences do not only depend on the weighting functions but also on the PSD of phase noise. Consequently, it is important to know the PSD of phase noise before carrying out the analysis. Although different oscillators have different PSD, they all share a very important property: big part of the power of its PSD is gathered around its central frequency \((f = 0\text{ in base-band signals})\). This property is very important because it makes the following analysis applicable to any oscillator. The analysis follows.

The shape of \(\tilde{W}_l^{CPE}(f)\) determines the value of \(P_{\tilde{N}_{CPE}}\). When \(\Delta f\) increases, the main lobe of \(\tilde{W}_l^{CPE}(f)\) is wider around \(f = 0\), implying higher values for \(P_{\tilde{N}_{CPE}}\). \(\tilde{W}_l^{CPE}(f)\) is identical \(\forall l \in I\), so \(P_{\tilde{N}_{CPE}}\) is identical \(\forall l \in I\). This can be seen in Figure 3.2.

The value of \(P_{\tilde{N}_{ICI}}\) depends on the shape of \(\tilde{W}_l^{ICI}(f)\). Its performance is opposite to the case of PN-CPE. When \(\Delta f\) increases the gap of \(\tilde{W}_l^{ICI}(f)\) around \(f = 0\) is wider, producing lower values for \(P_{\tilde{N}_{ICI}}\). This can be seen in Figure 3.3.

In Figure 3.3 we can see that \(\tilde{W}_l^{ICI}(f)\) has different shapes for different values of \(l\), so \(P_{\tilde{N}_{ICI}}\) also has different values. In the following sections we focus in the best and worst cases in terms of interference power (DC and EDGE sub-carrier respectively).
The combined total effect of PN-CPE and PN-ICI depends on the shape \( \tilde{W}_{\text{CPE}}(f) + \tilde{W}_{\text{ICI}}(f) \). For DC sub-carrier, if \( \Delta f \) is changed the differences in the shape of \( \tilde{W}_{\text{CPE}}(f) + \tilde{W}_{\text{ICI}}(f) \) are located in frequencies distant from \( f = 0 \) (where \( S_2 \approx 0 \)), so \( P_{\text{CPE}} + P_{\text{ICI}} \) does not change significantly. However, for the EDGE sub-carrier, the differences are located in frequencies around \( f = 0 \), so \( P_{\text{CPE}} + P_{\text{ICI}} \) increases when \( \Delta f \) increases.

Summarizing, when \( \Delta f \) increases \( P_{\text{CPE}} \) increases and \( P_{\text{ICI}} \) decreases \( \forall l \in I \). Also, \( P_{\text{CPE}} + P_{\text{ICI}} \) for the DC sub-carrier is nearly constant for different \( \Delta f \) but for the other sub-carriers it increases with the increasing of \( \Delta f \). These facts are shown in Figure 3.5.

Figure 3.2: Comparison of \( \tilde{W}_{\text{CPE}}(f) \) with \( B_W = 100 \text{ MHz} \) for different \( l \) and \( \Delta f \).

Figure 3.3: Comparison of \( \tilde{W}_{\text{ICI}}(f) \) with \( B_W = 100 \text{ MHz} \) for different \( l \) and \( \Delta f \).
3.5.2 Comparison between continuous and discrete time

In this section the comparison between the weighting functions for the continuous and discrete time is made. For the comparison we are going to use $\Delta f = 5$ MHz and $B_W = 100$ MHz. For the discrete time case $f_s = 110$ MHz is used.

The differences between both cases are due to: 1) The aliasing produced in the sampling. 2) The periodicity in the spectrum of the sampled signal (which modifies the interference between sub-carriers).
In Figure 3.6, $\tilde{W}$ and $W$ have been plotted. The differences in the shape of both weighting functions are insignificant. Taking into consideration that $S(f) \approx 0$ for $|f| > f_s$, we can state that $PN \approx P\tilde{N}$ for all $l$.

![Figure 3.6: Comparison of $\tilde{W}$ and $W$.](image)

In Figure 3.7 we have plotted $\tilde{W}$ and $W$ for the DC sub-carrier. As in the case of PN-CPE, the differences in the shape of $\tilde{W}$ and $W$ are negligible. Taking into consideration that $S(f) \approx 0$ for $|f| > f_s$, we can consider $PN \approx P\tilde{N}$ for the DC sub-carrier.

![Figure 3.7: Comparison of $\tilde{W}$ and $W$ in the DC sub-carrier.](image)

In Figure 3.8, $\tilde{W}$ and $W$ for the EDGE sub-carrier have been plotted. This case is the one in which the differences between the shapes of $\tilde{W}$ and $W$ are higher and more noticeable. Therefore, for the EDGE sub-carrier it is important to study the differences between $PN$ and $P\tilde{N}$ (which are produced by the periodicity of the sampled signal in the frequency...
domain).

Figure 3.8: Comparison of $\tilde{W}_l^{ICL}$ and $W_l^{ICL}$ in the EDGE sub-carrier.

In Figure 3.9, $P_{NICl}$ as a function of the sampling frequency is compared with $\tilde{P}_{NICl}$, both for the EDGE sub-carrier. From the graph it can be deduced that the sampling has a significant effect on $P_{NICl}$, and it can be seen how the value of $P_{NICl}$ is closer to $\tilde{P}_{NICl}$ when $f_s$ is increased. We can consider $P_{NICl} \approx \tilde{P}_{NICl}$ for $f_s \geq 1.1 B_W$ ($B_W$ is the signal bandwidth).

In conclusion, sampling has no significant effect for $P_{NICl}$ for $l \in I$ neither for $P_{NICl}$ in the DC sub-carrier. It has been shown that for $f_s \geq 1.1 B_W$ the effect of sampling in $P_{NICl}$ for the rest of sub-carriers is negligible. Therefore, from now on only the discrete time case is going to be studied and evaluated for the rest of waveforms. Also, the analysis of the shape of the weighing functions in Section 3.5.1 is applicable to the discrete time case.

Figure 3.9: Comparison of $P_{NICl}$ and $\tilde{P}_{NICl}$ as function of $f_s$ for $l = N_g$, $B_W = 100$ MHz, $\Delta f = 10$ kHz and $f_c = 82$ GHz.
3.6 Signal-to-Interference Ratio evaluations

In this section we evaluate the SIR using the derived expressions, also using simulations. The parameters $B_W = 100$ MHz and $f_s = 110$ MHz have been used.

3.6.1 Theoretical results

The theoretical evaluations for the Signal-to-Interference Ratio produced by phase noise are presented in this section. The results have been obtained with the expressions derived in Section 3.4.2 and taking as phase noise model the mmMagic project phase noise model introduced in Section 2.2.

**SIR due to phase noise** In Figure 3.10 the SIR as a function of $\Delta f$ including the effects of PN-CPE and PN-ICI has been plotted (for the EDGE and DC sub-carrier). In the graph it is obvious that degradations produced by phase noise are more important for high carrier frequencies. Also, the SIR does not change much in relation with $\Delta f$.

![Figure 3.10: SIR due to phase noise (including PN-CPE and PN-ICI effects) for three different $f_c$ and two different values of $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.](image)

In OFDM the interference term produced by the PN-CPE is easily removed (as was explained in Section 3.2). There are different methods to remove the PN-CPE (all of them make use of pilot symbols) as the one presented in [14]. So, as the PN-CPE correction is an easy process, it makes sense to study the effect of phase noise when PN-CPE is removed.

**Effect of removing PN-CPE** In Figure 3.11 the SIR as a function of $\Delta f$ including only the PN-ICI effect has been plotted (for the EDGE and DC sub-carrier). Removing the PN-CPE contributions has a high impact in the SIR, specially for high sub-carrier spacings in which $P_{N_{ICI}}$ is quite low compared with $P_{N_{CPE}}$. After removing the PN-CPE contribution, the SIR increases when $\Delta f$ increases.
3.6.2 Simulations results

In this subsection we present the SIR results obtained by Monte Carlo evaluations in order to check the accuracy of the theoretical evaluations. For the simulations the phase noise generator developed by mmMagic project has been used. The parameters used for the simulations are $B_W = 100 \text{ MHz}$ and $f_s = 110 \text{ MHz}$.

SIR due to phase noise

In Figure 3.12 the SIR as a function of $\Delta f$ for the theoretical and simulation results including the effects of the PN-CPE and PN-ICI has been plotted (for the EDGE and DC sub-carrier). In the figure it is obvious that the simulation results are in accordance with the theoretical ones.

Effect of removing PN-CPE

In Figure 3.13 the PN-SIR as a function of $\Delta f$ for the theoretical and simulation results including only the PN-ICI effect has been plotted (for the EDGE and DC sub-carrier). It is clear from the figure that the simulation results are coherent with the theoretical ones.
3.7 Symbol Error Rate simulations

The simulation results for the Symbol Error Rate (SER) produced by phase noise in an OFDM transmission are presented in this section\(^1\). For the simulations, we use the numerology for 5G.
proposed by Nokia Bells Labs and Ericsson Research in [15] and shown in Table 3.1.

<table>
<thead>
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<th>$f_c$</th>
<th>6 GHz</th>
<th>28 GHz</th>
<th>82 GHz</th>
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<td>60 kHz</td>
<td>480 kHz</td>
</tr>
<tr>
<td>$f_s$</td>
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<td>122.88 MHz</td>
<td>983.04 MHz</td>
</tr>
<tr>
<td>$N$</td>
<td>2048</td>
<td>2048</td>
<td>2048</td>
</tr>
<tr>
<td>$N_{act}$</td>
<td>1200</td>
<td>1200</td>
<td>1200</td>
</tr>
</tbody>
</table>

Table 3.1: 5G numerology for OFDM in carrier frequencies above 6 GHz proposed by Nokia Bells Labs and Ericsson Research.

The SER has been simulated for different values of Additive White Gaussian Noise (AWGN) and the influence of phase noise. Therefore, we show how limiting is phase noise in the performance of an OFDM transmission subject to AWGN.

In Figure 3.14 we have plotted simulation results of the SER as a function of the SNR produced by the AWGN using the parameters in Table 3.1. Three different curves have been plotted: The SER produced only by AWGN, the SER produced by AWGN and phase noise and the SER produced by AWGN and phase noise with the PN-CPE correction. Hence, we can see which is the effect of phase noise and the significance of the PN-CPE correction. Also, each curve has been plotted for two different symbol constellations, 16-QAM and 64-QAM.

From the presented results different facts can be observed. For the parameters given in Table 3.1 for low carrier frequencies in the mm-Band (as $f_c = 6 \text{ GHz}$) phase noise does not produce a significant degradation in the performance of OFDM (neither for 16-QAM nor for 64-QAM). For $f_c = 28 \text{ GHz}$ it is a significant degradation for high order symbol constellations (64-QAM), however when the PN-CPE component is removed the performance is very similar to the one with only the effect of AWGN (for both symbol constellations). For high carrier frequencies (as $f_c = 82 \text{ GHz}$) the SER increasing produced by phase noise is significant for SNR values higher than 15 dB. Specially for 64-QAM in which even after removing the PN-CPE component the degradation produced by phase noise is significant.

An important conclusion is that the performance of OFDM subject to phase noise and AWGN is not significantly degraded respect to the case with only AWGN thanks to the possibility of removing the PN-CPE contribution (which is an important advantage of OFDM respect to other waveforms in which none of the contributions produced by phase noise can be removed, as we will study in the following chapters).

---

1The probability density distribution (PDF) of $N_{CPE}$ and $N_{ICI}$ does not follow a Gaussian distribution as shown in [16]. Therefore in order to obtain theoretical results for the SER the mentioned PDF’s should be derived, but this topic is out of the scope of this thesis.
Figure 3.14: Simulation results for the SER due to phase noise and AWGN for the parameters in Table 3.1.
Chapter 4

QAM-FBMC

Quadrature Amplitude Modulation - Filter-Bank Multi-carrier (QAM-FBMC) is a non-orthogonal MCM proposed for 5G as alternative to OFDM. Its main features were introduced in Section 2.1.2. This chapter presents the derivations and evaluations carried out in order to study the effects of phase noise in QAM-FBMC. Also the effects produced by the self-interferences presented in Section 2.1.2 are studied. First we start deriving the expressions for the modulated and demodulated signals. Next we study the effects of phase noise and self-interferences in the demodulated signal, deriving expressions for the power of the interference terms. Following we derive the expressions for the Symbol-to-Interference Ratio (SIR). Once the expressions for the SIR are obtained, an analysis of the functions that determine the value of the SIR is carried out (we refer to these functions as weighting functions). Finally, the SIR is evaluated for some concrete implementation parameters.

4.1 Modulated and demodulated signal

Modulated signal

The discrete time modulated signal for a QAM-FBMC symbol with $N$ sub-carriers follows the expression

$$x[n] = \sum_{d \in D} \sum_{i \in I} X_{i,d} p_i[n - dN],$$

(4.1)

where $X_{i,d}$ is any symbol modulated in sub-carrier $i$ and overlap index $d$, $I$ is the set with the index of active sub-carriers, $D$ is the set with the index of the overlapping symbols ($D = \{-K + 1, \ldots, K - 1\}$) and $p_i[n]$ is the transmitter filter for sub-carrier $i$.

The filter for sub-carrier $i$ is defined as

$$p_i[n] := \begin{cases} p_{\text{even}}[n] e^{j2\pi i n} & \text{if } i \text{ is even} \\ p_{\text{odd}}[n] e^{j2\pi i n} & \text{if } i \text{ is odd} \end{cases},$$

(4.2)

where $p_{\text{even}}[n]$ and $p_{\text{odd}}[n]$ are the base filters for even and odd sub-carriers respectively. The base filters are FIR filters with $L = KN$ real-valued time coefficients and total energy equal to 1, i.e., $\sum_{n=-\infty}^{\infty} |p_{\text{even}}[n]|^2 = \sum_{n=-\infty}^{\infty} |p_{\text{odd}}[n]|^2 = 1$. 


The demodulation is performed using the matched filter demodulation. So, the demodulated signal in sub-carrier \( l \) follows from (2.2) and it is given by

\[
R_l = r[n] * g_l[n] \bigg|_{n=0} ,
\]

in which \( r[n] \) is the sampled base-band received signal, \( * \) is the convolution operator and \( g_l[n] \) is the matched filter for sub-carrier \( l \), which is defined as

\[
g_l[n] := p_i[-n]
\]

Note: Ideally the demodulation should produce \( R_l = X_{l,0} \), however this equality is not possible due to the intrinsic self-interferences of the waveform.

### 4.2 Demodulated signal subject to phase noise

The sampled base-band received signal with a sample rate \( f_s \) subject to phase noise is given by

\[
r[n] = x[n] (1 + j\phi[n]),
\]

which follows from the sampled version of (2.18) while \( \phi[n] \) is the sampled phase noise, defined as \( \phi[n] = \tilde{\phi}(t) \big|_{t=nf_s} \).

The demodulated signal subject to phase noise follows

\[
R_l = \bigg[ r[n] (1 + j\phi[n]) \bigg] * g_l[n] \bigg|_{n=0}
\]

\[\]

\[
\sum_{d \in D} \sum_{i \in I} X_{i,d} \bigg[ p_i[n-dN] * g_l[n] \bigg] \bigg|_{n=0} + \sum_{d \in D} \sum_{i \in I} X_{i,d} \bigg[ j\phi[n] p_i[n-dN] * g_l[n] \bigg] \bigg|_{n=0}
\]

\[\]

\[
\sum_{d \in D} \sum_{i \in I} X_{i,d} \lambda_{i,l,d} + \sum_{d \in D} \sum_{i \in I} X_{i,d} \beta_{i,l,d}
\]

\[\]

\[
X_{l,0} + \sum_{i \in I_{IC}} X_{i,0} \lambda_{i,l,0} + \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \lambda_{i,l,d} + \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \beta_{i,l,d}
\]

(a) follows from (4.3); (b) follows from (4.5); (c) follows from (4.1); (d) follows from the distributivity property of the convolution; (e) follows by defining \( \lambda_{i,l,d} := p_i[n-dN] * g_l[n] \bigg|_{n=0} \) and \( \beta_{i,l,d} := [j\phi[n] p_i[n-dN] * g_l[n] \bigg|_{n=0} \); (f) follows from separate the different terms according to its cause. Two new sets of indices have been introduced in (4.6), \( I_{IC} = I \setminus \{ l \} \) and \( D_{ISI} = D \setminus \{ 0 \} \), which contain the indices of the interfering sub-carriers and symbols respectively.
The terms in (4.6) that only depend on the impulse response of the filter are

\[ \lambda_{i,l,d} = p_i[n-dN] * g_l[n] \bigg|_{n=0}^{(a)} = \sum_{s=-\infty}^{\infty} p_i[s-dN] g_l[0-s] \bigg|_{n=0}^{(b)} = \sum_{s=-\infty}^{\infty} p_i[s-dN] p_l[s] \]  

(4.7)

(a) follows from the convolution definition evaluated in \( n = 0 \); (b) follows from (4.4). It is important that \( \forall i \in I, \forall l \in I \) and \( \forall d \in D \lambda_{i,l,d} \) is a deterministic parameter.

An important result that has been applied in (4.6) related with the term in (4.7) is

\[ \lambda_{l,l,0} = \sum_{s=-\infty}^{\infty} p_l[s] p_l[s] = \begin{cases} \sum_{s=-\infty}^{\infty} |p_{\text{even}}[s]|^2 = 1 & \text{if } l \text{ even} \\ \sum_{s=-\infty}^{\infty} |p_{\text{odd}}[s]|^2 = 1 & \text{if } l \text{ odd} \end{cases} \]  

(4.8)

The terms in (4.6) that not only depend on the impulse response of the filter but also in phase noise are

\[ \beta_{i,l,d} = [j \phi[n] p_i[n-dN] * g_l[n] \bigg|_{n=0} = \sum_{s=-\infty}^{\infty} j \phi[s] p_i[s-dN] g_l[0-s] = \sum_{s=-\infty}^{\infty} j \phi[s] p_i[s-dN] p_l[s] \]  

where the same steps than in (4.7) have been followed. In this case \( \forall i \in I, \forall l \in I \) and \( \forall d \in D \beta_{i,l,d} \) is a random variable because it depends on \( \phi[n] \).

The expression in (4.6) can be rewritten by defining a interference term for each of the effects produced by self-interferences and phase noise

\[ R_l = X_{i,0} + I_{ICI}^l + I_{ISI}^l + N_{CPE}^l + N_{ICI}^l + N_{ISI}^l, \]  

(4.10)

where \( I_{ICI}^l, I_{ISI}^l, N_{CPE}^l, N_{ICI}^l \) and \( N_{ISI}^l \) are defined next from (4.11) to (4.15).

The interference terms in (4.10) can be classified according to its cause:

- Interference terms produced by self-interferences

\[ I_{ICI}^l := \sum_{i \in I_{ICI}} X_{i,0} \lambda_{i,l,0} \]  

(4.11)

\[ I_{ISI}^l := \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \lambda_{i,l,d} \]  

(4.12)

which are the interference terms related with the effects presented in 2.1.2.

- Interference terms produced phase noise

\[ N_{CPE}^l := X_{i,0} \beta_{i,l,0} \]  

(4.13)

\[ N_{ICI}^l := \sum_{i \in I_{ICI}} X_{i,0} \beta_{i,l,0} \]  

(4.14)

\[ N_{ISI}^l := \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \beta_{i,l,d} \]  

(4.15)
As has been proved in this section, phase noise produces three different effects on the demodulated signal:

- **Phase noise Common Phase Error (PN-CPE).** It is identified by $N_{ICPE}^l$ and it is a rotated version of the ideally transmitted symbol $X_l$. The rotation is given by the term $\beta_{l,0}$, and it is identical for any even sub-carrier $l$ and identical for any odd sub-carrier $l$ of a QAM-FBMC symbol. This fact makes the PN-CPE correction a very easy process using pilot symbols.

- **Phase noise Inter-Carrier Interference (PN-ICI).** The introduction of phase noise worsens the orthogonality between sub-carriers. Therefore interference appears from the rest of the sub-carriers in the demodulated one, producing the interference term $N_{ICI}^l$.

- **Phase noise Inter-Symbol Interference (PN-ISI).** Phase noise also worsen the interference between overlapping symbols, producing the interference term $N_{ISI}^l$.

### 4.2.1 Interference power

In this subsection we are going to derive an expression for the power of the interference terms produced by self-interferences ($I_{ICl}^l$ and $I_{ISI}^l$) and phase noise ($N_{ICPE}^l$, $N_{ICI}^l$ and $N_{ISI}^l$).

The power of the interference terms due to self-interferences is given by

$$P_{I_{ICl}} := \mathbb{E} \left[ |I_{ICl}^l|^2 \right] = \mathbb{E} \left[ \sum_{\nu \in I_{ICl}} |X_{\nu,0} \lambda_{\nu,l,0}|^2 \right] = \mathbb{E}_X \left( \sum_{\nu \in I_{ICl}} |\lambda_{\nu,l,0}|^2 \right) \quad (4.16)$$

$$P_{I_{ISI}} := \mathbb{E} \left[ |I_{ISI}^l|^2 \right] = \mathbb{E} \left[ \sum_{\nu \in D_{ISI}} \sum_{\nu \in I} |X_{\nu,d} \lambda_{\nu,l,d}|^2 \right] = \mathbb{E}_X \left( \sum_{\nu \in D_{ISI}} \sum_{\nu \in I} |\lambda_{\nu,l,d}|^2 \right) \quad (4.17)$$

In (4.16) and (4.17) has been considered that $\forall \nu \in I$ and $\forall d \in D \{X_{\nu,d}\}$ is a sequence of zero-mean independent random variables. Moreover, $\mathbb{E}_X$ is the power of the mapped symbols and it is defined as $\mathbb{E}_X = \mathbb{E} \left[ |X_{\nu,d}|^2 \right] \forall \nu \in I$ and $\forall d \in D$.

The power of both self-interference terms is dependent on $|\lambda_{\nu,l,d}|^2$. According to Appendix B.1, it can be expressed as

$$|\lambda_{\nu,l,d}|^2 = W_{\nu,l,d}(\nu) \delta(\nu), \quad (4.18)$$

where

$$W_{\nu,l,d}(\nu) := \left| P_l(\nu) e^{-j2\pi \nu dN} \oplus \overline{P_l(\nu)} \right|^2, \quad (4.19)$$

being $P_l(\nu) := \mathcal{F}\{p[n]\}$ ($\mathcal{F}\{\}$ is the operator for the DTFT).

Using (4.18) in (4.16) and (4.17), we get

$$P_{I_{ICl}} = \mathbb{E}_X \left( \sum_{\nu \in I_{ICl}} W_{\nu,l,0}(\nu) \delta(\nu) \right) = \mathbb{E}_X \delta(\nu) W_{ICl}^l(\nu) \quad (4.20)$$

$$P_{I_{ISI}} = \mathbb{E}_X \left( \sum_{\nu \in D_{ISI}} \sum_{\nu \in I} W_{\nu,l,d}(\nu) \delta(\nu) \right) = \mathbb{E}_X \delta(\nu) W_{ISI}^l(\nu) \quad (4.21)$$
In (4.24), (4.25), and (4.26) has been considered that\(\forall\{\text{random variables}\}\) whose elements are independent of the elements of\(\forall\{\text{zero-mean independent random variables}\}\). Also it has been considered that\(\forall\{S\}\) where\(W\) is defined as

\[
W^I_{cl} = \sum_{i \in IcI} W_{i,l,0}(\nu)
\]

The power of the interference terms due to phase noise is given by

\[
P_{N_{CPE}} := E \left[ |N^I_{CPE}|^2 \right] = E \left[ |X_{i,0} \beta_{i,l,0}|^2 \right] = E_X \left[ |\beta_{i,l,0}|^2 \right] \tag{4.24}
\]

\[
P_{N_{cl}} := E \left[ |N^I_{cl}|^2 \right] = E \left[ \sum_{i \in IcI} X_{i,0} \beta_{i,l,0} \right]^2 = E_X \left( \sum_{i \in IcI} E \left[ |\beta_{i,l,0}|^2 \right] \right) \tag{4.25}
\]

\[
P_{N_{ISI}} := E \left[ |N^I_{ISI}|^2 \right] = E \left[ \sum_{d \in DIISI} \sum_{i \in I} X_{i,d} \beta_{i,l,d} \right]^2 = E_X \left( \sum_{d \in DIISI} \sum_{i \in I} E \left[ |\beta_{i,l,d}|^2 \right] \right) \tag{4.26}
\]

In (4.24), (4.25), and (4.26) has been considered that\(\forall i \in I\) and\(\forall d \in D\)\(\{X_{i,d}\}\) is a sequence of zero-mean independent random variables. Also it has been considered that\(\{\beta_{i,l,d}\}\) is a sequence of random variables whose elements are independent of the elements of\(\{X_{i,d}\}\)\(\forall i \in I, \forall l \in I\) \(\forall d \in D\).

The power of the interference components is dependent on\(E \left[ |\beta_{i,l,d}|^2 \right]\). According to Appendix (B.2), it is given by

\[
E \left[ |\beta_{i,l,d}|^2 \right] = \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,d}(\nu) d\nu \tag{4.27}
\]

where\(S_{\phi}(\nu)\) is the PSD of the sampled phase noise and\(W_{i,l,d}(\nu)\) was defined previously in (4.19).

Using (4.27) in (4.24), (4.25) and (4.26) we get

\[
P_{N_{CPE}} = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,0}(\nu) d\nu \right) = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W^I_{CPE}(\nu) d\nu \right) \tag{4.28}
\]

\[
P_{N_{cl}} = E_X \sum_{i \in IcI} \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,0}(\nu) d\nu \right) = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W^I_{cl}(\nu) d\nu \right) \tag{4.29}
\]

\[
P_{N_{ISI}} = E_X \sum_{d \in DIISI} \sum_{i \in I} \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,d}(\nu) d\nu \right) = E_X \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) W^I_{ISI}(\nu) d\nu \right) \tag{4.30}
\]

where\(W^I_{cl}(\nu)\) and\(W^I_{ISI}(\nu)\) were previously defined in (4.22) and (4.23) respectively.\(W^I_{CPE}(\nu)\) is defined as

\[
W^I_{CPE}(\nu) := W_{i,l,0}(\nu) \tag{4.31}
\]

\[\]

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4.2.2 Signal-to-Interference Ratio

In this subsection the effect of self-interferences and phase noise in the Signal-to-Interference Ratio (SIR) is studied. First, the expression with only the effect of self-interferences is derived. Next, the expression with both effects (self-interferences and phase noise) is obtained. Finally, we derive the expression for the SIR when the PN-CPE effect is removed.

SIR due to self-interferences From (4.10) the expression for the SIR in the demodulated signal in sub-carrier \( l \) produced by the self-interferences is easily. Taking into account only the interference terms produced by the self-interferences (\( I_{ICI}^l \) and \( I_{ISI}^l \)), the SIR is given by

\[
SIR^l = \frac{E_X}{P_{ICI}^l + P_{ISI}^l} \stackrel{(a)}{=} \frac{1}{\int_{-0.5}^{+0.5} W_{ICI}^l(\nu) S_\phi(\nu) d\nu + \int_{-0.5}^{+0.5} [W_{ICI}^l(\nu) + W_{ISI}^l(\nu)] \delta(\nu) + S_\phi(\nu) d\nu}
\]

\((a)\) follows from (4.20) and (4.21). \( P_{ICI}^l \) and \( P_{ISI}^l \) are the power of the interferences produced by S-ICI and S-ISI. \( E_X \) is the power of the transmitted symbol in sub-carrier \( l \), it is defined as \( E_X := \mathbb{E} \left| X_{l,0} \right|^2 \).

SIR due to self-interferences and phase noise From (4.10) the expression for the SIR in the demodulated signal in sub-carrier \( l \) produced by the combined effect of self-interferences and phase noise is obtained

\[
SIR^l = \frac{E_X}{P_{ICI}^l + P_{ISI}^l + P_{NPE} + P_{NICI}^l + P_{NISI}^l} \stackrel{(a)}{=} \frac{1}{\int_{-0.5}^{+0.5} W_{CPE}^l(\nu) S_\phi(\nu) d\nu + \int_{-0.5}^{+0.5} [W_{ICI}^l(\nu) + W_{ISI}^l(\nu)] [\delta(\nu) + S_\phi(\nu)] d\nu}
\]

\((a)\) follows from (4.20), (4.21), (4.28), (4.29), and (4.30). \( P_{NPE}, P_{NICI}^l \) and \( P_{NISI}^l \) are the power of PN-CPE, PN-ICI and PN-ISI respectively.

Effect of removing PN-CPE Assuming that the PN-CPE effect is completely removed, the expression for the SIR produced by the combined effect of self-interferences and phase noise in sub-carrier \( l \) is given by

\[
SIR^l = \frac{E_X}{P_{ICI}^l + P_{ISI}^l + P_{NICI}^l + P_{NISI}^l} \stackrel{(a)}{=} \frac{1}{\int_{-0.5}^{+0.5} [W_{ICI}^l(\nu) + W_{ISI}^l(\nu)] [\delta(\nu) + S_\phi(\nu)] d\nu}
\]

\((a)\) follows from (4.20), (4.21), (4.29) and (4.30).

4.3 Weighting functions

In Section 3.5 the concept of weighting function was presented. They are tools used in MCM compute the level of the interferences produced by phase noise. In QAM-FBMC, the weighting functions are also used to compute the level of the interferences produced by self-interferences.
In Section 4.2.1 we derived the expressions for the power of the interferences produced by phase noise, which are

\[
P_{N_{\text{CPE}}} = E_X \left( \int_{-0.5}^{0.5} S_\delta(\nu) W_{l_{\text{CPE}}}^{\nu}(\nu) \, d\nu \right)
\]

\[
P_{N_{\text{ICI}}} = E_X \left( \int_{-0.5}^{0.5} S_\delta(\nu) W_{l_{\text{ICI}}}^{\nu}(\nu) \, d\nu \right)
\]

\[
P_{N_{\text{ISI}}} = E_X \left( \int_{-0.5}^{0.5} S_\delta(\nu) W_{l_{\text{ISI}}}^{\nu}(\nu) \, d\nu \right)
\]

In the previous expression \(W_{l_{\text{CPE}}}^{\nu}(\nu), W_{l_{\text{ICI}}}^{\nu}(\nu)\) and \(W_{l_{\text{ISI}}}^{\nu}(\nu)\) are functions that shape \(S_\delta(\nu)\) in order to obtain the power of the interferences. Thus, \(W_{l_{\text{CPE}}}^{\nu}(\nu), W_{l_{\text{ICI}}}^{\nu}(\nu)\) and \(W_{l_{\text{ISI}}}^{\nu}(\nu)\) are the weighting functions for PN-CPE, PN-ICI and PN-ISI respectively. The expressions for the weighting functions can be found in (4.31), (4.22), and (4.23).

The expressions for the power of the interference terms produced by self-interferences were derived in (4.20) and (4.21), giving

\[
P_{l_{\text{ICI}}} = E_X \delta(\nu) W_{l_{\text{ICI}}}^{\nu}(\nu) \quad (4.35)
\]

\[
P_{l_{\text{ISI}}} = E_X \delta(\nu) W_{l_{\text{ISI}}}^{\nu}(\nu) \quad (4.36)
\]

The power of S-ICI and S-ISI are related with the value of the weighting functions in \(\nu = 0\). As \(P_{l_{\text{ICI}}}^{\nu}\) and \(P_{l_{\text{ISI}}}^{\nu}\) do not depend on \(S_\delta(\nu)\), the level of the self-interferences does not depend on \(f_c\).

**Note:** From now on we introduce some notation to refer to the positions of the sub-carriers. When we talk about the DC sub-carrier we will refer to the active sub-carrier in the middle of the spectrum \((l = \frac{N}{2})\) and when we talk about EDGE sub-carrier we will refer to the first active sub-carrier of the spectrum \((l = N_g)\).

### 4.3.1 Shape analysis

As has been studied, the power of the interference terms depends on the shape of the weighting functions. Hence, in this subsection an analysis of the shape of \(W_{l_{\text{CPE}}}^{\nu}(\nu), W_{l_{\text{ICI}}}^{\nu}(\nu)\) and \(W_{l_{\text{ISI}}}^{\nu}(\nu)\) is carried out. As in Section 3.5, it is important to have in mind that most of the power of \(S_\delta(\nu)\) is gathered in frequencies close to \(\nu = 0\). The analysis follows.

The shape of \(W_{l_{\text{CPE}}}^{\nu}(\nu)\) determines the value of \(P_{N_{\text{CPE}}}\). If \(\Delta f\) increases, \(W_{l_{\text{CPE}}}^{\nu}(\nu)\) has higher secondary lobes around \(\nu = 0\), so \(P_{N_{\text{CPE}}}\) increases. This effect can be seen in Figure 4.1. One important property of \(W_{l_{\text{CPE}}}^{\nu}(\nu)\) is that it is identical for all \(l\) even and identical for all \(l\) odd.

The analysis for the PN-ICI and PN-ISI is carried out together. The shape of \(W_{l_{\text{ICI}}}^{\nu}(\nu) + W_{l_{\text{ISI}}}^{\nu}(\nu)\) determines the value of \(P_{N_{\text{ICI}}} + P_{N_{\text{ISI}}}\). If \(\Delta f\) increases, \(W_{l_{\text{ICI}}}^{\nu}(\nu) + W_{l_{\text{ISI}}}^{\nu}(\nu)\) has lower values around \(\nu = 0\) \(\forall l \in I\), so \(P_{N_{\text{ICI}}} + P_{N_{\text{ISI}}}\) decreases. It is important that \(W_{l_{\text{ICI}}}^{\nu}(\nu) + W_{l_{\text{ISI}}}^{\nu}(\nu)\) has different shapes for different values of \(l\). For the DC sub-carrier it has a symmetrical shape around \(\nu = 0\), however for the rest of sub-carriers it is asymmetrical (this is the cause of the differences in \(P_{N_{\text{ICI}}} + P_{N_{\text{ISI}}}\) for different \(l\)). These effects can be seen in Figure 4.2.
Figure 4.1: Comparison of $W_{CPE}^l(\nu)$ for two different values of $l$ with $B_W = 100$ MHz and $f_s = 110$ MHz.

Figure 4.2: Comparison of $W_{ICI}^l(\nu) + W_{ISI}^l(\nu)$ for two different values of $l$ with $B_W = 100$ MHz and $f_s = 110$ MHz.

The value of $P_{ICl} + P_{ISI}$ produced by self-interferences is given by the value of $W_{ICI}^l(0) + W_{ISI}^l(0)$, which does not change for different $\Delta f$ (as can be seen in Figure 4.2).

In conclusion, if $\Delta f$ increases, $P_{CPE}$ increases and $P_{ICl} + P_{ISI}$ decreases. Therefore, for high $\Delta f$, $P_{CPE}$ is dominant over $P_{ICl} + P_{ISI}$ (as can be seen in Figure 4.3).
Figure 4.3: Power of the interference terms produced by phase noise with $f_c = 28$ GHz, $B_W = 100$ MHz and $f_s = 110$ MHz for two different sub-carrier locations.

4.4 Signal-to-Interference Ratio evaluations

In this section we evaluate the SIR using the derived expressions. The parameters $B_W = 100$ MHz and $f_s = 110$ MHz have been used. The base filters proposed in [17] for $K = 4$ are the ones used to obtain the results. As in Chapter 3, the mmMagic project phase noise model introduced in Section 2.2 has been used.

**SIR due to self-interferences** In Figure 4.4 the SIR produced by the self-interferences (S-ICI and S-ISI) as a function of $\Delta f$ is shown. The difference in the SIR for the DC and EDGE sub-carriers is explained because in the EDGE sub-carrier one of the two closest neighbour sub-carriers is a null band, however in the DC sub-carrier both closest sub-carriers are not null bands. The closest neighbours are the sub-carriers that higher level of interference produce, so for the EDGE sub-carrier the interference level is lower. As was explained in Section 4.3, the level of the self-interferences is constant for different values of $\Delta f$.

**SIR due to self-interferences and phase noise** Once the effect of the self-interferences has been studied, we present the evaluations with the combined effect of the self-interferences and phase noise. In Figure 4.5 the SIR with the effects of self-interferences and phase noise as a function of $\Delta f$ is shown. It can be seen how the effect of phase noise produces a decreasing in the SIR respect to the case in Figure 4.4. One important fact is that the effect of phase noise is not very significant because the power of the self-interferences is dominant. Only for high carrier frequencies, where the PSD of phase noise has higher values, the decreasing produced by phase noise in the SIR is significant.

**Effect of removing PN-CPE** As in the case of OFDM, in QAM-FBMC the interference term produced by the CPE is easily removed because it is a rotated version of the transmitted symbol. In Figure 4.6 the SIR with the effects of self-interferences and phase noise without PN-CPE as a function of $\Delta f$ is shown. Removing the effect of PN-CPE increases the SIR.
increasing of the SIR is higher for high $\Delta f$ because in these conditions, $P_{NCPE}$ is dominant over $P_{NICI} + P_{NISI}$, so removing it has a high impact in the SIR.

As was explained in Section 4.3, $P_{NICI} + P_{NISI}$ decreases when $\Delta f$ increases. This is the motivation for the performance in Figure 4.6.

It is important to clarify that for QAM-FBMC no simulation results are presented because the simulator provided by the mmMagic project does not include QAM-FBMC.

![Figure 4.4: SIR due to self-interferences (S-ICI and S-ISI) for two different values of $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.](image-url)
Figure 4.5: SIR due to self-interferences (S-ICI and S-ISI) and phase noise (PN-CPE, PN-ICI and PN-ISI) for three different $f_c$ and two different values of $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.

Figure 4.6: SIR due to self-interferences (S-ICI and S-ISI) and phase noise after PN-CPE removing (PN-ICI and PN-ISI) for three different $f_c$ and two different values of $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.
Chapter 5

OQAM-FBMC

Offset Quadrature Amplitude Modulation - Filter-Bank Multi-carrier (OQAM-FBMC) is a non-orthogonal MCM proposed as waveform for 5G which main features were introduced in Section 2.1.3. As in the case of OFDM and QAM-FBMC, it is very important to study the robustness of OQAM-FBMC against phase noises. This chapter presents the derivations and evaluations carried out in order to study the effects of phase noise in OQAM-FBMC. Also the effects produced by the self-interferences presented in the Section 2.1.3 are studied. First we start deriving the expressions for the modulated and demodulated signals. Next we study the effects of phase noise and self-interferences in the demodulated signal, deriving expressions for the power of the interference terms. Following we derive the expressions for the Symbol-to-Interference Ratio (SIR). Once the expressions for the SIR are obtained, an analysis of the functions that determine the value of the SIR is carried out (we refer to these functions as weighting functions). Finally, the SIR and SER are evaluated for some concrete implementation parameters.

5.1 Modulated and Demodulated signal

Modulated signal The discrete time modulated signal for an OQAM-FBMC symbol with $N$ sub-carriers is given by

$$x[n] = \sum_{d \in D} \sum_{i \in I} X_{i,d} \theta_{i,d} p_i \left[n - d \frac{N}{2}\right],$$  (5.1)

where $X_{i,d}$ is any mapped symbol in the real field which modulates sub-carrier $i$ and overlap index $d$ ($X_{i,d} \in \mathbb{R}$), $\theta_{i,d} := e^{j \pi d}$ is the phase related with the offset-mapping of sub-carrier $i$ and overlap index $d$, $I$ is the set with the indices of active sub-carriers, $D$ is the set with the indices of the overlapping symbols ($D = \{-2K+1, \ldots, 2K-1\}$) and $p_i[n]$ is the transmitter filter for sub-carrier $i$.

The expression for the filter in sub-carrier $i$ is given by

$$p_i[n] := p[n] e^{j 2\pi \frac{d}{N}},$$  (5.2)

in which $p[n]$ is the base filter for OQAM-FBMC. It is a FIR filter with $L := K N$ real-valued time coefficients and total energy equal to 1, i.e., $\sum_{-\infty}^{+\infty} |p[n]|^2 = 1$. 

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Demodulated signal  The demodulation is performed using the matched filter together with the offset-demapping. Therefore, the demodulated signal in sub-carrier $l$ follows from (2.2) modified to perform the offset demodulation

\[ R_l = \Re \left\{ r[n] \ast \theta_l g_l[n] \bigg|_{n=0} \right\} , \]

(5.3)

where $r[n]$ is the sampled base-band received signal, $\ast$ is the convolution operator, $\Re\{ \}$ is the real part operator, $\theta_l := e^{(-l)}$ is the phase for the offset-demapping and $g_l[n]$ is the matched filter for sub-carrier $l$, which is defined as

\[ g_l[n] := p_l[n] \]

(5.4)

Note: Ideally the demodulation should produce $R_l = X_{l,0}$, however this equality is not possible due to the intrinsic self-interferences of the waveform.

5.2 Demodulated signal subject to phase noise

The sampled base-band received signal with a sample rate $f_s$ and with the effects of phase noise is given by

\[ r[n] = x[n] \left( 1 + j\phi[n] \right) , \]

(5.5)

which follows from the sampled version of (2.18) while $\phi[n]$ is the sampled phase noise, defined as $\phi[n] = \tilde{\phi}(t)\bigg|_{t=nT_s}$.

The demodulated signal subject to phase noise is given by

\[
R_l \overset{(a)}{=} \Re \left\{ r[n] \ast \theta_l g_l[n] \bigg|_{n=0} \right\} \\
\overset{(b)}{=} \Re \left\{ x[n] \left( 1 + j\phi[n] \right) \ast \theta_l g_l[n] \bigg|_{n=0} \right\} \\
\overset{(c)}{=} \Re \left\{ \sum_{d \in D} \sum_{i \in I} X_{i,d} \theta_{i,d} p_i \left[ n - d \frac{N}{2} \right] \left( 1 + j\phi[n] \right) \ast \theta_l g_l[n] \bigg|_{n=0} \right\} \\
\overset{(d)}{=} \sum_{d \in D} \sum_{i \in I} X_{i,d} \Re \left\{ \theta_{i,d} p_i \left[ n - d \frac{N}{2} \right] \ast \theta_l g_l[n] \bigg|_{n=0} \right\} + \\
+ \sum_{d \in D} \sum_{i \in I} X_{i,d} \Re \left\{ j\phi[n] \theta_{i,d} p_i \left[ n - d \frac{N}{2} \right] \ast \theta_l g_l[n] \bigg|_{n=0} \right\} \\
\overset{(e)}{=} \sum_{d \in D} \sum_{i \in I} X_{i,d} \gamma_{i,l,d} + \sum_{d \in D} \sum_{i \in I} X_{i,d} \rho_{i,l,d} \\
\overset{(f)}{=} X_{l,0} + \sum_{d \in D_{I1}} \sum_{i \in I} X_{i,d} \gamma_{i,l,d} + \sum_{i \in I \backslash i \in I1} X_{i,0} \rho_{i,l,0} + \sum_{d \in D_{I2}} \sum_{i \in I} X_{i,d} \rho_{i,l,d} \\
\]  

(5.6)

\footnote{In OQAM-FBMC the demodulation is not equal for all the symbols, the even and odd time symbols are demodulated with $\Re\{ \}$ and $\Im\{ \}$ operator respectively. However, the time symmetry of the base filter makes that the expressions for both cases are equal. This is the motivation for analysing only the case with $\Re\{ \}$.}

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(a) follows from (5.3); (b) follows from (5.5); (c) follows from (5.1); (d) follows from the distributivity property of both the convolution operator and the $\Re\{\cdot\}$ operator; (e) follows by defining $\gamma_{i,l,d} := \Re\{\theta_{i,d} p_i[n - d \frac{N}{2}] * \theta_l g_l[n]\}_{n=0}$ and $\rho_{i,l,d} := \Re\{j \phi[n] \theta_{i,d} p_i[n - d \frac{N}{2}] * \theta_l g_l[n]\}_{n=0}$; (f) follows from separate the different terms according to its cause. Similarly to Chapter 4, two sets of indices have been introduced in (5.6), $I_{ICI} = I \setminus \{l\}$ and $D_{ISI} = D \setminus \{0\}$, containing the indices of the interfering sub-carriers and symbols respectively.

The terms in (5.6) that only depend on the impulse response of the filters and the phase terms are

$$\gamma_{i,l,d} = \Re\left\{ \theta_{i,d} p_i \left[ n - d \frac{N}{2} \right] * \theta_l g_l[n] \right\}_{n=0} \quad \text{(a)}$$

$$\gamma_{i,l,d} = \Re\left\{ \sum_{s=-\infty}^{\infty} \theta_{i,d} p_i \left[ s - d \frac{N}{2} \right] \theta_l g_l[0 - s] \right\} \quad \text{(b)}$$

$$\rho_{i,l,d} = \Re\left\{ \sum_{s=-\infty}^{\infty} j \phi[s] \theta_{i,d} p_i \left[ s - d \frac{N}{2} \right] \theta_l g_l[0 - s] \right\} \quad \text{(c)}$$

An important result that has been applied in (5.6) related with the term in (5.7) is

$$\gamma_{i,0,0} = \left\{ \theta_{0,d} p_i \left[ s \right] \theta_l p_l \left[ s \right] \right\} = \sum_{s=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} p_i \left[ s \right] \overline{p_l \left[ s \right]} \Re\left\{ j^{(i-l)} \right\} = \delta_{i,l} \quad \text{(d)}$$

The terms in (5.6) that not only depend on the impulse response of the filters and the phase terms but also in phase noise are

$$\rho_{i,l,d} = \Re\left\{ j \phi[n] \theta_{i,d} p_i \left[ n - d \frac{N}{2} \right] * \theta_l g_l[n] \right\}_{n=0} \quad \text{(e)}$$

in which the same steps than in (5.7) have been followed. In this case, $\forall l \in I$ and $\forall d \in D$ $\rho_{i,l,d}$ is a random variable because it depends on $\phi[n]$. 

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Another important result that has been applied in (5.6) related with the term in (5.8) is
\[
\rho_{l,l,0} \overset{(a)}{=} \sum_{s=-\infty}^{\infty} \Re \{ j\phi[n] \theta_{l,0} p_l[s] \theta_{l,0} \} \overset{(b)}{=} \sum_{s=-\infty}^{\infty} \Re \{ j\phi[n] |p_l[s]|^2 \} \overset{(c)}{=} \sum_{s=-\infty}^{\infty} \phi[n] |p_l[s]|^2 = 0
\]

(a) follows from (5.8); (b) follows from \( \theta_{l,0} \theta_l = j^{l+0} j^{-l} = 1 \); (c) follows from \( \phi[n] \) being a real-valued signal. The previous result, \( \rho_{l,l,0} = 0 \), is very important because it implies that OQAM-FBMC is not affected by the CPE effect produced by phase noise (because the transmitted \( X_{l,0} \) symbol and the degradation term of PN-CPE are always orthogonal). This is an important difference between OQAM-FBMC and QAM-FBMC and OFDM.

The expression in (5.6) can be rewritten by defining an interference term for each of the effects produced by self-interferences and phase noise
\[
R_l = X_{l,0} + I_{l}^{ISI} + N_{l}^{ICI} + N_{l}^{ISI}
\]
where \( I_{l}^{ISI} \), \( N_{l}^{ICI} \) and \( N_{l}^{ISI} \) are defined next from (5.10) to (5.12).

The interference terms in (5.9) can be classified according to its cause:

- **Interference terms produced by self-interferences**
  \[
  I_{l}^{ISI} := \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \gamma_{i,l,d}
  \]
  which is the interference term related with the effects presented in Section 2.1.3.

- **Interference terms produced by phase noise**
  \[
  N_{l}^{ICI} := \sum_{i \in I_{ICI}} X_{i,0} \rho_{i,l,0}
  \]
  \[
  N_{l}^{ISI} := \sum_{d \in D_{ISI}} \sum_{i \in I} X_{i,d} \rho_{i,l,d}
  \]

As has been proven in this section, phase noise produce two different effects on the demodulated signal:

- **Phase noise Inter-Carrier Interference (PN-ICI).** Phase noise worsens the orthogonality between sub-carriers. Therefore interference from the rest of the sub-carriers in the demodulated one appears, producing the interference term \( N_{l}^{ICI} \).

- **Phase noise Inter-Symbol Interference (PN-ISI).** Phase noise also worsens the interference between overlapping symbols, producing the interference term \( N_{l}^{ISI} \).

### 5.2.1 Interference power

In this subsection we are going to derive an expression for the power of the interference terms produced by self-interferences \( (I_{l}^{ISI}) \) and phase noise \( (N_{l}^{ICI} \) and \( N_{l}^{ISI}) \).
The power of the interference term due to self-interferences is given by

\[ P_{I_{SI}} := E \left[ |I_{SI}^l|^2 \right] = E \left[ \sum_{d \in D_{SI}} \sum_{i \in I} X_{i,d} \gamma_{i,l,d} \right]^2 = E_X \left( \sum_{d \in D_{SI}} \sum_{i \in I} |\gamma_{i,l,d}|^2 \right) \]  

(5.13)

In (5.13) has been considered that \( \forall i \in I \) and \( \forall d \in D \) \( \{X_{i,d}\} \) is a sequence of zero-mean independent random variables. Moreover, \( E_X \) is the power of any mapped symbol and it is defined as \( E_X = E \left[ |X_{i,d}|^2 \right] \ \forall i \in I \) and \( \forall d \in D \).

The power of both Self-Interference types is dependent on \( |\gamma_{i,l,d}|^2 \). According to Appendix C.1, it can be expressed as

\[ |\gamma_{i,l,d}|^2 = W_{S_{I_{SI}}}^i (\nu) \delta(\nu), \]

(5.14)

where

\[ W_{S_{I_{SI}}}^i (\nu) := \frac{1}{4} \left[ V_d \left( \nu - \frac{i - l}{N} \right) \right]^2 + \left| V_d \left( \nu + \frac{i - l}{N} \right) \right|^2 + 2 \Re \{ Y_{i,l,d}(\nu) \}, \]

(5.15)

in which \( V_d(\nu) := F \{ p[s] p[s - dN/2] \} \) and \( Y_{i,l,d}(\nu) := e^{j\pi(i - l + d)\nu} V_d \left( \nu - \frac{i - l}{N} \right) V_d^* \left( \nu + \frac{i - l}{N} \right) \).

Using (5.14) in (5.13), we get

\[ P_{I_{SI}} = E_X \left( \sum_{d \in D_{SI}} \sum_{i \in I} W_{S_{I_{SI}}}^i (\nu) \delta(\nu) \right) = E_X \delta(\nu) W_{I_{SI}}^S (\nu) \]

(5.16)

where

\[ W_{I_{SI}}^S (\nu) := \sum_{d \in D_{SI}} \sum_{i \in I} W_{S_{I_{SI}}}^i (\nu) \]

(5.17)

The power of the interference terms due to phase noise is given by

\[ P_{N_{\beta CI}} := E \left[ |N_{\beta CI}^l|^2 \right] = E \left[ \sum_{i \in I_{\beta CI}} X_{i,0} \rho_{i,0} \right]^2 = E_X \left( \sum_{i \in I_{\beta CI}} E \left[ |\rho_{i,0}|^2 \right] \right) \]

(5.18)

\[ P_{N_{\beta SI}} := E \left[ |N_{\beta SI}^l|^2 \right] = E \left[ \sum_{d \in D_{\beta SI}} \sum_{i \in I} X_{i,d} \rho_{i,l,d} \right]^2 = E_X \left( \sum_{d \in D_{\beta SI}} \sum_{i \in I} E \left[ |\rho_{i,l,d}|^2 \right] \right) \]

(5.19)

In (5.18) and (5.19) has been considered that \( \forall i \in I \) and \( \forall d \in D \) \( \{X_{i,d}\} \) is a sequence of zero-mean independent random variables. Also it has been considered that \( \{\rho_{i,l,d}\} \) is a sequence of random variables whose elements are independent of the elements of \( \{X_{i,d}\} \) \( \forall i \in I, \forall l \in I \forall d \in D \).
The power of the interference components is dependent on \( E \left[ |\rho_{i,l,d}|^2 \right] \). According to Appendix C.2, it can be expressed as

\[
E \left[ |\rho_{i,l,d}|^2 \right] = \int_{-0.5}^{+0.5} S_\phi(\nu) W_{i,l,d}(\nu) d\nu
\]  

(5.20)

where \( S_\phi(\nu) \) is the PSD of the sampled phase noise and \( W_{i,l,d}(\nu) \) is defined as

\[
W_{i,l,d}(\nu) := \frac{1}{4} \left[ V_d \left( \nu - \frac{i - l}{N} \right)^2 + V_d \left( \nu + \frac{i - l}{N} \right)^2 - 2 \mathfrak{R} \{ Y_{i,l,d}(\nu) \} \right],
\]  

(5.21)

in which \( V_d(\nu) := \mathcal{F} \{ p[s] p[s-d \frac{N}{2}] \} \) and \( Y_{i,l,d}(\nu) := e^{j\pi(i-l+d)} V_d(\nu - \frac{i - l}{N}) V_d^*(\nu + \frac{i - l}{N}) \).

Using (5.20) in (5.18) and (5.19), we get

\[
P_{N_{CLI}} = E_X \sum_{i \in I}\left( \int_{-0.5}^{+0.5} S_\phi(\nu) W_{i,l,d}(\nu) d\nu \right) = E_X \left( \int_{-0.5}^{+0.5} S_\phi(\nu) W_{SCI}^{CLI}(\nu) d\nu \right)
\]  

(5.22)

\[
P_{N_{SISI}} = E_X \sum_{d \in D_{SISI}} \sum_{i \in I}\left( \int_{-0.5}^{+0.5} S_\phi(\nu) W_{i,l,d}(\nu) \right) = E_X \left( \int_{-0.5}^{+0.5} S_\phi(\nu) W_{SISI}^{CLI}(\nu) d\nu \right)
\]  

(5.23)

where

\[
W_{SCI}^{CLI}(\nu) := \sum_{i \in I_{CLI}} W_{i,l,0}(\nu)
\]  

(5.24)

\[
W_{SISI}^{CLI}(\nu) := \sum_{d \in D_{SISI}} \sum_{i \in I} W_{i,l,d}(\nu)
\]  

(5.25)

The functions in (5.24) and (5.25) are the functions that later we will call weighting functions for the phase noise degradations.

5.2.2 Signal-to-Interference Ratio

In this subsection the effect of self-interferences and phase noise in the Signal-to-Interference Ratio (SIR) is studied. First, the expression with only the effect of self-interferences is derived. Finally, the expression with both effects (self-interferences and phase noise) is obtained.

SIR due to self-interferences From (5.9) the expression for the SIR in the demodulated signal in sub-carrier \( l \) produced by the self-interferences is easily derived. Taking into account only the interference term produced by the self-interferences (\( I_{SISI}^{CLI} \)), the SIR is given by

\[
\text{SIR}_l = \frac{E_X}{P_{I_{SISI}}^{(a)}} \frac{1}{W_{I_{SISI}}^{S-I_{SISI}}(\nu) \delta(\nu)}
\]  

(5.26)

(a) follows from (5.16). \( P_{I_{SISI}} \) is the power of the interference produced by S-ISI. \( E_X \) is the power of the transmitted symbol in sub-carrier \( l \), it is defined as \( E_X := E \left[ |X_{l,0}|^2 \right] \).
SIR due to self-interferences and phase noise  From (5.9) the expression for the SIR in the demodulated signal in sub-carrier \( l \) produced by the combined effect of self-interferences and phase noise is obtained

\[
\text{SIR}_l = \frac{E_X}{P_l^{\text{ISI}} + P_{N l}^{\text{ICI}} + P_{N l}^{\text{ISI}}}
\]

\[
\frac{1}{W_l^{\text{ISI}}(\nu) \delta(\nu) + \int_{-0.5}^{0.5} \left[ W_l^{\text{ICl}}(\nu) + W_l^{\text{ISI}}(\nu) \right] S_\phi(\nu) d\nu}
\]

(5.27)

(a) follows from (5.16), (5.22), and (5.23). \( P_{N l}^{\text{ISI}} \) and \( P_{N l}^{\text{ISI}} \) are the power of PN-ICI and PN-ISI respectively.

5.3 Weighting functions

In Section 3.5 the concept of weighting function was presented. As was previously explained, they are tools used in MCM to compute the level of the interferences produced by phase noise. In OQAM-FBMC, contrary to QAM-FBMC, the weighting functions used for the phase noise degradations can not be used for the self-interferences degradations.

In Section 5.2.1 we derived the expressions for the power of the interferences produced by phase noise, which were

\[
P_{N l}^{\text{ICl}} = E_X \left( \int_{-0.5}^{0.5} S_\phi(\nu) W_l^{\text{ICl}}(\nu) d\nu \right)
\]

\[
P_{N l}^{\text{ISI}} = E_X \left( \int_{-0.5}^{0.5} S_\phi(\nu) W_l^{\text{ISI}}(\nu) d\nu \right)
\]

In the previous expression \( W_l^{\text{ICl}}(\nu) \) and \( W_l^{\text{ISI}}(\nu) \) are functions that shape \( S_\phi(\nu) \) in order to obtain the power of the interferences. Thus, \( W_l^{\text{ICl}}(\nu) \) and \( W_l^{\text{ISI}}(\nu) \) are the weighting functions for PN-ICI and PN-ISI respectively. The expressions for the weighting functions can be found in (5.24) and (5.25).

Note: From now on we introduce some notation to refer to the positions of the sub-carriers. When we talk about the DC sub-carrier we refer to the active sub-carrier in the middle of the spectrum \( l = \frac{N}{2} \) and when we talk about EDGE sub-carrier we refer to the first active sub-carrier of the spectrum \( l = N_g \).

5.3.1 Shape analysis

As has been studied, the power of the interference terms depend on the shape of the weighting functions. Next we are going to analyse the shape of \( W_l^{\text{ICl}}(\nu) \) and \( W_l^{\text{ISI}}(\nu) \). As in Section 3.5 and Section 4.3, it is important to have in mind that most of the power of \( S_\phi(\nu) \) is gathered in frequencies close to \( \nu = 0 \). The analysis follows.

The shape of \( W_l^{\text{ICl}}(\nu) \) determines the value of \( P_{N l}^{\text{ICl}}. \) If \( \Delta f \) increases, the gap of \( W_l^{\text{ICl}}(\nu) \) around \( \nu = 0 \) is wider, so \( P_{N l}^{\text{ICl}} \) decreases \( \forall l \in I. \) This effect can be seen in Figure 5.1.
The value of $P_{N_{fSI}}$ is dependent on the shape of $W_{f}^{fSI}(\nu)$. If $\Delta f$ increases, the main lobe of $W_{f}^{fSI}(\nu)$ around $\nu = 0$ is wider, so $P_{N_{fSI}}$ increases $\forall l \in I$. This effect can be seen in Figure 5.2.

One important property of $W_{f}^{fCI}(\nu)$ and $W_{f}^{fSI}(\nu)$ is that they have different shapes for different values of $l$, so $P_{N_{fCI}}$ and $P_{N_{fSI}}$ also have different values. We focus in the best and worst case in terms of interference power (DC and EDGE sub-carrier respectively).

The shape of $W_{f}^{fCI}(\nu) + W_{f}^{fSI}(\nu)$ has been plotted in Figure 5.3. If $\Delta f$ increases, the values of $W_{f}^{fCI}(\nu) + W_{f}^{fSI}(\nu)$ for the DC sub-carrier around $\nu = 0$ are not modified, so the change in $P_{N_{fCI}} + P_{N_{fSI}}$ is not significant. Contrary, in the EDGE sub-carrier the lobe of $W_{f}^{fCI}(\nu) + W_{f}^{fSI}(\nu)$

Figure 5.1: Comparison of $W_{f}^{fCI}(\nu)$ for two different $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.

Figure 5.2: Comparison of $W_{f}^{fSI}(\nu)$ for two different $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.

Figure 5.3: Comparison of $W_{f}^{fCI}(\nu) + W_{f}^{fSI}(\nu)$.
around $\nu = 0$ is wider, producing an increasing in $P_{N_{I1}} + P_{N_{I1}}$.

Figure 5.3: Comparison of $W_{I_{I1}}(\nu) + W_{I_{I1}}(\nu)$ for two different $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.

Summarizing, when $\Delta f$ increases $P_{N_{I1}}$ increases and $P_{N_{I1}}$ decreases $\forall l \in I$. Also, $P_{N_{I1}} + P_{N_{I1}}$ for the DC sub-carrier is nearly constant for different $\Delta f$ but for other sub-carriers it increases if $\Delta f$ increases. These facts can be seen in Figure 5.4.

Figure 5.4: Power of the interference terms produced by phase noise for $f_c = 28$ GHz, $B_W = 100$ MHz and $f_s = 110$ MHz.
5.4 Signal-to-Interference Ratio evaluations

In this section we evaluate the SIR using the derived expressions and simulations. The parameters $B_W = 100$ MHz and $f_s = 110$ MHz have been used. The base filters proposed in [10] for $K = 2$, $K = 3$ and $K = 4$ are the ones used to obtain the results.

5.4.1 Theoretical results

The results presented in this subsection have been obtained with the expressions derived in Section 5.2.2. The mmMagic project phase noise model introduced in Section 2.2 has been used to obtain the results.

SIR due to self-interferences Before showing the results with the effects of phase noise, it is essential to show which is the best result that can be achieve without any external degradation (as the channel or the RF impairments). So it is important to obtain the SIR with the effects of self-interferences only. In Figure 5.5 the SIR produced by the self-interferences (S-ISI) as a function of $\Delta f$ is shown.

![Figure 5.5: SIR due to self-interferences (S-ISI) for three different overlapping factors and two different $l$, with $B_W = 100$ MHz and $f_s = 110$ MHz.](image)

In order to give a better understanding of Figure 5.5, the power of each interference produced by self-interferences is plotted in Figure 5.6 and Figure 5.7. The graph is a grid in which each square represents the interference of one symbol and its color indicates the power of the interference. The demodulated symbol is painted in yellow.

In Figure 5.6 and Figure 5.7 for $K = 2$ the number of significant interferences $^2$ and its power changes for each $\Delta f$. So the total power of self-interferences changes for different values of $\Delta f$, increasing if $\Delta f$ increases (in coherence with Figure 5.5). In Figure 5.5, for the case with $K = 2$ there is not significant difference between the case for the DC and EDGE sub-carrier because the significant interferences are distributed around all the sub-carriers and overlapping indices.

$^2$We consider the interference that have higher power than $-80$ dBW as significant interference.
In Figure 5.6 and Figure 5.7 for $K = 4$ the number of significant interferences and its power does not change for different $\Delta f$ (leading to the constant curve for $K = 4$ in Figure 5.5, contrary to the case with $K = 2$). The significant interferences are located only in the sub-carriers that are close to the one demodulated. Thus, the total interferences power for EDGE sub-carrier is significantly lower than for the DC sub-carrier, because the proportion of interfering symbols that are removed by the null bands is high. As can be seen in Figure 5.5 the difference in the SIR for the DC and EDGE sub-carrier is significant and constant for $K = 4$.

![Figure 5.6: Power of self-interferences in the DC sub-carrier for two different $\Delta f$ and two different $K$, with $B_W = 100$ MHz and $f_s = 110$ MHz.](image)

SIR due to self-interferences and phase noise Once the effect of the self-interferences has been studied, we present the evaluations with the combined effect of the self-interferences and phase noise. In Figure 5.8 the SIR with the effects of self-interferences and phase noise as a function of $\Delta f$ is shown for different overlapping factors and sub-carrier indices. Phase noise produces a decreasing in the SIR respect to the cases in Figure 5.5.

For low carrier frequencies the differences in the SIR between the different overlapping factors is significant. It is because the power of the interference terms produced by phase noise are low.
Figure 5.7: Power of self-interferences in the EDGE sub-carrier for two different $\Delta f$ and two different $K$, with $B_W = 100$ MHz and $f_s = 110$ MHz.

in comparison to the power of the interferences produced by self-interferences. However, for high frequencies (as 82 GHz), the power of the interferences produced by phase noise are dominant over the ones produced by self-interferences, resulting in very similar SIR results for the three overlapping factors (because the power of the interference produced by phase noise is similar for the three cases, but not the power of the interferences produced by self-interferences).
Figure 5.8: SIR due to self-interferences (S-ISI) and phase noise (PN-ICI and PN-ISI) for three different $f_c$ and three different overlapping factors for the DC and EDGE sub-carrier, with $B_W = 100$ MHz and $f_s = 110$ MHz.
5.4.2 Simulation results

The results presented in this subsection have been obtained by Monte Carlo evaluations using the phase noise generator developed by mmMagic project. The simulation results have been obtained in order to check the accuracy of the theoretical evaluations.

SIR due to self-interferences

In Figure 5.10 the SIR as a function of $\Delta f$ for the theoretical and simulation results including the effects of self-interferences. In the figure it is obvious that the simulation results are in accordance with the theoretical ones.

\[ \Delta f \text{ [kHz]} \]
\[ \text{SIR} \text{ dB} \]

\[ K = 4 - \text{THEO} \]
\[ K = 4 - \text{SIM} \]
\[ K = 3 - \text{THEO} \]
\[ K = 3 - \text{SIM} \]
\[ K = 2 - \text{THEO} \]
\[ K = 2 - \text{SIM} \]

SIR due to self-interferences and phase noise

In Figure 5.10 the SIR as a function of $\Delta f$ for the theoretical and simulation results including the effects of self-interferences and phase noise. In the figure it is clear that the simulation results are in accordance with the theoretical ones.

\[ \Delta f \text{ [kHz]} \]
\[ \text{SIR} \text{ dB} \]

\[ K = 4 - \text{THEO} \]
\[ K = 4 - \text{SIM} \]
\[ K = 3 - \text{THEO} \]
\[ K = 3 - \text{SIM} \]
\[ K = 2 - \text{THEO} \]
\[ K = 2 - \text{SIM} \]

Figure 5.9: Comparison between the theoretical and simulation results for the SIR due to self-interferences (S-ISI) for three different overlapping factors and for the DC and EDGE sub-carrier, with $B_w = 100$ MHz and $f_s = 110$ MHz.

SIR due to self-interferences and phase noise

In Figure 5.10 the SIR as a function of $\Delta f$ for the theoretical and simulation results including the effects of self-interferences and phase noise. In the figure it is clear that the simulation results are in accordance with the theoretical ones.
Figure 5.10: Comparison between the theoretical and simulation results for the SIR due to self-interferences (S-ISI) and phase noise (PN-ICI and PN-ISI) for three different $f_c$ and three different overlapping factors for the DC and EDGE sub-carrier, with $B_W = 100$ MHz and $f_s = 110$ MHz.
5.5 Symbol Error Rate evaluations

In this section we present the simulation evaluations for the Symbol Error Rate (SER) produced by phase noise. For the simulations we use the numerology presented in [15] and shown in Table 5.1, as in Chapter 3. The SER has been simulated for different values of AWGN and the influence of phase noise and self-interferences.

<table>
<thead>
<tr>
<th>$f_c$</th>
<th>6 GHz</th>
<th>28 GHz</th>
<th>82 GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta f$</td>
<td>15 kHz</td>
<td>60 kHz</td>
<td>480 kHz</td>
</tr>
<tr>
<td>$f_s$</td>
<td>30.72 MHz</td>
<td>122.88 MHz</td>
<td>983.04 MHz</td>
</tr>
<tr>
<td>$N$</td>
<td>2048</td>
<td>2048</td>
<td>2048</td>
</tr>
<tr>
<td>$N_{act}$</td>
<td>1200</td>
<td>1200</td>
<td>1200</td>
</tr>
</tbody>
</table>

Table 5.1: 5G numerology in frequencies above 6 GHz proposed by Nokia Bells Labs and Ericsson Research.

In Figure 5.11 we have plotted the results of the simulations of the SER as a function of the SNR produced by the AWGN using the parameters in Table 5.1. Two different curves have been plotted: the SER produced by AWGN and self-interferences and the SER produced by AWGN, self-interferences and phase noise. Each curve has been plotted for $K = 2$ and $K = 4$, which are the best overlapping factors in terms of achievable SIR. Also, two different symbol constellations have been used, 16-QAM and 64-QAM.

From the presented evaluations different facts can be observed. For the parameters in Table 5.1, for low carrier frequencies in the mm-Band (as $f_c = 6$ GHz) phase noise does not produce a significant degradation in the performance of OQAM-FBMC (neither for 16-QAM nor for 64-QAM). For $f_c = 28$ GHz it is a significant degradation for high order symbol constellations (64-QAM) in values of SNR higher than 20 dB. For high carrier frequencies (as $f_c = 82$ GHz) the SER increasing produced by phase noise is significant for both symbol constellations in SNR values higher than 15 dB, specially for 64-QAM.

The evaluation results show how the differences in the performance for $K = 2$ and $K = 4$ are not very significant. Only for the case of Figure 5.11d the differences between both cases are significant, being the performance of the case with $K = 4$ slightly better than with $K = 2$. 
Figure 5.11: Simulation results for the SER due to phase noise (PN-ICI and PN-ISI), self-interferences (S-ISI) and AWGN for the parameters in Table 5.1.
Chapter 6

Comparison of waveforms subject to phase noise

In Chapters 3, 4, and 5 we derived weighting functions to evaluate the effects of phase noise in OFDM, QAM-FBMC, and OQAM-FBMC, respectively. At the end of each chapter these weighting functions were used to evaluate the performance of the three waveforms using common parameters. In this chapter we use the obtained results to compare the effects of phase noise in the studied waveforms and determine which one offers better performance for the mmMagic phase noise model.

On the one hand, we will compare the achievable SIR after PN-CPE compensation. It is the most relevant case since PN-CPE is easily removed using pilot symbols, without increasing the complexity or the overhead of the system (because pilot symbols are needed anyway for channel estimation). It is important that OQAM-FBMC is not affected by PN-CPE, as explained in Chapter 5. On the other hand, we will compare the achievable SIR after PN-CPE compensation without taking into account the effect of self-interferences. This case shows which is the true effect of phase noise in the achievable SIR (because it does not show the degradations due to the non-orthogonalities of QAM-FBMC and OQAM-FBMC).

Achievable SIR after PN-CPE compensation  In Figure 6.1 we show a comparison of the achievable SIR after PN-CPE compensation for the three waveforms. It includes the effect of phase noise, but also the effect of the self-interferences for the non-orthogonal waveforms (QAM-FBMC and OQAM-FBMC). In the graph it can be seen how QAM-FBMC offers the worst performance in all the cases, mainly because its high level of self-interferences. For the case of OQAM-FBMC, we can find big differences in the performance for each overlapping factor. These differences are produced by the different level of self-interferences that OQAM-FBMC has for each overlapping factor (as studied in Chapter 5). The case with $K = 4$ outperforms the cases with $K = 2$ and $K = 3$. Finally, it can be seen that OFDM outperforms QAM-FBMC and OQAM-FBMC, mainly because it is an orthogonal MCM which is not affected by self-interferences, but also because it offers a very good sensitivity against phase noise (as we will show next).

Achievable SIR after PN-CPE compensation excluding Self-Interferences  In Figure 6.2 we have plotted a comparison of the achievable SIR after PN-CPE compensation ex-
cluding the effects of self-interferences. This graph shows the true effect of phase noise in the studied waveforms, letting know which is the sensitivity of each waveform against this degradation. From the graph, it is clear that OQAM-FBMC is the waveform with worst sensitivity against phase noise (with very similar results for the different overlapping factors). Also, it is important to mention that QAM-FBMC offers a good sensitivity against phase noise (with values quite close to the OFDM ones), however it is affected by a high level of self-interferences which degrades significantly its performance (as mentioned previously). In conclusion, OFDM is the studied waveform with best sensitivity against phase noise, with high values of SIR even for high carrier frequencies.

Finally, in Figure 6.3 we have plotted the results of the simulations for the SER as a function of the SNR produced by the AWGN (using the parameters in Table 6.1). Three different curves have been plotted: the SER produced by AWGN and phase noise for OFDM, the SER produced by AWGN and phase noise for OFDM after PN-CPE compensation and the SER produced by AWGN and phase noise for OQAM-FBMC with $K = 4$. Each curve has been plotted for 16-QAM and 64-QAM. In the graph it can be seen how for low frequencies in the mm-band (as $f_c = 6$ GHz) the differences between the performance between OFDM and OQAM-FBMC are negligible. For $f_c = 28$ GHz and high order constellation (64-QAM) the performance of OFDM is sligly better than OQAM-FBMC. For high carrier frequencies the differences in the performance between OFDM and OQAM-FBMC are quite high for both constellations, specially when the PN-CPE contribution of OFDM is removed. Concretely, for $f_c = 82$ GHz and 64-QAM the SER error-floor for OQAM-FBMC is higher than $10^{-2}$, for OFDM without CPE correction it is slightly lower than $10^{-2}$ and for OFDM with CPE compensation it is lower than $10^{-5}$. Thus, OFDM outperforms OQAM-FBMC for high carrier frequencies even without PN-CPE correction.

<table>
<thead>
<tr>
<th>$f_c$ (GHz)</th>
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<th>28 GHz</th>
<th>82 GHz</th>
</tr>
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<tr>
<td>$N_{act}$</td>
<td>1200</td>
<td>1200</td>
<td>1200</td>
</tr>
</tbody>
</table>

Table 6.1: 5G numerology in frequencies above 6 GHz proposed by Nokia Bells Labs and Ericsson Research.
Figure 6.1: Comparison of the SIR due to the interferences produced by self-interferences and phase noise after PN-CPE correction for three different $f_c$ and two sub-carrier locations, with $B_W = 100$ MHz and $f_s = 110$ MHz.
Figure 6.2: Comparison of the SIR due to the interferences produced by phase noise excluding the self-interferences and PN-CPE for three different $f_c$ and two sub-carrier locations, with $B_W = 100$ MHz and $f_s = 110$ MHz.
Figure 6.3: Simulation results for the SER due to phase noise, self-interferences and AWGN for the parameters in Table 6.1.
Chapter 7

Conclusions and future work

7.1 Conclusions

In this thesis we have derived the weighting functions for OFDM, QAM-FBMC and OQAM-FDMC, three of the most promising waveform candidates for 5G. The weighting functions allow us to evaluate the effects of phase noise in the performance of the different waveforms in a very intuitive way. Thus, the derived weighting functions have been used together with the mmMagic phase noise model to obtain different results for the achievable SIR of OFDM, QAM-FBMC and OQAM-FBMC subject to phase noise. Also, Monte Carlo evaluations have been done to verify the accuracy of the results obtained with the weighting functions. Moreover, the effects of phase noise in the Symbol Error Rate have been obtained by simulation for OFDM and OQAM-FBMC with $K = 4$. Finally, the results for all the waveforms have been compared.

After analysing the results obtained in this thesis, we can conclude that phase noise is only a significant degradation for very high frequencies in the millimetre-wave bands. We have shown that QAM-FBMC offers good sensitivity against phase noise (with values quite similar to the OFDM ones). However, it is a non-orthogonal MCM highly degraded by its self-interferences and it offers very poor performance (even without the effect of phase noise). In addition, OQAM-FBMC is a non-orthogonal MCM with worst sensitivity against phase noise than QAM-FBMC but it is less degraded by its self-interferences, leading to a better performance than QAM-FBMC for the mmMagic phase noise model. It is important to highlight that the performance of OQAM-FBMC is related with the selected overlapping factor. The sensitivity against phase noise is very similar for different overlapping factors, but the power of the self-interferences is different for each one. It has been shown that OQAM-FBMC with $K = 4$ outperforms the cases with $K = 2$ and $K = 3$. Finally, OFDM is an orthogonal MCM and it is not affected by self-interferences. Moreover, it has been shown that its sensitivity against phase noise is very good, mainly because the PN-CPE can be easily removed using pilot symbols (without increasing the complexity of the system). Thus, OFDM offers a very good performance in the presence of phase noise. We have exposed that for 64-QAM and 82 GHz the error-floor in the case of OFDM with CPE correction is lower than $10^{-5}$, but in the case of OQAM-FBMC with $K = 4$ it is higher than $10^{-2}$ (for the mmMagic phase noise model). After the analysis of all the results, we can conclude that OFDM outperforms OQAM-FBMC and QAM-FBMC in terms of phase noise effects (for the mmMagic phase noise model) but OQAM-FBMC and QAM-FBMC offer higher spectral density.
7.2 Future work

The study presented in this thesis might be continued by extending the comparison of the effects of phase noise to other waveforms, using similar derivations to the ones presented in Chapter 3, 4 and 5. Concretely it might be suitable to derive tools to study the effect of phase noise in Window-OFDM and DFTS-OFDM (promising waveforms for 5G). It would be also interesting to study the shape of the filters used by the different waveforms and how to optimize them to be more robust against phase noise. Also, the study and analysis of different phase noise correction methods may be interesting (not only for the PN-CPE, but also for PN-ICI and PN-ISI). Finally, particularly interesting might be to study the probability distribution function of the interference terms produced by phase noise in order to obtain a method to theoretically evaluate the Symbol Error Rate, since the Gaussian assumption is not accurate in this case.
Appendices
Appendix A

OFDM Derivations

A.1 Derivation of $E \left[ |\tilde{\varphi}_{i,l}|^2 \right]$

In this appendix an expression for $E \left[ |\tilde{\varphi}_{i,l}|^2 \right]$ is derived. First, we are going to define a new filter that will simplify the following derivations

$$\tilde{h}_{i,l}(t) := \tilde{p}_i(t) \tilde{p}_l(t) = \tilde{p}^2(t) e^{j2\pi \frac{i-l}{T} t},$$

(A.1)

in which the definition in (2.8) has been used.

Next, the derivation of $E \left[ |\tilde{\varphi}_{i,l}|^2 \right]$ follows.

$$E \left[ |\tilde{\varphi}_{i,l}|^2 \right] \overset{(a)}{=} E \left[ j \int_{-\infty}^{+\infty} \tilde{\phi}(\tau) \tilde{p}_i(\tau) \tilde{p}_l(\tau) d\tau \right]^2$$

$$\overset{(b)}{=} E \left[ j \int_{-\infty}^{+\infty} \tilde{\phi}(\tau) \tilde{h}_{i,l}(\tau) d\tau \right]^2$$

$$\overset{(c)}{=} E \left[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\phi}(\tau) \tilde{h}_{i,l}(\tau) \tilde{\phi}(\tau + t) \tilde{h}_{i,l}(\tau + t) d\tau dt \right]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E \left[ \tilde{\phi}(\tau)\tilde{\phi}(\tau + t) \right] \tilde{h}_{i,l}(\tau) \tilde{h}_{i,l}(\tau + t) d\tau dt$$

$$\overset{(d)}{=} \int_{-\infty}^{+\infty} R_{\tilde{\phi}}(-t) \left( \int_{-\infty}^{+\infty} \tilde{h}_{i,l}(\tau) \tilde{h}_{i,l}(\tau + t) d\tau \right) dt$$

$$\overset{(e)}{=} \int_{-\infty}^{+\infty} R_{\tilde{\phi}}(-t) \left( \tilde{h}_{i,l}(t) \ast \tilde{h}_{i,l}(-t) \right) dt$$

$$\overset{(f)}{=} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} S_{\tilde{\phi}}(f) e^{-j2\pi tf} df \right) \left( \tilde{h}_{i,l}(t) \ast \tilde{h}_{i,l}(-t) \right) dt$$

$$= \int_{-\infty}^{+\infty} S_{\tilde{\phi}}(f) \left( \int_{-\infty}^{+\infty} \left( \tilde{h}_{i,l}(t) \ast \tilde{h}_{i,l}(-t) \right) e^{-j2\pi tf} dt \right) df$$

$$\overset{(g)}{=} \int_{-\infty}^{+\infty} S_{\tilde{\phi}}(f) \tilde{W}_{i,l}(f) df$$

(A.2)
(a) follows from the definition of $\tilde{\varphi}_{i,l}$ in (3.6); (b) follows from (A.1); (c) follows from the application of the absolute value operator ($|\cdot|$); (d) follows from the definition of the autocorrelation of a stochastic process, i.e., $R_X(t) = E[X(\tau)X(\tau - t)]$ (being $X$ an stochastic process); (e) follows from the definition of the convolution ($\ast$ is the convolution operator); (f) follows from the definition of the autocorrelation of a WSS stochastic process as the Fourier transform of its PSD ($S_{\tilde{\varphi}}(f)$ is the PSD of phase noise, which is a WSS stochastic process); (g) follows from the definition of a new function $\tilde{W}_{i,l}(f)$, which is defined in (A.6).

What follows next is the definition and simplification of $\tilde{W}_{i,l}(f)$

$$\tilde{W}_{i,l}(f) := \int_{-\infty}^{+\infty} \left( \tilde{h}_{i,l}(t) \ast \tilde{h}_{i,l}(-t) \right) e^{-j2\pi ft} dt$$

$$= \mathcal{F} \left\{ \tilde{h}_{i,l}(t) \ast \tilde{h}_{i,l}(-t) \right\}$$

$$= \mathcal{F} \left\{ \tilde{h}_{i,l}(t) \right\} \mathcal{F} \left\{ \tilde{h}_{i,l}(-t) \right\}$$

$$= \lvert \mathcal{F} \left\{ \tilde{h}_{i,l}(t) \right\} \rvert^2$$

$$= \lvert \mathcal{F} \left\{ \tilde{p}^2(t) e^{j2\pi \frac{i-l}{T} T} \right\} \rvert^2$$

$$= \lvert \mathcal{F} \left\{ \tilde{p}^2(t) \right\} \ast \delta \left( f - \frac{i-l}{T} \right) \rvert^2$$

$$= \lvert \mathcal{F} \left\{ \tilde{p}^2(t) \right\} \ast \delta \left( f - \frac{i-l}{T} \right) \rvert^2$$

$$= \lvert \mathcal{F} \left\{ \left( f - \frac{i-l}{T} \right) \right\} \rvert^2$$

(A.3)
A.2 Derivation of $E\left[|\varphi_{i,l}|^2\right]$

In this appendix an expression for $E\left[|\varphi_{i,l}|^2\right]$ is derived. First, we are going to define a new filter that will simplify the following derivations

$$h_{i,l}[n] := p_i[n] \overline{p_l[n]} = p^2[n] e^{j2\pi \frac{n}{N}},$$  \hspace{1cm} (A.4)

in which the definition in (3.22) has been used.

Next, the derivation of $E\left[|\varphi_{i,l}|^2\right]$ follows

\begin{align*}
E\left[|\varphi_{i,l}|^2\right] &\xlongequal{(a)} E \left[ \left| \sum_{s=-\infty}^{+\infty} \phi[s] p_i[s] \overline{p_l[s]} \right|^2 \right] \\
&\xlongequal{(b)} E \left[ \left| \sum_{s=-\infty}^{+\infty} \phi[s] h_{i,l}[s] \right|^2 \right] \\
&\xlongequal{(c)} E \left[ \sum_{n=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \phi[s] h_{i,l}[s] \phi[s+n] h_{i,l}[s+n] \right] \\
&= \sum_{n=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} E[\phi[s]\phi[s+n]] h_{i,l}[s] h_{i,l}[s+n] \\
&\xlongeual{(d)} \sum_{n=-\infty}^{+\infty} R_{\phi}[n] \left( \sum_{s=-\infty}^{+\infty} h_{i,l}[s] h_{i,l}[s+n] \right) \\
&\xlongeual{(e)} \sum_{n=-\infty}^{+\infty} R_{\phi}[n] \left( h_{i,l}[n] * h_{i,l}[-n] \right) \\
&\xlongeual{(f)} \sum_{n=-\infty}^{+\infty} \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu) e^{-j2\pi \nu n} \mathrm{d}\nu \right) h_{i,l}[n] * h_{i,l}[-n] \\
&= \int_{-0.5}^{+0.5} S_{\phi}(\nu) \left( \sum_{n=-\infty}^{+\infty} \left( h_{i,l}[n] * h_{i,l}[-n] \right) e^{-j2\pi \nu n} \right) \mathrm{d}\nu \\
&\xlongeual{(g)} \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l}(\nu) \mathrm{d}\nu \hspace{1cm} (A.5)
\end{align*}

(a) follows from (3.30); (b) follows from (A.4); (c) follows from the definition of the absolute value; (d) follows from the definition of the autocorrelation of a discrete stochastic process, i.e., $R_X[n] = E[X[s]X[s-n]]$ (being $X$ a discrete stochastic process); (e) follows from the definition of the convolution ($*$ is the convolution operator); (f) follows from the definition of the autocorrelation of a discrete WSS stochastic process as the Inverse Discrete Time Fourier Transform (IDTFT) of its PSD ($S_{\phi}(\nu)$ is the PSD of sampled phase noise, which is a WSS discrete stochastic process); (g) follows from the definition of a new function $W_{i,l}(\nu)$. 

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What follows next is the definition and simplification of $W_{i,l}(\nu)$

$$W_{i,l}(\nu) := \sum_{n=-\infty}^{+\infty} \left( h_{i,l}[n] * h_{i,l}[-n] \right) e^{-j2\pi \nu n}$$

(a) follows from the definition of the DTFT (denoted by the operator $F\{\}$); (b) follows from the convolution theorem [18, pp. 297]; (c) follows from the complex conjugate symmetry of the real signals and $X$ denotes complex conjugate of $X$ ($h_{i,l}[N]$ is a real-valued signal); (d) follows from the definition of absolute value; (e) follows from the definition of $h_{i,l}[n]$ in (A.4); (f) follows from the convolution theorem ($\star$ is the operator for the circular convolution); (g) follows from the DTFT of $p^2[n]$ (which is a discrete rectangular pulse with $N$ samples and amplitude $N$).

\begin{align*}
(a) & \quad F\{h_{i,l}[n] * h_{i,l}[-n]\} \\
(b) & \quad F\{h_{i,l}[n]\} F\{h_{i,l}[-n]\} \\
(c) & \quad F\{h_{i,l}[n]\} F\{h_{i,l}[n]\} \\
(d) & \quad |F\{h_{i,l}[n]\}|^2 \\
(e) & \quad \left| F\{p^2[n] e^{j2\pi \frac{i-l}{N}}\} \right|^2 \\
(f) & \quad \left| F\{p^2[n]\} \star \delta \left( \nu - \frac{i-l}{N} \right) \right|^2 \\
(g) & \quad \left( \frac{1}{N} \sin(\pi N) \right)^{e^{2\pi (N-1)}} \star \delta \left( \nu - \frac{i-l}{N} \right)^2 \\
& \quad = \frac{1}{N^2} \left| \sin \left( \pi \left( \nu - \frac{i-l}{N} \right) \right) \right|^2 (A.6)
\end{align*}

$$W_{i,l}(\nu) := \left| F\{p[n] e^{j2\pi \frac{i-l}{N}}\} \right|^2 = \frac{1}{N^2} \left| \sin \left( \pi \left( \nu - \frac{i-l}{N} \right) \right) \right|^2 (A.7)$$
Appendix B

QAM-FBMC Derivations

B.1 Derivation of $|\lambda_{i,l,d}|^2$

In this appendix an expression for $|\lambda_{i,l,d}|^2$ is derived. First, we are going to define a new filter that will simplify the following derivations

$$h_{i,l,d}[n] := p_i[n - dN] \overline{p_l[n]} \quad \text{(B.1)}$$

The derivation of $|\lambda_{i,l,d}|^2$ follows.

$$|\lambda_{i,l,d}|^2 = \left| \sum_{s=-\infty}^{\infty} p_i[s - dN] \overline{p_l[s]} \right|^2 \quad \text{(a)}$$

$$= \left| \sum_{s=-\infty}^{\infty} h_{i,l,d}[s] \right|^2 \quad \text{(b)}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} h_{i,l,d}[s] h_{i,l,d}[s + n] \quad \text{(c)}$$

$$= \sum_{n=-\infty}^{\infty} (h_{i,l,d}[n] * h_{i,l,d}[-n]) \quad \text{(d)}$$

$$= \mathcal{F}\{h_{i,l,d}[n] * h_{i,l,d}[-n]\}(\nu = 0) \quad \text{(e)}$$

$$= W_{i,l,d}(\nu) \delta(\nu) \quad \text{(f)}$$

(a) follows from (4.7); (b) follows from (B.1); (c) follows from definition of the absolute value (whose operator is $| |^2$); (d) follows from convolution’s definition (whose operator is $*$); (e) follows from the interpretation of the infinite sum as a DTFT in $\nu = 0$ (where $\nu := \frac{f}{f_s}$ and $\mathcal{F}\{ \}$ is the DTFT operator); (f) follows from the definition of a new function $W_{i,l,d}(\nu) := \mathcal{F}\{h_{i,l,d}[n] * h_{i,l,d}[-n]\}$. 

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What follows next is the definition and simplification of $W_{i,l,d}(\nu)$

$$W_{i,l,d}(\nu) := F\{h_{i,l,d}[n] * h_{i,l,d}[-n]\}$$

(a) $F\{h_{i,l,d}[n]\} F\{h_{i,l,d}[-n]\}$

(b) $F\{h_{i,l,d}[n]\} F\{h_{i,l,d}[n]\}$

(c) $|F\{h_{i,l,d}[n]\}|^2$

(d) $|F\{p_i[n - dN] \bar{p}_l[n]\}|^2$

(e) $|F\{p_i[n - dN]\} \oplus F\{\bar{p}_l[n]\}|^2$

(f) $|F\{p_i[n]\} e^{-j2\pi\nu dN} \oplus F\{\bar{p}_l[n]\}|^2$

(g) $|P_i(\nu) e^{-j2\pi\nu dN} \oplus \bar{P}_l(\nu)|^2$  \hspace{1cm} (B.3)

(a) follows from the convolution theorem; (b) follows from the complex conjugate symmetry of the DTFT for a real signal; (c) follows from the definition of absolute value; (d) follows from (B.1); (e) follows from the convolution theorem (where $\oplus$ is the circular convolution operator) [18, pp. 297]; (f) follows from the DTFT of a time shift [18, pp. 296]; (g) follows from defining $P_i(\nu) = F\{p_i[n]\}(\nu)$.

Next, a detailed expression for $P_i(\nu)$ is given

$$P_i(\nu) = F\{p_i[n]\}(\nu) \overset{(a)}{=} \begin{cases} F\{p_{even}[n] e^{j2\pi \frac{n}{N}}\}(\nu) = P_{even}(\nu - \frac{i}{N}) & \text{if } i \text{ even} \\ F\{p_{odd}[n] e^{j2\pi \frac{n}{N}}\}(\nu) = P_{odd}(\nu - \frac{i}{N}) & \text{if } i \text{ odd} \end{cases} \hspace{1cm} (B.4)$$

where the frequency shift property of the DTFT has been applied [18, pp. 300]. Also, we define $P_{even}(\nu) := F\{p_{even}[n]\}$ and $P_{odd}(\nu) := F\{p_{odd}[n]\}$. 

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B.2 Derivation of $E[|\beta_{i,l,d}|^2]$

In this appendix an expression for $E[|\beta_{i,l,d}|^2]$ is derived. The derivation follows.

\[
E[|\beta_{i,l,d}|^2] \overset{(a)}{=} E\left[ j \sum_{s=-\infty}^{+\infty} \phi[s] p_i[s - dN] \overline{p_i[s]} \right]^2
\]

\[
\overset{(b)}{=} E\left[ j \sum_{s=-\infty}^{+\infty} \phi[s] h_{i,l,d}[s] \right]^2
\]

\[
\overset{(c)}{=} E\left[ \sum_{n=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \phi[s] h_{i,l,d}[s] \phi[s + n] h_{i,l,d}[s + n] \right]
\]

\[
= \sum_{n=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} E[\phi[s]\phi[s + n]] h_{i,l,d}[s] h_{i,l,d}[s + n]
\]

\[
\overset{(d)}{=} \sum_{n=-\infty}^{+\infty} R_{\phi}[-n] \left( \sum_{s=-\infty}^{+\infty} h_{i,l,d}[s] h_{i,l,d}[s + n] \right)
\]

\[
\overset{(e)}{=} \sum_{n=-\infty}^{+\infty} R_{\phi}[-n] \left( h_{i,l,d}[n] * h_{i,l,d}[-n] \right)
\]

\[
\overset{(f)}{=} \sum_{n=-\infty}^{+\infty} \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu)e^{-j2\pi\nu n} d\nu \right) \left( h_{i,l,d}[n] * h_{i,l,d}[-n] \right)
\]

\[
= \int_{-0.5}^{+0.5} S_{\phi}(\nu) \left( \sum_{n=-\infty}^{+\infty} \left( h_{i,l,d}[n] * h_{i,l,d}[-n] \right)e^{-j2\pi\nu n} \right) d\nu
\]

\[
\overset{(g)}{=} \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,d}(\nu) d\nu
\]

\[
\overset{(h)}{=} \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l,d}(\nu) d\nu
\]

(a) follows from (4.9); (b) follows from (B.1); (c) follows from the definition of the absolute value; (d) follows from the definition of the autocorrelation of a discrete stochastic process, i.e., $R_X[n] = E[X[s]X[s - n]]$ (being $X$ a discrete stochastic process); (e) follows from the definition of the autocorrelation of a discrete WSS stochastic process as the IDTFT of its PSD ($S_{\phi}(\nu)$ is the PSD of sampled phase noise, which is a WSS discrete stochastic process); (f) follows from the definition of the autocorrelation of a discrete WSS stochastic process as the IDTFT of its PSD ($S_{\phi}(\nu)$ is the PSD of sampled phase noise, which is a WSS discrete stochastic process); (g) follows from the definition of $W_{i,l}(\nu) := F \{h_{i,l,d}[n] * h_{i,l,d}[-n]\}$ previously derived in (B.3).
Appendix C

OQAM-FBMC Derivations

C.1 Derivation of $|\gamma_{i,l,d}|^2$

In this appendix an expression for $|\lambda_{i,l,d}|^2$ is derived. First, we are going to define a new filter that will simplify the following derivations

$$h_{i,l,d}^R[n] := \Re\left\{\theta_{i,d} \ p_i \left[n - d \frac{N}{2}\right] \ \theta_{l,p[l]}\right\}$$

$$= \Re\left\{j^{i-l+d} \ p_i[n] \ p \left[n - d \frac{N}{2}\right] e^{j2\pi \frac{i-l}{N} n}\right\}$$

$$= \Re\left\{p[n] \ p \left[n - d \frac{N}{2}\right] e^{j(2\pi \frac{i-l}{N} n + (i-l+d) \frac{\pi}{2})}\right\}$$

$$= \Re\left\{\Omega_{i,l,d}[n]\right\}$$

$$= p[n] \ p \left[n - d \frac{N}{2}\right] \ \cos(\Omega_{i,l,d}[n]) \quad (C.1)$$

(a) follows from $\theta_{i,d} \theta_l = j^{i-l+d}$ and from (5.2); (b) follows from $j^{i-l+d} = e^{j(i-l+d) \frac{\pi}{2}}$; (c) follows by defining $\Omega_{i,l,d}[n] := 2\pi \frac{i-l}{N} n + (i-l+d) \frac{\pi}{2}$ and takes into account that $p[n] \in \mathbb{R}, \forall n \in \mathbb{R}$; (d) follows from Euler’s formula, i.e., $e^{ix} = \cos(x) + j \sin(x) \ \forall x \in \mathbb{R}$.
The derivation of $|\lambda_{i,t,d}|^2$ follows

\[
|\gamma_{i,t,d}|^2 \overset{(a)}{=} \sum_{s=-\infty}^{\infty} \Re \left\{ \theta_{i,t,d} p_s \left[ s - d \frac{N}{2} \right] \theta_{t,s}[s] \right\}^2
\]

\[
\overset{(b)}{=} \sum_{s=-\infty}^{\infty} h_{i,t,d}^R[s]^2
\]

\[
\overset{(c)}{=} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} h_{i,t,d}^R[s] h_{i,t,d}^R[s+n]
\]

\[
\overset{(d)}{=} \sum_{n=-\infty}^{\infty} (h_{i,t,d}^R[n] * h_{i,t,d}^R[-n])
\]

\[
\overset{(e)}{=} \mathcal{F} \{ h_{i,t,d}^R[n] * h_{i,t,d}^R[-n] \} (\nu = 0)
\]

\[
\overset{(f)}{=} W_{i,t,d}^S(\nu) \delta(\nu)
\]

(a) follows from (5.7); (b) follows from (C.1); (c) follows from definition of the absolute value; (d) follows from convolution’s definition (whose operator is $*$); (e) follows from the interpretation of the infinite sum as a DTFT in $\nu = 0$ (where $\nu := \frac{t}{d}$ and $\mathcal{F} \{ \}$ is the DTFT operator); (f) follows by defining $W_{i,t,d}^S(\nu) := \mathcal{F} \{ h_{i,t,d}^R[n] * h_{i,t,d}^R[-n] \}$ which is derived in (C.3).

What follows next is the definition and simplification of $W_{i,t,d}^S(\nu)$

\[
W_{i,t,d}^S(\nu) := \mathcal{F} \{ h_{i,t,d}^R[n] * h_{i,t,d}^R[-n] \}
\]

\[
\overset{(a)}{=} \mathcal{F} \{ h_{i,t,d}^R[n] \} \mathcal{F} \{ h_{i,t,d}^R[-n] \}
\]

\[
\overset{(b)}{=} \mathcal{F} \{ h_{i,t,d}^R[n] \} \mathcal{F} \{ h_{i,t,d}^R[n] \}
\]

\[
\overset{(c)}{=} \mathcal{F} \{ h_{i,t,d}^R[n] \}^2
\]

\[
\overset{(d)}{=} \mathcal{F} \{ p[n] \} \mathcal{F} \{ n - d \frac{N}{2} \cos(\Omega_{i,t,d}[n]) \}
\]

\[
\overset{(e)}{=} \mathcal{F} \{ p[n] \} \mathcal{F} \{ n - d \frac{N}{2} \} \mathcal{F} \{ \cos(\Omega_{i,t,d}[n]) \}
\]

\[
\overset{(f)}{=} V_d(\nu) \mathcal{F} \left\{ \cos \left( 2\pi \frac{i - l}{N} n + (i - l + d) \frac{\pi}{2} \right) \right\}
\]

\[
\overset{(g)}{=} V_d(\nu) \mathcal{F} \left\{ \delta \left( \nu - \frac{i - l}{N} \right) e^{j(i-l)d} \frac{\pi}{2} \delta \left( \nu + \frac{i - l}{N} \right) e^{-j(i-l)d} \frac{\pi}{2} \right\}
\]

\[
\overset{(h)}{=} \left| V_d \left( \nu - \frac{i - l}{N} \right) e^{j(i-l+d)} \frac{\pi}{2} + V_d \left( \nu + \frac{i - l}{N} \right) e^{-j(i-l+d)} \frac{\pi}{2} \right|^2
\]

\[
\overset{(i)}{=} \frac{1}{4} \left[ \left| V_d \left( \nu - \frac{i - l}{N} \right) \right|^2 + \left| V_d \left( \nu + \frac{i - l}{N} \right) \right|^2 + 2 \Re \{ Y_{i,t,d}(\nu) \} \right]
\]
(a) follows from the convolution theorem [18, pp. 297]; (b) follows from the complex conjugate symmetry of the DTFT for a real signal; (c) follows from the definition of absolute value; (d) follows from (C.1); (e) follows from the convolution theorem (where ⊙ is the circular convolution operator); (f) follows by defining $V_d(\nu) := \mathcal{F} \{ p[n] p[n - d \frac{N}{2}] \} = P(\nu) \odot \left( P(\nu)e^{-2\pi\nu\frac{d}{N}} \right)$, being $P(\nu) := \mathcal{F} \{ p[n] \}$; (g) follows from the DTFT of a cosine; (h) follows from the application of the circular convolution operator; (i) follows from the application of the absolute value operator and $Y_{i,l,d}(\nu)$ is defined as

$$Y_{i,l,d}(\nu) := e^{j\pi(i-l+d)} \left( \nu - \frac{i-l}{N} \right) \left( \nu + \frac{i-l}{N} \right)$$
C.2 Derivation of $E[|\rho_{i,l,d}|^2]$ 

In this appendix an expression for $E[|\rho_{i,l,d}|^2]$ is derived. First, we are going to define a new filter that will simplify the following derivations

$$h_{i,l,d}[n] := \Re \left\{ j \theta_{i,l,d} p_i \left[ n - d \frac{N}{2} \right] e^{j2\pi \frac{n}{N}} \right\}$$

(a) follows from $\theta_{i,l,d} = j^{i-l+d}$ and from (5.2); (b) follows from $j^{i-l+d} = e^{j(i-l+d)\frac{\pi}{2}}$; (c) follows by defining $\Omega_{i,l,d}[n] := 2\pi \frac{i-l+d}{N} n + (i-l+d) \frac{\pi}{2}$ and takes into account that $p[n] \in \mathbb{R}, \forall n \in \mathbb{R}$; (d) follows from Euler’s formula, i.e., $e^{jx} = \cos(x) + j \sin(x) \forall x \in \mathbb{R}$.
Next, the derivation of a suitable expression for $E\left[|\rho_{i,l,d}|^2\right]$ follows.

$$E\left[|\rho_{i,l,d}|^2\right]\overset{(a)}{=} E\left[j \sum_{s=-\infty}^{+\infty} \Re\left\{ j \phi[s] \theta_{i,d} \left[ s - \frac{dN}{2} \right] \theta_{i,l}[s] \right\}^2\right]$$

$$\overset{(b)}{=} E\left[j \sum_{s=-\infty}^{+\infty} \phi[s] h_{i,l,d}^I[s]\right]^2$$

$$\overset{(c)}{=} E\left[ \sum_{s=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \phi[s] h_{i,l,d}^I[s] \phi[s+n] h_{i,l,d}^I[s+n]\right]$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} E[\phi[s]\phi[s+n]] h_{i,l,d}^I[s] h_{i,l,d}^I[s+n]$$

$$\overset{(d)}{=} \sum_{n=-\infty}^{+\infty} R_{\phi}[-n] \left( \sum_{s=-\infty}^{+\infty} h_{i,l,d}^I[s] h_{i,l,d}^I[s+n] \right)$$

$$\overset{(e)}{=} \sum_{n=-\infty}^{+\infty} R_{\phi}[-n] \left( h_{i,l,d}^I[n] * h_{i,l,d}^I[-n] \right)$$

$$\overset{(f)}{=} \sum_{n=-\infty}^{+\infty} \left( \int_{-0.5}^{+0.5} S_{\phi}(\nu)e^{-j2\pi\nu n} d\nu \right) \left( h_{i,l,d}^I[n] * h_{i,l,d}^I[-n] \right)$$

$$= \int_{-0.5}^{+0.5} S_{\phi}(\nu) \left( \sum_{n=-\infty}^{+\infty} \left( h_{i,l,d}^I[n] * h_{i,l,d}^I[-n] \right)e^{-j2\pi\nu n} \right) d\nu$$

$$\overset{(g)}{=} \int_{-0.5}^{+0.5} S_{\phi}(\nu) F \left\{ h_{i,l,d}^I[n] * h_{i,l,d}^I[-n] \right\} d\nu$$

$$\overset{(h)}{=} \int_{-0.5}^{+0.5} S_{\phi}(\nu) W_{i,l}(\nu) d\nu$$

Equations (a) follows from (5.8); (b) follows from (C.4); (c) follows from the definition of the absolute value; (d) follows from the definition of the autocorrelation of a discrete stochastic process, i.e., $R_X[n] = E[X[s]X[s-n]]$ (being $X$ a discrete stochastic process); (e) follows from the definition of the convolution ($*$ is the convolution operator); (f) follows from the definition of the autocorrelation of a discrete WSS stochastic process as the IDTFT of its PSD ($S_{\phi}(\nu)$ is the PSD of sampled phase noise, which is a WSS discrete stochastic process); (g) follows by defining $W_{i,l}(\nu) := F \left\{ h_{i,l,d}^I[n] * h_{i,l,d}^I[-n] \right\}$, which is derived next in (C.6).
What follows next is the definition and simplification of $W_{i,l,d}(\nu)$

$$W_{i,l,d}(\nu) := \mathcal{F} \{ h_{i,l,d}[n] * h_{i,l,d}[-n] \}$$

(a) follows from the convolution theorem [18, pp. 297];
(b) follows from the complex conjugate symmetry of the DTFT for a real signal;
(c) follows from the definition of absolute value;
(d) follows from (C.4);
(e) follows from the convolution theorem (where $\ast$ is the circular convolution operator);
(f) follows by defining $V_d(\nu) := \mathcal{F} \{ p[n] p [n - dN/2] \} = P(\nu) \ast (P(\nu)e^{-2\pi \nu \frac{d}{N}})$, being $P(\nu) := \mathcal{F} \{ p[n] \}$;
(g) follows from the DTFT of a sine;
(h) follows from the application of the circular convolution operator;
(i) follows from the application of the absolute value operator and $Y_{i,l,d}(\nu)$ is defined as

$$Y_{i,l,d}(\nu) := e^{\pi l(i-l)}V_d \left( \nu - \frac{i-l}{N} \right) V_d^* \left( \nu + \frac{i-l}{N} \right)$$
Bibliography


