The Euclidean Distance
Degree of Conics

LUKAS GUSTAFSSON
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Abstract - English

The Euclidean Distance Degree (EDD) of a variety is the number of critical points of the squared distance function of a general point outside the variety. In this thesis we give a classification of conics based on their EDD, originally attributed to Cayley. We show that circles and parabolas have EDD 2 and 3 respectively while all other conics have EDD 4. We reduce the computation of the EDD to finding solutions of the determinant of a certain generalized matrix, called the hyperdeterminant of type $2 \times 3 \times 3$. This determinant is computed using the celebrated Schlafli decomposition.
Abstract - Svenska

Euclidean Distance Degree (EDD) av en algebraisk varietet är antalet kritiska punkter hos kvadraten av avståndsfunktionen givet en generell punkt utanför varieteten. I detta examensarbete ger vi en klassifikation av kägelsnitt utifrån deras EDD, som originellt gjordes av Cayley. Vi visar att cirklar och paraboler har EDD 2 och 3 respektive medan alla andra kägelsnitt har EDD 4. Vi reducerar beräkningen av EDD till att hitta nollställen till determinanten av en generaliserad matris, den s.k. hyperdeterminanten av typ $2 \times 3 \times 3$. Denna determinant beräknas med hjälp av Schläfli-dekomposition.
Introduction

This thesis deals with an important invariant of algebraic varieties, namely the Euclidean Distance Degree (EDD). The EDD is an important factor in estimating the distance of an algebraic variety to a generic point in the ambient space. Let \( X \subset \mathbb{C}^n \) be a non-singular algebraic variety and let \( u \in \mathbb{C}^n \) be a generic point. The EDD of \( X \) over \( \mathbb{C}^n \) is defined as the number of critical points of the squared Euclidean distance function

\[
d_u(x) = \sqrt{(x-u)^2 + (y-v)^2}
\]

As an example, when fixing a point outside the circle, there are always 2 critical points of the squared distance function.

For more on the EDD see section 1.5. In this thesis we study the complex EDD of conics defined by real polynomials. We prove that:

**Theorem:** (see theorem 2 section 2.2)

Let \( C \) be an irreducible conic. Then:

- \( EDD(C) = 2 \) if \( C \) is a circle
- \( EDD(C) = 3 \) if \( C \) is a parabola
- \( EDD(C) = 4 \) otherwise

For generic conics this confirms the value proved for generic hypersurfaces in [2]. The complete classification for conics is attributed to Cayley. The theorem reproves his result, which we were unable to completely recover from existing literature. The main tools for proving the theorem are the hyperdeterminant and the Schläfli method, introduced in 1.6.

Further studies

The classification of EDD for higher degree curves or higher dimensional surfaces can be related to duality theory and discriminants. In chapter 3 we discuss some possible generalizations.

Acknowledgements

I would like to thank my supervisor Sandra Di Rocco for the vital guidance and extensive feedback on the thesis draft. Tianfang Zhang is acknowledged for valuable discussions and William Eriksson for listening to my rambling.
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Chapter 1

Theoretical Background

This thesis is about the Euclidean Distance Degree (EDD) of conics. We shall start by defining the two main concepts: conics and EDD. To define the EDD one needs the basic framework of regular points and tangent space of an algebraic variety which we define. We also define the hyperdeterminant which will allow us to calculate the EDD.

1.1 Notation

Before we begin we need to clarify some of the notation that will be used. The sign \( \setminus \) will denote set difference, \( V \) will denote the algebraic set defined as the zero locus of an ideal or the ideal generated by a certain polynomial. If we are working over the field \( K \) then
\[
V(I) = \{ x \in K^n | \forall f \in I, f(x) = 0 \}
\]
In most cases \( I \subset K[x_1, \ldots, x_n] \) will be represented by its generators, for example \( V(x^2 + y^2 - 1) \) is the unit circle.

For a given algebraic set \( X \) we will have statements being true for a generic point. This means that the statement is true for all points except for those in a set of zero Lebesgue measure, for example the zero locus of some polynomials.

Recall that, if \( f \in \mathbb{C}[x_1, \ldots, x_n] \), the gradient of \( f \) is the vector
\[
\nabla f = \begin{bmatrix}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{bmatrix}
\]

1.2 Conics

First we give an intuitive definition of what a circular cone and conic sections are and work it into something more useful.

Definition 1. A circular cone is an algebraic set in \( \mathbb{R}^3 \) that up to orthogonal transformation can be defined by the polynomial equation
\[
Ax^2 + By^2 + Cz^2 = 0
\]

Proposition 1. An algebraic set \( X \subset \mathbb{R}^3 \) is a circular cone if and only if it is the zero locus of a quadratic form on \( \mathbb{R}^3 \) given by some real symmetric matrix \( M \).

Proof. One direction is trivial. Assume we are given a quadratic form \( f \) on \( \mathbb{R}^3 \). Then by the Spectral theorem there is an orthonormal basis in which it can be represented by a diagonal matrix. By taking the zero locus of this quadratic form we obtain the desired result. \( \square \)
Remark 1. Note that if all eigenvalues of a symmetric matrix are non-zero and all have the same sign, the corresponding circular cone would be the origin. We also see that when in the situation where one of the eigenvalues $A, B, -C$ are 0 and the other 2 arbitrary, w.l.o.g we can assume $-C = 0$, then the variety is $Ax^2 + By^2 = 0$, is either a point when $A, B$ have the same sign or in the other case we obtain a union of two hyperplanes

$$(\sqrt{|A|}x + \sqrt{|B|}y)(\sqrt{|A|}x - \sqrt{|B|}y) = 0$$

as there is no restriction on $z$. This leads us to the following definition.

Definition 2. A circular cone is said to be degenerate if and only if in every orthonormal basis of $\mathbb{R}^3$ it is represented by a matrix with determinant 0.

Definition 3. A conic section is the equivalence class of algebraic sets in $\mathbb{R}^3$ that up to affine transformation is defined by the intersection of a circular cone and a plane, given by the system of polynomial equations

$$\begin{align*}
\mathbf{x}^T M \mathbf{x} &= 0 \\
ax + by + cz &= \lambda
\end{align*}$$

(1.3)

where $M$ is symmetric, $\lambda \in \mathbb{R}$ and $(a, b, c) \neq 0 \in \mathbb{R}^3$ is a unit vector. We denote the set of conic sections by $\mathcal{C}$.

Definition 4. A conic section is defined to degenerate if and only if one of its representatives is given by the intersection of a degenerate circular cone and some hyperplane. Non-degenerate conic sections are referred to as proper.

The polynomials of degree at most 2 in $\mathbb{R}[x, y]$ are called conics. If $f$ is a conic then it is of the form

$$f(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

These conics will often be identified with their zero locus.

Remark 2. These polynomials are called conics because of their relation to the conic sections as we are about to demonstrate.

Definition 5. The conics are divided into 4 classes depending on the coefficients

- Circles: $a_{20} = a_{02}, \quad a_{11} = 0$
- Ellipse: $a_{20}a_{02} - a_{11} > 0$
- Parabolas: $a_{20}a_{02} - a_{11} = 0$
- Hyperbolas: $a_{20}a_{02} - a_{11} < 0$

**Definition 6.** We define the **matrix corresponding to the conic** $f$ to be

$$M_f = \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ a_{11} & a_{02} & a_{01} \\ a_{20} & a_{01} & a_{00} \end{bmatrix}$$

Note that every symmetric $3 \times 3$ matrix $M_f$ defines a circular cone of the form

$$\pi^T M_f \pi = 0$$

**Definition 7.** A conic $f$ is called degenerate if its corresponding circular cone is degenerate.

Consider the map

$$\mathcal{C}(f) = \mathcal{V}(\pi^T M_f \pi, z - 1)$$

Here we send $f$ to an algebraic set that is a representative of some conic section. We call $\mathcal{C}(f)$ the **conic section corresponding to** $f$. We will refer to $\mathcal{V}(\pi^T M_f \pi, z - 1)$ as the canonical representative of $\mathcal{C}(f)$

**Remark 3.** For every conic $f$ there is a set bijection between those of the form $\mathcal{V}(f) \subset \mathbb{R}^2$ and the canonical representatives $\mathcal{V}(f, z - 1) \subset \mathbb{R}^3$. This bijection is translation in the $z$-direction. These algebraic sets will be identified unless the difference plays an important role.

**Remark 4.** We say that a conic $f$ **corresponds to the conic section** $A$ if $f$ lies in the pre-image of $A$ under the correspondence map.

$$f \in \mathcal{C}^{-1}(A)$$

**Proposition 2.** The **correspondence map** $\mathcal{C}$ is surjective.

**Proof.** Given an arbitrary conic section $C$ and representative given as the solution set to 1.3 we can always change basis to an orthonormal one where the two first basis vectors span the plane $ax + by + cz = 0$ and the third is $(a, b, c)$. Then in this new basis with coordinates $(x', y', z')$ we have that the cone is intersected with the hyperplane

$$z' = \lambda$$

along this plane

$$\pi' = \begin{bmatrix} x' \\ y' \\ \lambda \end{bmatrix}$$

Every conic has a representative algebraic set of the form

$$\begin{cases} g(\pi) = \pi^T M \pi = 0 \\ z = \lambda \end{cases}$$

(1.5)

where $M$ is a symmetric matrix and $\lambda \in \mathbb{R}$. Let $f(x, y) = g(x, y, \lambda)$. Clearly then

$$\mathcal{V}(g, z - \lambda) = \mathcal{V}(f, z = \lambda)$$

(1.6)

After translating this representative algebraic set along the $z$-axis to $z = 1$, we have that

$$\begin{cases} f(x, y) = 0 \\ z = 1 \end{cases} \iff \begin{cases} \pi^T M_f \pi = 0 \\ z = 1 \end{cases}$$

(1.7)

This is the canonical representative of $f$. So $C$ lies in the image of $f$ under the correspondence map.

**Proposition 3.** A conic is irreducible over $\mathbb{C}$ iff it is proper.
Proof. Let
\[ f(x, y) = ax^2 + 2cxy + by^2 + 2dx + 2ey + l \]
To show one direction. Assume \( f \) is a reducible conic.
\[ f(x, y) = (k_1x + k_2y + k_3)(q_1x + q_2y + q_3) \] \tag{1.8}

The following code in Macauley2 verifies that the determinant of the corresponding matrix \( M_f \) vanishes and therefore \( f \) is degenerate (non-proper).

\begin{verbatim}
R = QQ[k1, k2, k3, q1, q2, q3]
a = k1*q1
b = k2*q2
l = k3*q3
c = 1/2*(k1*q2 + k2*q1)
d = 1/2*(k1*q3 + k3*q1)
e = 1/2*(k2*q3 + k3*q2)
M = matrix({{a,c,d}, {c,b,e}, {d,e,l}})
det(M)
\end{verbatim}

To show the other direction, assume that the determinant of \( M_f \) is 0. This means that the system
\[
\begin{align*}
ax + cy + dz &= 0 \\
(cx + by + ez) &= 0 \\
dx + ey + lz &= 0
\end{align*}
\] \tag{1.9}
Has a non-trivial solution.

Case 1:

Assume the solution is of the form \( X' = \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} \). Then we have that
\[
\begin{align*}
ax_0 + cy_0 &= -d \\
cx_0 + by_0 &= -e \\
dx_0 + ey_0 &= -l
\end{align*}
\] \tag{1.10}
This means that
\[
\begin{align*}
f(x, y) &= ax^2 + 2cxy + by^2 - 2(ax_0 + cy_0)x - 2(cx_0 + by_0)y + (ax_0^2 + 2cx_0y_0 + by_0^2) \\
f(x, y) &= a(x^2 - 2xx_0 + x_0^2) + b(y^2 - 2yy_0 + y_0^2) + 2c(xy - y_0x - x_0y + x_0y_0) \\
a(x - x_0)^2 + b(y - y_0)^2 + 2c(x - x_0)(y - y_0)
\end{align*}
\] \tag{1.11}
This is a quadratic form in the variables \( x' = x - x_0, y' = y - y_0 \) given by the upper left \( 2 \times 2 \) submatrix of \( M_f \). We can use the spectral theorem to do a linear change of variables again to \( s, t \) where we obtain
\[
f(x, y) = At^2 + Bs^2 = (\sqrt{A}t + \sqrt{B}s)(\sqrt{A}t - \sqrt{B}s)
\] \tag{1.14}
where we have factored over \( \mathbb{C} \). Since \( s, t \) are linear polynomials in \( x, y \) we have factored \( f \).

Case 2:
Assume that the solution is of the form \( \mathbf{X} = \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} \). We may assume \( x_0, y_0 \neq 0 \) w.l.o.g. because otherwise this would mean that \( f(x, y) \) only depends on \( x \) or \( y \), and then one can use the standard quadratic formula to find the roots. Now we have a solution

\[
\begin{align*}
ax_0 + cy_0 &= 0 \\
ax_0 + by_0 &= 0 \\
dx_0 + ey_0 &= 0 
\end{align*}
\]
\[
\iff \begin{cases} 
\begin{aligned} 
a &= t_0c = 0 \\
b &= \frac{1}{t_0}c \\
d &= t_0e 
\end{aligned} 
\end{cases} \quad (1.15)
\]

We obtain that

\[
f(x, y) = t_0c x^2 + 2cxy + \frac{c}{t_0}y^2 + 2t_0ex + 2cy + l \quad (1.16)
\]

substitute \( y = t_0 y' \)

\[
f(x, y) = t_0c(x^2 + 2xy' + y'^2) + t_0 2c(x + y') + l \quad (1.17)
\]

Let \( s = x + y' \), then

\[
f(x, y) = As^2 + Bs + l \quad (1.18)
\]

which clearly has two roots w.r.t. \( s \) and we have factored \( f \).

**Corollary 1.** Irreducible conics over \( \mathbb{C} \) are non-singular.

**Proof.** The conic \( f \) having a singular point means exactly that there exists a solution \( \mathbf{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \) to the system

\[
M_f \mathbf{X} = 0 \quad (1.19)
\]

The calculation proving this result is done in proposition 10. A necessary condition for such a solution to exist is that the determinant of \( M_f \) is zero. Therefore \( f \) must be reducible if it has a singular point. \( \square \)

### 1.3 Regular and singular points of an algebraic variety

Let \( I_X = (f_1, \ldots, f_s) \) and \( f_i \in \mathbb{C}[x_1, \ldots, x_n] \). Let \( X = \mathbb{V}(I_X) \). Given \( f_i \) we can construct the map

\[
f : \mathbb{C}^n \to \mathbb{C}^s \quad f(x) = (f_1(x), f_2(x), \ldots, f_s(x)) \quad (1.20)
\]

Consider the Jacobian \( s \times n \) matrix

\[
J_f(x) = \begin{bmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \vdots \\ \nabla f_s(x) \end{bmatrix} \quad (1.21)
\]

where the gradients are seen as row vectors so that \((J_f)_{ij} = \frac{\partial f_i}{\partial x_j}\), the formal partial derivatives. Observe that the matrix \( J_f(x) \) has a generic rank i.e. the rank of \( J_f(x) \) is constant over a Zariski open set and may decrease over a Zariski closed subset.

**Definition 8.** A point \( x \in X \) is **regular** if rank \( J_f(x) \) is maximal. Non-regular points are called **singular**.
**Definition 9.** The set of singular points is called the **singular locus** and is denoted by $X_{\text{Sing}}$.

As observed earlier $X_{\text{Sing}}$ is a Zariski closed subset and more specifically defined by the ideal

$$I_{\text{Sing}} = (f_1, \ldots, f_s, \{r \times r \text{ minors of } J_f\}) \quad X_{\text{Sing}} = V(I_{\text{Sing}})$$

where $r$ is the generic rank of the variety.

**Proposition 4.** When $f_i \in \mathbb{R}[x_1, \ldots, x_n]$. The tangent space $T_x(X \setminus X_{\text{Sing}})$ is given by the kernel of $J_f(x)$

$$T_x(X \setminus X_{\text{Sing}}) = \ker J_f(x)$$

**Proof.** $J_f$ is the matrix of the differential $df$ of the smooth map $f$. The differential of a restriction of a smooth map is the restriction of the differential of the same map. When regarding tangent vectors as paths its is clear that every path is sent to 0 by $df|_x$ because $X \setminus X_{\text{Sing}}$ is contained in a levelset of $f$. Since the kernel of $J_f(x)$ has the same dimension as the tangent space at $x$ they must be equal, translation by $x$ fixes the basepoint. □

**Remark 5.** In accordance to Proposition 4, when $X$ is defined by complex polynomials $f_i$, the proof doesn’t change and we can regard it as a "complex submanifold". We define the **tangent space** at a point $x$ to be kernel of the Jacobian $J_f$ evaluated at $x$. Whenever the tangent space of a variety at a point $x$ is mentioned from now on, it is $\ker J_f(x)$ we refer to.

**Definition 10.** Suppose we have two algebraic sets defined by one polynomial each,

$$X = V(f_a), Y = V(f_b) \subset \mathbb{C}^n$$

We define a **point of tangency between** $X$ and $Y$ to be a point $x \in X \cap Y$ such that $T_x(X) \subseteq T_x(Y)$ or $T_x(X) \supseteq T_x(Y)$. In other words the gradients of $f_a$ and $f_b$ should be linearly dependent. This can be formulated as a solution to the following system of equations,

$$\exists \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \text{ s.t. } \begin{cases} f_a(x) = 0 \\ f_b(x) = 0 \\ s_1 \nabla f_a(x) + s_2 \nabla f_b(x)(x) = 0 \end{cases}$$

(1.22)

We call the point of tangency **regular** if $x$ is not a singular point of $X$ or $Y$ i.e.

$$\nabla f_a(x), \nabla f_b(x) \neq 0$$

(1.23)
Figure 1.3: The origin is a point of tangency for $V((x - 1)^2 - y^2 - 1)$ and $V(y^2 - x^3)$

Source: [1]

Figure 1.4: The origin is a regular point of tangency for $V((x - 1)(y - 1))$ and $V((x - 0.5)^2 + (y - 0.5)^2 - 0.5)$

Source: [1]
1.4 The Euclidean Distance Degree

Given an algebraic set $X \subset \mathbb{R}^n$ and a generic point $u \in \mathbb{R}^n \setminus X$, a concept of interest is the set of regular points of $X$ that are critical for the distance function from $u$, restricted to $X$ that is

$$d_u(x) = \sqrt{\sum_i (x_i - u_i)^2}$$  \hspace{1cm} (1.24)

Now the partial derivative of the distance function along $X$ at a regular point is just the scalar product of the gradient of $d_u$ with the corresponding tangent vector of $X$ at $x$. This means that this derivative along $X$ vanishes when the gradient of the distance function is orthogonal to the tangent space at $x$.

$$\nabla d_u(x) = \frac{(x - u)}{\sqrt{\sum_i (x_i - u_i)^2}}$$  \hspace{1cm} (1.25)

So we can w.l.o.g. use the vector $x - u$, a factor $\frac{1}{2}$ times the gradient of the squared distance function, instead of the gradient of $d_u$. We seek solutions to the following system

$$x \not\in X_{Sing} \quad (x - u) \perp T_xX$$  \hspace{1cm} (1.26)

We say that two vectors $x, y$ are normal iff

$$x \cdot y := \sum_i x_i y_i = 0$$  \hspace{1cm} (1.27)

This is denoted by $x \perp y$ since for real vectors it corresponds to being orthogonal.

Observe that for a given $p \in \mathbb{C}^n \setminus \{0\}$, the set of solutions to

$$p \cdot x = 0$$  \hspace{1cm} (1.28)

is a vector subspace of codimension 1.

We also want to study these points for complex varieties $X$ and points $u$. We will therefore define

**Definition 11.** For an algebraic set $X$ (complex or real), the **EDD w.r.t. a point $u$** is the cardinality of the solution set to

$$x \not\in X_{Sing} \quad (x - u) \perp T_xX$$  \hspace{1cm} (1.29)

**Example 1.** Let’s look at the curve defined by $y^3 - x^4 + x = 0$ and let $u = (-2.323, 0)$. The algebraic set has no singular points and there are at least three real solutions: $(-1, 1.26), (0, 0), (1, 0)$ to the set of equations 1.29.
Remark 6. The following proposition will be main ingredient in the proof of the main result. Also note that the number of regular points of tangency between a variety $X$ and the set of spheres around $u$ is the total number of points of tangency, minus the number of singular points on $X$. From now on a sphere centered at $u$ refers to the zero locus of a polynomial
\[
\sum_i (x_i - u_i)^2 - r^2 \in \mathbb{C}[x_1, \ldots, x_n]
\]

Proposition 5. The EDD w.r.t. $u$ is equal to the total number of regular points of tangency between $X$ and spheres centered at $u$.

Proof. Given a solution $p$ to the system 1.29 then let $r^2 = \sum_i (p_i - u_i)^2$. The point $p$ is a regular tangent point of tangency between $X$ and $V(\sum_i (x_i - u_i)^2 - r^2)$, when $u$ is generic. And if there is a regular point of tangency, $p_i$ between $X$ and a circle centered at $u$ then it must contribute to the EDD w.r.t. the center of the circle. \qed

Let $X = V(I_X)$ and $I_X = (f_1, \ldots, f_s)$, where $f_i$ may have complex or real coefficients. Let
\[
J_f(x) = \begin{bmatrix}
    x - u \\
    \nabla f_1(x) \\
    \nabla f_2(x) \\
    \vdots \\
    \nabla f_s(x)
\end{bmatrix}
\]

the matrix built by $x - u$ and the gradients as row vectors.

Lemma 1.
\[(x - u) \perp T_xX \iff \dim \ker J_f(x) = \dim \ker J_f'(x) \quad (1.31)\]

Proof. The kernel of $J_f'(x)$ is always contained in the kernel of $J_f(x)$. If $x - u \perp \ker J_f(x)$ and $v \in \ker J_f(x)$, then $v \in \ker J_f'(x)$ as well. If the kernels are equal, then $x - u \perp \ker J_f(x)$. \qed

Lemma 2. Let $V(I), V(J) \subset \mathbb{C}^n$, then
\[V(I : J^\infty) = (V(I) \setminus V(J)) \quad (1.32)\]

where $(I : J^\infty)$ is the saturated ideal quotient.

Proof. See B.1 in Appendix B. \qed

Proposition 6. Given a point $u$, let
\[I_T = (f_1, \ldots, f_s, \{(c+1) \times (c+1) \text{ minors of } J_f^1\}) \quad (1.33)\]
\[I_{X_{\text{Sing}}} = (f_1, \ldots, f_s, \{c \times c \text{ minors of } J_f\}) \quad (1.34)\]

Then the Zariski closure of the set of solutions, $S$, to the equation
\[x \notin X_{\text{Sing}} \quad (x - u) \perp T_xX \quad (1.35)\]
is given by
\[S = V(I_T : X_{\text{Sing}}^\infty) \quad (1.36)\]
The ideal $(I_T : X_{\text{Sing}}^\infty)$ will be referred to as the critical ideal and its zero locus is referred to as the critical points.

Proof. We use Lemma 1 to reformulate equation 1.29 such that $x \in S \subset X$ iff
\[J_f(x) \text{ has maximal rank} \quad \& \quad \text{rank } J_f'(x) = \text{rank } J_f(x) \quad (1.37)\]
The sets of points where these conditions are true are Zariski open and closed respectively, when regarded as subsets of $X$. Let the maximal rank of $J_f(x)$ be $c$. The points where the rank of $J_f(x)$ is $c$ is Zariski open because they are the complement of the singular locus $X_{\text{Sing}}$, which is closed.
Note that it is always true that \( \text{rank } J'_f(x) \geq \text{rank } J_f(x) \). Let us therefore study the set
\[
T = \{ x \in X : \text{rank } J'_f(x) \leq c \}
\]
(1.38)
By the same argument, \( T \) is the intersection of \( X \) with the zero loci of the \( (c+1) \times (c+1) \) minors of \( J'_f \). So what we want is \( J_f(x) \) to have maximal rank and \( J'_f(x) \) to have rank strictly lower than \( c+1 \). So the support of the EDD is
\[
S = T \setminus X_{\text{Sing}}
\]
(1.39)
So we are searching for the set difference of two Zariski closed sets. Note that \( I_T \) and \( I_{X_{\text{Sing}}} \) are the ideals that correspond to \( T \) and \( X_{\text{Sing}} \). We want the points contained in the set difference of these algebraic sets in \( \mathbb{C}^n \). The Zariski closure of the set difference is given by the saturated ideal quotient by Lemma 2.

**Proposition 7.** When \( X \) is regarded as a complex algebraic set, for generic \( u \in \mathbb{C}^n \), the cardinality \( |S| \), is finite and constant. This constant is what we define to be the EDD of the algebraic set \( X \).

**Proof.** This is proven in Lemma 2.1 in [2].

**Corollary 2.** For generic \( u \in \mathbb{C}^n \), \( \overline{S} = S \)

**Proof.** By proposition 7 \( \overline{S} \) is finite and finite sets are their own Zariski closure.

### 1.5 Computing the EDD

The topic that will be discussed in this paper is foremost the ways determining the EDD from geometry. We want a way of quickly determining the EDD without computing it explicitly. Before we begin we study some examples of how one might approach the problem of calculating the EDD without the use of further theory.

#### 1.5.1 Using symbolic code

Given an explicit ideal with coefficients in the field \( \mathbb{Q} \) one can compute the EDD in Macauley 2 with the code in Appendix A section A.1.

#### 1.5.2 Example: The circle

Here we prove that the EDD of the unit circle is 2.

Let \( X = \mathcal{V}(x^2 + y^2 - 1) \). Notice how this circle has no singular points. Let \( (u,v) \neq (0,0) \), then by eq. 1.29 the EDD is simply the number of solutions \( (x,y) \) such that the system
\[
\begin{cases}
x^2 + y^2 - 1 = 0 \\
\det \begin{bmatrix} 2x & 2y \\ x-u & y-v \end{bmatrix} = 0 \\
\end{cases}
\]
(1.40)
Where the second equation signifies that the vector \( (x-u, y-v) \) is normal to the tangent space at \( x \), i.e. a multiple of the gradient of the polynomial defining the circle. This is an easy system to solve, assume w.l.o.g. that \( u \neq 0 \)
\[
\begin{cases}
x^2 + y^2 - 1 = 0 \\
uy - vx = 0 \\
\end{cases} \iff \begin{cases}
u^2x^2 + (uy)^2 - u^2 = 0 \\
uy - vx = 0 \\
\end{cases}
\]
(1.41)
The upper equation has exactly 2 solutions for \( x \) except when \( u^2 + v^2 = 0 \) but that is considered a non-generic choice of \( (u,v) \). We conclude that the EDD is 2. Also note that when when \( (u,v) \in \mathbb{R}^2 \) then the points of tangency will also be real!
1.5.3 Attempt for arbitrary irreducible conic

Now let us study the example where the number of variables is \( n = 2 \) and \( X = V(f) \), where \( f \) is an arbitrary irreducible conic i.e. a polynomial of degree 2 as discussed in section 1.2. Now \( J_f(x) \) is simply the gradient \( \nabla f(x) \) as a row vector. This matrix having full rank is the same as being non-zero. Also no polynomial of degree 2 has a trivial gradient so, the dimension of \( X \) is equal to its codimension, 1. Let \( p = (u, v) \in \mathbb{C}^n \). Now \( J_f(x) \) has rank \( c = 1 \) if and only

\[
\det \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}
\end{bmatrix} = 0
\]

Note that \( p \) being a generic point means that \( x - p \) can be assumed to be non-zero.

The most naive approach to calculating the EDD of such a plane curve would be to let \( f \) have arbitrary coefficients

\[
f(x, y) = a_1 + a_2 x + a_3 y + a_4 x y + a_5 x^2 + a_6 y^2
\]

(1.43)

Then one could try to compute the critical ideal \( I_S \) symbolically seeing \( f \) as a polynomial in \( \mathbb{C}[a_1, \ldots, a_6, x, y] \) (only taking derivatives w.r.t. \( x \) and \( y \)). This is however difficult to do since \( I_S \) is computed by checking divisibility of polynomials and one can only check this for explicit values of \( a_i \).

Writing a system of equations

Let \( g = X^T M X = 0 \) be a irreducible conic. Let’s try to find the critical points in \( \mathbb{R}^2 \) w.r.t. \( u \in \mathbb{R}^2 \). Let

\[
a x^2 + b y^2 + c x y + d x + e y + f = 0
\]

(1.44)

Now consider the associated matrix

\[
M = \begin{bmatrix}
a & c/2 & d/2 \\
c/2 & b & e/2 \\
d/2 & e/2 & f
\end{bmatrix}
\]

(1.45)

Clearly the determinant is 0 if \( a = b = c = 0 \). So one of the degree 2 terms must be non-zero when \( g \) is non-degenerate. Affine transformations do not change the EDD so by rotation, reflection and translation one can reduce the problem to finding the number of solutions to the system

\[
\begin{cases}
x^2 + by^2 + cy + d = 0 \\
(x - u)(2by + c) - (y - v)(2x) = 0
\end{cases}
\]

(1.46)

the first equation signifies that \( (x, y) \) is on the conic \( g = 0 \) and the second that the gradient \( \nabla f \) is parallel to \( (u - x, v - y) \).

**Remark 7.** Solving this system by brute force is tedious and will teach us nothing about the general problem. We will use more advanced tools to help us solve this and many other problems. In particular we will make use the hyperdeterminant and Schlafli decomposition for computing it.

1.6 Hyperdeterminant of a tensor

The hyperdeterminant of a tensor is a generalisation of the ordinary determinant of a matrix. We will be defining it through the following analytic property described in [4].

**Definition 12.** Let \( 1 \leq i \leq n \) and \( x^{(i)} = (x_0^{(i)}, \ldots, x_{k_i}^{(i)}) \in \mathbb{C}^{k_i+1} \) then every tensor/multilinear form

\[
f : \mathbb{C}^{k_1+1} \times \ldots \times \mathbb{C}^{k_n+1} \to \mathbb{C}
\]

can be represented as a hypermatrix \((f_{i_1,\ldots,i_n})\). We define \( x = (x^{(1)}, \ldots, x^{(n)}) \) to be non-trivial if and only if each \( x^{(i)} \neq 0 \) is a non-zero vector. Then the multilinear form corresponding to the hypermatrix is

\[
f(x) = \sum_{i_1,\ldots,i_n} f_{i_1,\ldots,i_n} x^{(1)}_{i_1} \cdots x^{(n)}_{i_n}
\]

(1.47)
The Hyperdeterminant of format \((k_1 + 1) \times \ldots \times (k_n + 1)\) is, if it exists, a polynomial in \(\mathbb{Z}[i_1, \ldots, i_n]\) that is irreducible over \(\mathbb{Z}\) such that

\[
\text{Hyperdet}_{(k_1 + 1) \times \ldots \times (k_n + 1)}(i_1, \ldots, i_n) = 0 \iff f(x) \text{ has a non-trivial multiple root } x \quad (1.48)
\]

Remark 8. The existence and uniqueness of the hyperdeterminant is quite remarkable. Existence and uniqueness is defined in [4].

Proposition 8. The hyperdeterminant of format \((k_1 + 1) \times \ldots \times (k_n + 1)\) exists if and only if

\[
\forall j \quad k_j \leq \sum_{i \neq j} k_i
\]

Proof. See chapter 14 of [4].

Proposition 9 (Schläfl). The hyperdeterminant of format \(2 \times 3 \times 3\) is given by the formula

\[
\text{Hyperdet}_{2 \times 3 \times 3} = \text{Disc}(\det(M + tN)) \quad (1.49)
\]

Where the discriminant is the usual 1-variable discriminant w.r.t. \(t\).

Proof. See Chapter 14 of [4].
Chapter 2

Main Results

In this section we will characterize the EDD of all irreducible conics \( f = 0 \). As far as we know this was originally proven by Cayley [5], but we could not find an exhaustive proof.

2.1 Matrix equations for points of tangency

Recall that an arbitrary conic is given by the matrix product

\[
X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad f(x, y) = X^T M_f X
\]

where \( M_f \) is the matrix corresponding to \( f \) as defined in Section 1.2.

2.1.1 The system defining a regular point of tangency

Here we prove a few results that will allow us to write down the system defining a regular point of tangency between a circle and an arbitrary irreducible conic using matrices.

**Proposition 10.** A point \((x_0, y_0)\) is singular on the variety \( f = X^T M_f X \) iff it satisfies the equation

\[
M_f X = 0
\]

**Proof.** This is a special case of proposition 19 and 18.

A point is critical iff the following system is solved

\[
\begin{align*}
\frac{\partial f}{\partial x}(x_0, y_0) &= 0 \\
\frac{\partial f}{\partial y}(x_0, y_0) &= 0 \\
f(x_0, y_0) &= 0
\end{align*}
\]

\[
\iff \begin{cases}
2a_{20}x_0 + a_{11}y_0 + a_{10} = 0 \\
a_{11}x_0 + 2a_{02}y_0 + a_{01} = 0 \\
a_{20}x_0^2 + a_{11}x_0y_0 + a_{02}y_0^2 + a_{10}x_0 + a_{01}y_0 + a_{00} = 0
\end{cases}
\]

We now deduce from the lower two equations that

\[
a_{11}x_0y_0 = a_{11}x_0 \frac{y_0}{2} + a_{11}y_0 \frac{x_0}{2} = -(a_{01} + 2a_{02}y_0) \frac{y_0}{2} - (a_{10} + 2a_{20}x_0) \frac{x_0}{2}
\]

We now deduce from the lower two equations that

\[
\frac{{\partial f}}{{\partial x}}(x_0, y_0) = 0 \iff \begin{cases}
2a_{20}x_0 + a_{11}y_0 + a_{10} = 0 \\
a_{11}x_0 + 2a_{02}y_0 + a_{01} = 0 \\
a_{20}x_0^2 + a_{11}x_0y_0 + a_{02}y_0^2 + a_{10}x_0 + a_{01}y_0 + a_{00} = 0
\end{cases}
\]

Plugging this into the bottom equation we get

\[
\begin{cases}
2a_{20}x_0 + a_{11}y_0 + a_{10} = 0 \\
a_{11}x_0 + 2a_{02}y_0 + a_{01} = 0 \\
a_{20}x_0^2 + a_{11}x_0y_0 + a_{02}y_0^2 = 0
\end{cases}
\]

This is a system of linear equations in \( x_0, y_0 \) so we can rewrite this system with \( X = \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} \) as

\[
M_f X = 0
\]
Proposition 11. A regular point \((x_0, y_0)\) is a point of tangency between conics \(f_1\) and \(f_2\) if and only if for some some \((s_1, s_2)\)
\[
(s_1M_{f_1} + s_2M_{f_2})\mathcal{X} = 0
\]  
(2.7)
and \(f_1(x_0, y_0) = 0\) or \(f_2(x_0, y_0) = 0\).

Proof. This is also a special case of proposition 19 and 18. For algebraic sets defined by one equation, such as conics, being a point of tangency means that the point lies on both sets and that
the gradients are linearly dependent. If we let \(a_{ij}\) be the coefficients of \(f_1\) and \(b_{ij}\) for \(f_2\) then
\[
\begin{cases}
s_1\nabla f_1 + s_2\nabla f_2 = 0 \\
f_1 = 0 \\
f_2 = 0 
\end{cases}
\leftrightarrow
\begin{cases}
s_1(2a_{20}x_0 + a_{11}y_0 + a_{10}) + s_2(2b_{20}x_0 + b_{11}y_0 + b_{10}) = 0 \\
s_1(2a_{02}y_0 + a_{11}x_0 + a_{01}) + s_2(2b_{02}y_0 + b_{11}x_0 + b_{01}) = 0 \\
a_{20}x_0^2 + a_{11}x_0y_0 + a_{02}y_0^2 + a_{10}x_0 + a_{01}y_0 + a_{00} = 0 \\
b_{20}x_0^2 + b_{11}x_0y_0 + b_{02}y_0^2 + b_{10}x_0 + b_{01}y_0 + b_{00} = 0 
\end{cases}
(2.8)
\]
Since the point by assumption is regular on both varieties \(s_1, s_2 \neq 0\). Notice that
\[
\begin{cases}
(s_1a_{11} + s_2b_{11})y_0 = -[(s_1a_{20} + s_2b_{20})x_0 + (s_1a_{10} + s_2b_{10})] \\
(s_1a_{11} + s_2b_{11})x_0 = -[(s_1a_{02} + s_2b_{02})y_0 + (a_{01} + b_{01})] 
\end{cases}
\Rightarrow
(2.9)
\]
\[
(s_1a_{11} + s_2b_{11})x_0y_0 = -\frac{1}{2}[(s_1a_{20} + s_2b_{20})x_0^2 + (s_1a_{10} + s_2b_{10})x_0 + (s_1a_{02} + s_2b_{02})y_0^2 + (s_1a_{01} + s_2b_{01})y_0] 
\]  
(2.10)
Plug this into the equation system 2.8 to obtain
\[
\begin{cases}
(s_1a_{20} + s_2b_{20})x_0 + (s_1a_{11} + s_2b_{11})y_0 + (s_1a_{10} + s_2b_{10}) = 0 \\
(s_1a_{02} + s_2b_{02})y_0 + (s_1a_{11} + s_2b_{11})x_0 + (s_1a_{01} + s_2b_{01}) = 0 \\
\frac{1}{2}(s_1a_{10} + s_2b_{10})x_0 + \frac{1}{2}(s_1a_{01} + s_2b_{01})y_0 + (s_1a_{00} + s_2b_{00}) = 0 \\
b_{20}x_0^2 + b_{11}x_0y_0 + b_{02}y_0^2 + b_{10}x_0 + b_{01}y_0 + b_{00} = 0 
\end{cases}
(2.11)
\]
This can be reformulated as the system
\[
\begin{cases}
(s_1M_{f_1} + s_2M_{f_2})\mathcal{X} = 0 \\
f_2 = 0 
\end{cases}
(2.12)
\]
Similarly \(f_1\) can be swapped for \(f_2\). \(\square\)

Proposition 12. Let \(M_1\) be a real symmetric non-singular \(3 \times 3\) matrix. Moreover let
\[
M_2 = \begin{bmatrix}
1 & 0 & -u \\
0 & 1 & -v \\
-u & -v & u^2 + v^2 - r^2
\end{bmatrix}
\]
With generic \((u, v) \in \mathbb{C}^2\) and \(r^2 \in \mathbb{C}\). Let
\[
\mathcal{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \neq 0
\]
and
\[
f_1 = \mathcal{X}^T M_1 \mathcal{X}
\]
Given $M_1, M_2$, a regular point of tangency is a point $(x, y)$, solving the system

$$
\exists s \text{ s.t. } \begin{cases} 
X^T M_1 X = 0 \\
X^T M_2 X = 0 \\
(s_1 M_1 + s_2 M_2) X = 0
\end{cases} \quad (T)
$$

Proof. Here the first two equations imply that $(x, y)$ is a point on both the conics. The third equation says that the gradients are linearly dependent by 11. The more general system that defines a regular point of tangency is actually

$$
\begin{cases} 
X^T M_1 X = 0 \\
X^T M_2 X = 0 \\
(s_1 M_1 + s_2 M_2) X = 0 \\
M_1 X \neq 0 \\
M_2 X \neq 0
\end{cases} \quad (2.14)
$$

The two last equations are equivalent to the point $X$ not being singular by proposition 10. These two last equations are redundant because we are studying an irreducible conic which are non-singular by corollary 1. Also the gradient of $f_2$ only vanishes at $(u, v)$ but that point will never be a common point because it is chosen to be generic.\qed

Note that we allow the solutions to these systems to be complex.

2.1.2 The system related to the hyperdeterminant

As defined in Section 1.6 the hyperdeterminant vanishes when there exists a multiple root of the corresponding multilinear form. Let us get familiar with multilinear forms and the relation to its partial derivatives in order to write down a compact system of equations that corresponds to this.

Proposition 13. A multilinear form $F(x^{(1)}, \ldots, x^{(n)})$, or in short $F(x)$, is 0 whenever all of its partial derivatives w.r.t. one of the vector arguments are 0.

Proof. Every multilinear form is a polynomial in the entries of its vector arguments. Moreover each monomial in this polynomial contains exactly one factor from each vector argument. It is then clear that

$$
\frac{\partial F}{\partial x^{(j)}_i} = F(x^{(1)}, \ldots, x^{(j-1)}, e_i, x^{(j+1)}, \ldots, x^{(n)})
$$

where $e_i$ is the $i$'th unit vector in $\mathbb{C}^{k+1}$. Now $F$ is linear in this argument so

$$
\forall j \leq n \quad F(x^{(1)}, \ldots, x^{(n)}) = \sum_i \frac{\partial F}{\partial x^{(j)}_i} e_i^{(j)}
$$

So clearly if for some $j$, $\frac{\partial F}{\partial x^{(j)}_i} = 0$ then $F = 0$.\qed

Definition 13. We define the $j$'th partial gradient $\nabla_j F$ of a multilinear form $F(x^{(1)}, \ldots, x^{(n)})$ to be the vector of derivatives w.r.t. the components of the $j$'th vector argument $x^{(j)}$.

Proposition 14. The $j$'th partial gradient of a multilinear form at a point $x$ is given by

$$
\nabla_j F = F(x^{(1)}, \ldots, x^{(j-1)}, \_\_\_, x^{(j+1)}, \ldots, x^{(n)})
$$

where $\nabla_j F$ is seen as a point in the dual space of $\mathbb{C}^{k+1}$.

Proof. For every fix set of $x^{(m)}$, $m \neq j$,

$$
F(x^{(1)}, \ldots, x^{(j-1)}, \_\_\_, x^{(j+1)}, \ldots, x^{(n)})
$$

is a linear form on $\mathbb{C}^{k+1}$. By equation 2.15 its values at the standard basis is the partial derivative of $F$ w.r.t. $x^{(j)}$. This is a point in the dual $(\mathbb{C}^{k+1})^*$ that acts on $\mathbb{C}^{k+1}$ identically to the partial gradient. They are therefore equal.\qed

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Example 2. We shall study a small example of proposition 14. Let \( n = 2 \) and \( k_1 = k_2 = 1 \) such that
\[
F(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = f_{11}x_1y_1 + f_{12}x_1y_2 + f_{21}x_2y_1 + f_{22}x_2y_2 = [x_1 \quad x_2] \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]
then by definition
\[
\nabla_x F = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \end{bmatrix} = \begin{bmatrix} f_{11}y_1 + f_{12}y_2 \\ f_{21}y_1 + f_{22}y_2 \end{bmatrix},
\]
\[
\nabla_y F = \begin{bmatrix} \frac{\partial F}{\partial y_1} \\ \frac{\partial F}{\partial y_2} \end{bmatrix} = \begin{bmatrix} f_{11}x_1 + f_{21}x_2 \\ f_{12}x_1 + f_{22}x_2 \end{bmatrix} = \begin{bmatrix} (x_1 \quad x_2) \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \end{bmatrix} = F(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ldots)
\]
(2.19)

Proposition 15. Let \( H = [M_1, M_2] \) be a \( 2 \times 3 \times 3 \) hypermatrix with \( M_1, M_2 \) symmetric. Consider for \( \bar{\tau} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \),
\[
H(s, \bar{\tau}, \bar{\tau}) = s_1\bar{\tau}^T M_1 \bar{\tau} + s_2\bar{\tau}^T M_2 \bar{\tau}
\]
(2.20)

\( \text{Hyperdet}(H) = 0 \) if and only if the following system has at least one complex solution
\[
\begin{align*}
(\bar{\tau}^T M_1 \bar{\tau}) = 0 \\
(\bar{\tau}^T M_2 \bar{\tau}) = 0 \\
(s_1M_1 + s_2M_2) \bar{\tau} = 0 \\
(\bar{\tau}^T M_1 + s_2M_2) \bar{\tau} = 0
\end{align*}
\]
(2.21)

where
\[
\bar{\tau} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq 0 \quad \bar{\tau}' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \neq 0 \quad s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \neq 0
\]

Proof. Proposition 13 implies that \( H(s, \bar{\tau}, \bar{\tau}') = 0 \) is redundant in the definition of the hyperdeterminant. The two first matrix equations in (\#) are simply the partial derivatives w.r.t. \( s_1, s_2 \) which is easily seen looking at eq. 2.20. The next two equations are given by proposition 14. These are the partial gradients, obtained by not inserting any argument for the variables we are differentiating with respect to. The last equation is a bit tricky, what we actually get as an equation for the partial gradient is
\[
\bar{\tau}'^T (s_1M_1 + s_2M_2) = 0
\]
but by assumption \( M_1, M_2 \) were symmetric so we may transpose the equation and we are done. \( \square \)

2.2 Points of tangency via hyperdeterminant

If the system (\( T \)) from 2.13 has a solution \( (s, \bar{\tau}) \) then \( (s, \bar{\tau}, \bar{\tau}) \) is clearly a solution to (\#) from equation 2.21.

This tells us that using the hyperdeterminant when searching for the existence of points of tangency will never give a false negative. However it might give a false positive. We shall now show that this never happens.

Theorem 1. Let \( M_1 \) be a real non-singular symmetric matrix and
\[
M_2 = \begin{bmatrix}
1 & 0 & -u \\
0 & 1 & -v \\
-u & -v & u^2 + v^2 - r^2
\end{bmatrix}
\]

With generic \( u, v \in \mathbb{R} \) and \( r^2 \in \mathbb{C} \).

Then the hyperdeterminant of the hypermatrix \( H = [M_1, M_2] \) is 0 if and only if there exists a regular point of tangency for the algebraic sets that correspond to \( M_1 \) and \( M_2 \).
Proof. By proposition 15 and proposition 12 the statement is equivalent to showing that there exists a solution to the system (T) iff there exists a solution to (\#). Equation (T) describes regular points of tangency between conics.

One direction is trivial, namely (T) gives a solution to (\#). We prove that a solution \((s, \bar{x}, x)\) to (\#) produces a solution to (T). These solutions can be complex.

We divide the proof into three cases defined by the dimension of the kernel of \(M := (s_1M_1 + s_2M_2)\) for fix solution to (\#). Note that if the kernel has dimension 0 there exist no solutions to (\#).

**Case 1: dim ker \(M = 1\)**

In this scenario it is clear that \(\bar{x}, \bar{x}'\) both non-zero then \(\exists \lambda \in \mathbb{C} \setminus \{0\}\) such that \(\bar{x}' = \lambda \bar{x}\). Therefore can therefore w.l.o.g assume \(\bar{x} = \bar{x}'\). The only complication is if \(\bar{x}\) is not of the desired form with a 1 as the third argument. We must do a proof by contradiction to ensure this never happens.

Assume that \(ker(M) = \text{Span} \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} \neq 0\). This also implies \(x_0, y_0 \neq 0\).

Then \(x\) must solve the equation

\[
\bar{x}^T M_2 \bar{x} = [x_0 \ y_0 \ 0] \begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & -v \\ -u & -v & u^2 + v^2 - r^2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} = x_0^2 + y_0^2 = 0 \tag{2.22}
\]

This results in the necessary criterion

\[
y_0 = \pm ix_0 \tag{2.23}
\]

Because we allow complex solutions at this stage. It must also solve the equation

\[
\bar{x}^T M_1 \bar{x} = [x_0 \ y_0 \ 0] \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} = ax_0^2 + 2bx_0y_0 + dy_0^2 = 0 \tag{2.24}
\]

then by eq. 2.23 we have that

\[
x_0^2(a - d \pm i2b) = 0 \tag{2.25}
\]

Since \(a, b, c, d, e, f \in \mathbb{R}\) and \(x_0 \neq 0\) we conclude that then

\[
M_1 = \begin{bmatrix} a & 0 & c \\ 0 & a & e \\ c & e & f \end{bmatrix} \tag{2.26}
\]

Here \(a \neq 0\) or \(M_1\) would be degenerate. We can also w.l.o.g. \(a = 1\) since the algebraic set remains the same when we divide by \(a\). The gradient is only multiplied by a non-zero scalar.

Now study the equation

\[
(s_1M_1 + s_2M_2)x = \begin{bmatrix} s_1 + s_2 & 0 & s_1c - s_2u \\ 0 & s_1 + s_2 & s_1e - s_2v \\ s_1c - s_2u & s_1e - s_2v & s_1f + s_2(u^2 + v^2 - r^2) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} = 0 \tag{2.27}
\]

by eq. 2.23 and \(x_0 \neq 0\) we have that

\[
\begin{cases}
  s_1 + s_2 = 0 \\
  (c + u) \pm i(e + v) = 0
\end{cases} \tag{2.28}
\]
Thus by \( c, e, u, v \in \mathbb{R} \) we must have that

\[
M_1 = \begin{bmatrix}
1 & 0 & -u \\
0 & 1 & -v \\
-u & -v & f
\end{bmatrix}
\]

(2.29)

and thus we can let \( s_1 = 1, \ s_2 = -1 \) and obtain

\[
M = M_1 - M_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & f - (u^2 + v^2 - r^2)
\end{bmatrix}
\]

(2.30)

But then \( M \) has a kernel of at least dimension 2. This is a contradiction to our very first assumption in this case. □

Every solution of (\( \# \)) is a valid solution of (\( T \)) in this case.

**Case 2:** \( \text{dim ker } M = 2 \)

Now given a solution \( \mathbf{x}, \mathbf{x}' \) to (\( \# \)), \( \mathbf{x} \) and \( \mathbf{x}' \) lie in the kernel of \( M \).

**Subcase 1)** \( \mathbf{x}, \mathbf{x}' \) linearly independent

Now \( \ker(M) \) is spanned by \( \mathbf{x}, \mathbf{x}' \) and we let

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix}
x'_1 \\
x'_2 \\
x'_3
\end{bmatrix}
\]

(2.31)

**Subsubcase a:** \( x_3 = x'_3 = 0 \)

This means that kernel of \( M \) must be the \( x_1, x_2 \) plane. Therefore \( M \), being a symmetric matrix must be of the form

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & a
\end{bmatrix}, \quad a \neq 0
\]

(2.32)

Then by the formula \( M = s_1 M_1 + s_2 M_2 \), we obtain that \( M_1 \) is a matrix of the form

\[
\frac{1}{s_1} \begin{bmatrix}
-s_2 & 0 & s_2 u \\
0 & -s_2 & s_2 v \\
s_2 u & s_2 v & -s_2(u^2 + v^2 - r^2) - a
\end{bmatrix}
\]

(2.33)

This means that \( M_1 \) will give rise to the exact same algebraic set as the matrix

\[
M'_1 = \begin{bmatrix}
1 & 0 & -u \\
0 & 1 & -v \\
-u & -v & u^2 + v^2 - r^2 + \frac{a}{s_1^2}
\end{bmatrix}
\]

(2.34)

This is a circle centered at \((u, v)\). This means that \((u, v)\) is not a generic point w.r.t. the variety defined by \( M_1 \).

**Subsubcase b:** w.l.o.g \( x_3 \neq 0 \)

This means that there is a vector with \( z = 1 \) in the kernel of \( M \). So the intersection \( \ker(M) \cap \{z = 1\} \) is actually a line parametrised by

\[
\mathcal{X}(t) = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}, \quad \begin{bmatrix}
x'_1 \\
x'_2 \\
x'_3
\end{bmatrix} = \begin{bmatrix}
x'_3 t + \frac{1}{x_3} \\
x'_2 t \\
x'_3
\end{bmatrix}
\]

(2.35)

Then clearly

\[
\mathcal{X}(t) M_1 \mathcal{X}(t) = 0
\]

(2.36)
has a solution \( t_0 \in \mathbb{C} \) by the fundamental theorem of algebra. Note that by construction \( X = X(t_0) \in \ker(M) \). What we have left to verify is that

\[
0 = M X = X^T M X = X^T(s_1 M_1 + s_2 M_2) X = s_1 X^T M_1 X + s_2 X^T M_2 X = s_2 X^T M_2 X \quad (2.37)
\]

We have now verified that \( X \) is a valid solution to \( T \).

**Subcase 2) \( \pi, \pi' \) linearly dependent**

W.l.o.g. we set \( \pi = \pi' \). This is just a repetition of the case where \( \dim(\ker(M)) = 1 \). The only possible complication is if \( \pi \) has a \( 0 \) at the third component. Then however \( M_1 \) is forced to be of the form

\[
M_1 = \begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & -v \\ -u & -v & f \end{bmatrix} \quad (2.38)
\]

and then clearly it corresponds to a circle with center \((u,v)\), so then \((u,v)\) is not generic w.r.t. to this algebraic set.

**Case 3: \( \dim \ker M = 3 \)**

This means that \( M_1 \) and \( M_2 \) are multiples of each other and induce the same algebraic set. This means that \((u,v)\) is not chosen generically w.r.t. the algebraic set \( f_1 = \pi^T M_1 \pi = 0 \)

### 2.3 The EDD of conics

By using the results in the previous section we shall now classify all irreducible conics by their EDD. We will now use Schläfli’s method for computing the hyperdeterminant from proposition 9. We do this by computing the number of circles/spheres around a generic point \((u,v)\) that have a regular point of tangency with the conic. This method for computing the EDD was discussed in section 1.5 proposition 5.

**Proposition 16.** Let

\[
M_1 = \begin{bmatrix} a & c & d \\ c & b & e \\ d & e & f \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & -v \\ -u & -v & (u^2 + v^2 - r^2) \end{bmatrix}
\]

Let

\[
C_1 = f + (a + b)(u^2 + v^2) - av^2 + 2cv + 2du - bu^2 + 2cuv
\]

\[
C_2 = (a + b)f + ab(u^2 + v^2) + 2aev - e^2 + 2bdu - d^2 + c^2 r^2 - c^2(u^2 + v^2) - 2c(ev + dv) \quad (2.40)
\]

\[
D = abf - ac^2 - bde^2 - c^2 f + 2cued
\]

then the hyperdeterminant of \( H = [M_1, M_2] \) is

\[
\begin{align*}
&\quad [(a + b)^2(c^2 - ab)^2 + 4(c^2 - ab)^3]r^6 + \\
&+ 2(a + b)^2(c^2 - ab)C_2 - 2(a + b)C_1(c^2 - ab)^2 + \\
&+ 12(c^2 - ab)^2 C_2 + 4D(a + b)^3 + 18(a + b)(c^2 - ab) D r^5 + \\
&- [(a + b)^2 C_2^2 + (c^2 - ab)^2 C_1^2 - 4(a + b)(c^2 - ab) C_1 C_2 + 12(c^2 - ab) C_2^2] \\
&- 12(a + b)^2 C_1 - 27 D^2 + 18(a + b) C_2 D - 18(c^2 - ab) C_1 D r^4 + \\
&+ [2(c^2 - ab) C_2 C_1^2 - 2(a + b) C_1 C_2^2 + 4C_2^2 + 12 D(a + b) C_2^2 - 18 C_1 C_2 D] r^2 + \\
&+ [C_1^2 C_2^2 - 4 D C_1 C_2]
\end{align*}
\]

**Proof.** We will use Theorem proposition 9 to calculate it. Let us begin by calculating

\[
det(M_1 + t M_2) = \begin{bmatrix} a + t & c & d - ut \\ c & b + t & e - vt \\ d - ut & e - vt & f + [(u^2 + v^2) - r^2] t \end{bmatrix}
\]
Let’s calculate each term separately. The first term is

\[(a+t)(b+t)(f + [(u^2 + v^2) - r^2]t) - (a+t)(e - vt)^2 =

(ab + (a + b)t + t^2)(f + (a^2 + v^2)t - r^2t) - (a + t)(e^2 - 2evt + v^2t^2) =

(abf + (a + b)ft + t^2f) + (ab(u^2 + v^2)t + (a + b)(u^2 + v^2)t^2 +

(u^2 + v^2)t^3) - (abr^2t + (a + b)r^2t^2 + r^2t^3) - (a + t)(e^2 - 2evt + v^2t^2) =

\[(u^2 + v^2) - r^2]t^3 +

[f + (a + b)(u^2 + v^2) - (a + b)r^2 - av^2 + 2evt]t^2 +

[(a + b)f + ab(u^2 + v^2) - abr^2t + 2ae = c^2t - 2ae^2t^2 + av^2t^3)]

Next we have a term of the form:

\[-c^2(f + [(u^2 + v^2) - r^2]t) = [c^2r^2 - c^2(u^2 + v^2)]t - c^2f \]

Then we have a term of the form

\[2c(e - vt)(d - ut) = 2c(ed - (eu + dv)t + uv^2t^2) = [2cuv]t^2 + [-2c(eu + dv)]t + [2ced] \]

Lastly we have a term of the form

\[-(b + t)(d - ut)^2 = -(b + t)(d^2 - 2dut + u^2t^2) = (-bd^2 + 2bdut - bu^2t^3) + (-d^2t + 2dut^2 - u^2t^3) =

\[-u^2t^3 +

[2du - bu^2]t^2 +

[2bduto - d^2t] +

[-bd^2] \]

Now by adding all of these together we obtain:

\[det(M + tN) =

[-r^2]t^3 +

[f + (a + b)(u^2 + v^2) - (a + b)r^2 - av^2 + 2evt + 2du - bu^2 + 2cuv]t^2 +

[(a + b)f + ab(u^2 + v^2) - abr^2t - abu^2v - c^2f + 2ced]

Before we calculate the discriminant let us write the original polynomial \(det(M + tN)\) as

\[det(M + tN) =

[-r^2]t^3 +

[-(a + b)r^2 + C_1]t^2 +

[(c^2 - ab)r^2 + C_2] +

D \]
where

\[ C_1 = f + (a + b)(u^2 + v^2) - av^2 + 2cv + 2du - bu^2 + 2cuv \]
\[ C_2 = (a + b)f + ab(u^2 + v^2) + 2acev - (a^2 + 2bdu - d^2 + c^2)r^2 - c^2(u^2 + v^2) - 2c(cev + dv) \]

\[ D = abf - ac^2 - bd^2 - c^2f + 2ced \]

\[ C_1, C_2, D \] do not depend on \( r \). Note that \( D \) is the determinant of \( M \).

Now the discriminant of a degree 3 polynomial \( At^3 + Bt^2 + Ct + D \) is

\[ B^2C^2 - 4AC^3 - 4B^2D - 27A^2D^2 + 18ABCD \]

This will in turn be viewed as a polynomial in the variable \( r^2 \) since the rest are fixed as the variety \( X \) is fixed when we calculate the EDD. Let us now calculate the terms of the discriminant

\[ B^2C^2 = \left[(a + b)^2r^4 - 2(a + b)C_1r^2 + C_1^2\right]\left[(c^2 - ab)^2r^4 + 2(c^2 - ab)C_2r^2 + C_2^2\right] = \]

\[ B^2C^2 = \]

\[ \left[(a + b)^2(c^2 - ab)^2\right]r^8 + \]

\[ 2(a + b)^2(c^2 - ab)C_2 - 2(a + b)C_1(c^2 - ab)^2\] \[ r^6 + \]

\[ (a + b)^2C_1^2 + (c^2 - ab)^2C_1^2 - 4(a + b)(c^2 - ab)C_1C_2\] \[ r^4 + \]

\[ 2(c^2 - ab)C_2C_1^2 - 2(a + b)C_1C_2^2\] \[ r^2 + \]

\[ C_1^2C_2^2 \]

\[ -4AC^3 = (-4)(-r^2)\left[(c^2 - ab)^3r^6 + 3(c^2 - ab)^2C_2r^4 + 3(c^2 - ab)C_2^2r^2 + C_2^3\right] = \]

\[ 4(c^2 - ab)^3r^8 + \]

\[ 12(c^2 - ab)^2C_2^2\] \[ r^6 + \]

\[ 12(c^2 - ab)C_2^3\] \[ r^4 + \]

\[ 4C_2^4\] \[ r^2 + \]

\[ [0] \]

\[ -4B^3D = (-4)D(-1)\left[(a + b)^3r^6 - 3(a + b)^2C_1r^4 + 3(a + b)C_1^2r^2 - C_1^3\right] = \]

\[ 0\] \[ r^8 + \]

\[ 4D(a + b)^3\] \[ r^6 + \]

\[ -12D(a + b)^2C_1\] \[ r^4 + \]

\[ 12D(a + b)C_1^2\] \[ r^2 + \]

\[ -4DC_1^3 \]

The term \(-27A^2D^2\) is simply expressed as \(-27D^2r^4\)

\[ 18ABCD = 18(-r^2)(-1)\left[(a + b)r^2 - C_1\right]\left[(c^2 - ab)r^2 + C_2\right]D = \]

\[ 0\] \[ r^8 + \]

\[ 18(a + b)(c^2 - ab)D\] \[ r^6 + \]

\[ 18(a + b)C_2D - 18(c^2 - ab)C_1D\] \[ r^4 + \]

\[ -18C_1C_2D\] \[ r^2 + \]

\[ [0] \]
Now we add them together to obtain the discriminant \( \text{DISC} \)

\[
\text{Hyperdet}([M_1, M_2]) = \text{Disc}(\det(M_1 + tM_2)) = \\
[(a + b)^2(c^2 - ab)^2 + 4(c^2 - ab)^3]r^8 + \\
[2(a + b)^2(c^2 - ab)C_2 - 2(a + b)C_1(c^2 - ab)^2 + \\
12(c^2 - ab)^2C_2^2 + 4D(a + b)^3 + 18(a + b)(c^2 - ab)D]r^6 + \\
[(a + b)^2C_2^2 + (c^2 - ab)^2C_1^2 - 4(a + b)(c^2 - ab)C_1C_2 + 12(c^2 - ab)C_2^2 \\
- 12D(a + b)^2C_1 - 27D^2 + 18(a + b)C_2D - 18(c^2 - ab)C_1D]r^4 + \\
[2(c^2 - ab)C_2C_1^2 - 2(a + b)C_1C_2^2 + 4C_2^4 + 12D(a + b)C_1^2 - 18C_2D]r^2 + \\
[C_1^4C_2^2 - 4DC_1^3]
\] (2.53)

**Lemma 3.** Fix some real symmetric non-singular matrix \( M_2 \) and let \((u, v) \in \mathbb{C}^2\) be generic. Let \( M_2 \) be as in theorem 1. Then there is a one-to-one correspondence of roots, w.r.t. \( r^2 \), of the hyperdeterminant calculated in proposition 16 and points of tangency between the two varieties given by \( M_1, M_2 \).

**Proof.** By theorem 1, the hyperdeterminant is 0 if and only if there is a regular point of tangency for a given value of \( r^2 \) when \((u, v)\) is chosen generically. However if for one value of \( r^2 \) there exist two points of tangency we could not count the number of points of tangency. We must show that for generic \((u, v)\) the values of \( r^2 \) for which the hyperdeterminant vanishes, there is only one point of tangency and that all roots w.r.t. \( r^2 \) of the hyperdeterminant are distinct.

Fix the matrix \( M_1 \) and let \( r^2 = R \). Then

\[
H(R) := \text{Hyperdet}([M_1, M_2]) = A_4R^4 + A_3(u, v)R^3 + A_2(u, v)R^2 + A_1(u, v)R + A_0(u, v) \in \mathbb{R}[u, v][R]
\] (2.54)

Notice that \( A_4 \) never depends on the choice of \((u, v)\). The familiar one-variable discriminant is a polynomial in the coefficients of the original polynomial that tells us if there exists a double root or not. We use this to obtain that

\[
\text{Disc}(H(R))(u, v) = 0 \in \mathbb{R} \iff H(R) \text{ has a multiple root w.r.t } R
\] (2.55)

We verify in section A.2 that for irreducible conics \( \text{Disc}(H(R)) \) does indeed depend on \((u, v)\) and therefore all the roots are distinct for generic \((u, v)\).

The biduality theorem from [4] implies that if there are two multiple roots (distinct in \( x_i\mathbb{P}^4_{\mathbb{C}} \)) of a multilinear form, the corresponding hypermatrix \( H \) is a singular point of hyperdeterminant. However, we just proved that one of the partial derivatives of the hyperdeterminant does not vanish and therefore the points we are interested in are not singular. We conclude that there is a one-to-one correspondence between points of tangency and roots of the hyperdeterminant.

**Theorem 2.** Given an irreducible conic, the EDD is

- 2 if it is a circle
- 3 if it is a parabola
- 4 otherwise

**Proof.** We will use Theorem 1 along with proposition 16 and lemma 3. That is because as mentioned in proposition 5 the EDD w.r.t. a point \( u \) can be calculated by finding tangent solutions with complex circles/spheres centered at \( u \). We simply use the hyperdeterminant in proposition 16 and look at the degree of the hyperdeterminant as a polynomial over the variable \( r^2 \). Every value of \( r^2 \) that makes the hyperdeterminant vanish will give us exactly one point of tangency that in turn contributes to the EDD.
A parabola is defined by \( c^2 - ab = 0 \), and for a circle it is \( a = b, c = 0 \). Study the coefficient of \( r^3 \).

\[
(a + b)^2(c^2 - ab)^2 + 4(c^2 - ab)^3 = (c^2 - ab)^2[(a + b)^2 + 4(c^2 - ab)] = (c^2 - ab)^2[(a - b)^2 + 4c^2]
\]

With our assumption that all constants are real it is clear that this coefficient only disappears for parabolas and circles. However the possibility of a generic conic to be one of these is 0. A generic conic in two variables has 4 values of \( r^2 \) such that the hyperdeterminant vanishes corresponding to points of tangency.

Now what degree is the discriminant for parabolas and circles respectively. Assume that \( c^2 - ab = 0 \) (a parabola). Then we have that the coefficient for \( r^6 \) is:

\[
4D(a + b)^3
\]

This is never 0 because we assume \( D \neq 0 \) is necessary for the conic to be irreducible. Clearly \( (a + b) = 0 \) would also imply \( c^2 + a^2 = 0 \) and this is not possible. We conclude that for parabolas the coefficient of \( r^6 \neq 0 \), always.

For circles we have that \( a = b, c = 0 \) and we obtain that the coefficient of \( r^6 \) is:

\[
2(4a^3)(-a^3)C_2 - 2(2a)C_1a^4 + 12a^4C_2 + 32Da^3 + 36a(-a^2)D = \\
-8a^4C_2 - 4C_1a^5 + 12C_2a^4 + 32Da^3 - 36Da^3 = \\
a^3(-4C_1a^2 + 4C_2a - 4D)
\]

And now we just plug \( C_1, C_2, D \) into the formula. We start by simplifying them under the assumption that we have a circle \( (a=b, c=0) \):

\[
C_1 = f + a(u^2 + v^2) + 2ev + 2du \\
C_2 = 2af + a^2(u^2 + v^2) + 2aev - e^2 + 2adu - d^2 \\
D = a^2f - a(e^2 + d^2)
\]

Notice how in this case \( C_2 = aC_1 + af - (e^2 + d^2) = aC_1 + D/a \)

Plug these into the formula and we obtain that the coefficient for \( r^6 \) is

\[
a^3(-4C_1a^2 + 4(aC_1 + D/a)a - 4D) = a^3 \cdot 0 = 0
\]

So indeed this coefficient disappears for circles. Let’s check the coefficients for \( r^4 \) instead. This will be non-zero. Let’s plug in \( a = b, c = 0 \) into the expression for it

\[
4a^2C_2^2 + a^4C_1^2 + 8a^3C_1C_2 - 12a^2C_2^2 - 48DC_1a^2 - 27D^2 + 36C_2Da + 18C_1Da^2 = \\
a^6C_1^2 + 8a^4C_1C_2 - 8a^3C_2^2 - 48DC_1 - 27D^2 + 36C_2Da + 18C_1Da^2 = \\
a^4C_1^2 + 8a^3C_1(aC_1 + D/a) - 8a^2(aC_1 + D/a)^2 - 48DC_1 - 27D^2 + 36(aC_1 + D/a)Da + 18C_1Da^2 = \\
C_1^2[a^4 + 8a^3 - 8a^4] + C_1D[8a^2 - 16a^2 - 48a^2 + 36a^2 + 18a^2] + D^2[-8 - 27 + 36] = \\
C_1^2 [a^4] + C_1D[-2a^2] + D^2 = (a^2C_1 - D)^2
\]

And this expression is 0 iff \( (a^2C_1 - D) \) is 0. We compute

\[
a^2C_1 - D = a^2[f + a(u^2 + v^2) + 2(ev + du)] - [a^2f - a(e^2 + d^2)] = \\
a^2(u^2 + v^2) + 2a^2(2ev + du) + a(e^2 + d^2)
\]

If this expression is 0 then

\[
(au)^2 + (av)^2 + 2(au)d + 2(av)e + e^2 + d^2 = 0 \implies (au + d)^2 + (av + e)^2 = 0 \implies \\
d = -au \quad e = -av
\]

Meaning that the equation for the circle defined by \( M_1 \) is

\[
aux^2 + av^2 - 2axy - 2avy + f = 0 \implies (x - u)^2 + (y - v)^2 + f/a = 0
\]

This means that \((u, v)\) is at the center of the circle. Intuitively this clearly has a degenerate number of critical points on the circle. This \((u, v)\) is not generic.
Chapter 3

Discussion

We want to study plane curves defined as the zero locus of higher degree polynomials. We hope their EDD can be classified in a similar way to the conics. We first reintroduce the concept of tensors, multilinear forms and present some propositions relating higher degree curves to multilinear forms.

3.1 Higher degree curves

We start off by studying a degree 3 example,

Example 3. Here we introduce a low degree example of the relation between polynomials and their multilinear counterparts so that we may see how to generalize these concepts.

Let $f$ be a degree 3 plane curve with real coefficients.

$$f(x, y) = f_{111}x^3 + f_{222}y^3 + f_{333} + 3f_{112}x^2y + 3f_{122}xy^2 + 3f_{113}x^2z + 3f_{133}xz^2 + 3f_{223}y^2z + 3f_{233}yz^2 + 6f_{123}xyz$$

It can be seen that it is the following homogenous polynomial evaluated at $z = 1$

$$F(x, y, z) = f_{111}x^3 + f_{222}y^3 + f_{333}z^3 + 3f_{112}x^2y + 3f_{122}xy^2 + 3f_{113}x^2z + 3f_{133}xz^2 + 3f_{223}y^2z + 3f_{233}yz^2 + 6f_{123}xyz$$

where $f_{ijk} \in \mathbb{R}$. Notice that this polynomial is generated by a "symmetric" $3 \times 3 \times 3$ tensor $(f_{ijk})$ in the sense that you may permute the indices

$$\forall \sigma \in S_3 \quad f_{ijk3.1} = f_{\sigma(1)\sigma(2)\sigma(3)}$$

and

$$F(x_1, x_2, x_3) = \sum_{ijk} f_{ijk}x_ix_jx_k$$

Now we see that the homogenous polynomial $F$ is just the multilinear form $F$ evaluated along the diagonal where we define $F$ to be

$$F(\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}, \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}, \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix}) := \sum_{ijk} f_{ijk}x_i^{(1)}x_j^{(2)}x_k^{(3)}$$

To end this example we conclude that for any degree 3 plane curve we can write

$$f(x, y) = F(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix})$$

where $F$ is a symmetric multilinear form.
Proposition 17. Any polynomial \( f \in \mathbb{R}[x, y] \) of degree at most \( n \) can be written on the form

\[
f(x, y) = F(\chi, \ldots, \chi) \quad \chi = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

where \( F : (\mathbb{R}^3)^n \to \mathbb{R} \) is a unique symmetric multilinear form of this format if \( f \) is of exactly degree \( n \). We call this \( F \) the multilinear form corresponding to \( f \).

Proof. All we have to do is to construct a symmetric tensor of format \( 3 \times \ldots \times 3 \) that evaluates to \( f \) along the diagonal and when \( z = 1 \). We simply repeat the process of Example 3. Let

\[
f(x, y) = \sum_{a, b \geq 0 \atop a + b \leq n} k_{a,b} x^a y^b \quad (3.5)
\]

Then let \( J = \{1, 2, 3\}^n \subset \mathbb{R}^n \). We let \( i \in J \) and define \( S_n \cdot i \) to be the orbit of \( i \) when we let the symmetric group \( S_n \) act on \( J \) by permuting the indices. For example if \( n = 3 \) then

\[
S_3 \cdot (1, 2, 3) = \{(1, 2, 3), (3, 1, 2), (2, 3, 1), (3, 2, 1), (1, 3, 2), (2, 1, 3)\} \\
S_3 \cdot (1, 1, 2) = \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\} \quad (3.6)
\]

Every such orbit is uniquely defined by the number of 1:s, 2:s and 3:s there are in the elements of the orbit. We define these numbers of 1:s, 2:s and 3:s by \( a(i), b(i), c(i) \) respectively for reasons that will soon be clear. For example

\[
a(1, 1, 2) = 2, \quad b(1, 1, 2) = 1, \quad c(1, 1, 2) = 0 \quad (3.7)
\]

We may denote the set of orbits by \( P \) and denote the orbits by \( S_n \cdot i = p(a(i), b(i), c(i)) \). As an example

\[
p(1, 1, 1) = S_3 \cdot (1, 2, 3) = S_3 \cdot (1, 3, 2) = \ldots = S_3 \cdot (3, 2, 1) \\
p(2, 1, 0) = S_3 \cdot (1, 1, 2) = S_3 \cdot (2, 1, 1) = S_3 \cdot (1, 2, 1) \quad (3.8)
\]

We define the tensor \( f_i \) by

\[
f_i = \frac{k_{a(i), b(i)}}{p(a(i), b(i), 3 - a(i) - b(i))} \quad (3.9)
\]

We constructed \( f_i \) such that the sum over every orbit is exactly

\[
\sum_{i \in p(a, b, c)} f_i = \sum_{i \in J \atop a(i) = a \atop b(i) = b \atop c(i) = 3 - a - b} f_i = k_{a,b} \quad (3.10)
\]

and every term in this sum is equal. By construction this tensor is symmetric. We define for the vectors \( x^{(k)} \in \mathbb{C}^3 \)

\[
F(x^{(1)}, \ldots, x^{(n)}) = \sum_{i \in J} f_i x^{(1)}_{i_1} \cdots x^{(n)}_{i_n} \quad (3.11)
\]

Now we just verify what happens when we evaluate it along the diagonal

\[
\forall k \quad x^{(k)}_{i_k} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cdot \epsilon_{i_k} = \chi_{i_k} \quad (3.12)
\]

\[
F(\chi, \ldots, \chi) = \sum_{i \in J} f_i \chi_{i_1} \cdots \chi_{i_n} = \sum_{p(a,b,c) \in P} \left( \sum_{i \in p(a,b,c)} f_i x^a y^b \chi^c \right) = \sum_{p(a,b,c) \in P} \left( \sum_{i \in p(a,b,c)} f_i x^a y^b \chi^c \right) = \sum_{a,b,c \geq 0 \atop a + b + c = n} k_{a,b} x^a y^b \chi^c = f(x, y)
\]

It is clear that the conditions posed on \( f \) forced \( F \) to be of the following form and that \( f_i \) is the unique tensor that describes \( f \).
Remark 9. Note that the degree 2 polynomial describing a circle around \((u, v)\) can always be written on this form. It corresponds to the following symmetric hypermatrix in \((\mathbb{R}^3)^{\otimes n}\) satisfying
\[
c_{i_1, \ldots, i_n} = \begin{cases} 
    c_{113 \ldots 3} = \frac{1}{(3)} & \text{if } n = 3 \\
    c_{223 \ldots 3} = \frac{1}{(2)} & \text{if } n = 2 \\
    c_{13 \ldots 3} = -2u \frac{1}{(1)} = -2u & \text{if } n = 1 \\
    c_{23 \ldots 3} = -2v \frac{1}{(1)} = -2v & \text{if } n = 1 \\
    c_{3 \ldots 3} = u^2 + v^2 - r^2 & \text{if } n = 0 \\
    0 & \text{otherwise}
\end{cases}
\] (3.13)

Proposition 18. Given a polynomial \(f \in \mathbb{R}[x, y]\) of degree \(n \geq 2\) and its corresponding multilinear form \(F\). Then if \(x = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\) then
\[
\begin{align*}
\frac{\partial f}{\partial x} &= nF(\chi, \ldots, \chi, e_1) \\
\frac{\partial f}{\partial y} &= nF(\chi, \ldots, \chi, e_2)
\end{align*}
\] (3.14)

Proof. Let \(x, y\) depend on the parameter \(t\). Note that we can simplify the derivatives of \(f_1\) and \(f_2\) by using the chain rule
\[
\frac{\partial f}{\partial t} = \frac{\partial F}{\partial t}(\chi, \ldots, \chi) = \sum_{j,k} \frac{\partial F}{\partial x_k^{(j)}} \frac{\partial \chi_k}{\partial t} = \sum_{j=1}^{n} \nabla_j F \cdot \frac{\partial \chi}{\partial t} = \sum_{j=1}^{n} F(\chi, \ldots, \frac{\partial \chi}{\partial t}, \ldots, \chi)
\] (3.15)

We now use that \(F\) is symmetric, so we obtain that
\[
\frac{\partial f}{\partial t} = nF(\chi, \ldots, \chi, \frac{\partial \chi}{\partial t})
\] (3.16)

Choosing \(x(t) = x_0 + t\) or \(y(t) = y_0 + t\) with the other coordinate set to a constant we obtain the result. \(\square\)

Definition 14. Given a multilinear form \(F : (\mathbb{R}^m)^n \to \mathbb{R}, n \geq 2\), we define \(\overline{F}\) to be the multilinear form
\[
\overline{F} : (\mathbb{R}^m)^{(n-1)} \to \mathbb{R}^m
\] (3.17)

\[
\overline{F}(x^{(1)}, \ldots, x^{(n-1)}) = [F(x^{(1)}, \ldots, x^{(n-1)}, e_1) \ldots F(x^{(1)}, \ldots, x^{(n-1)}, e_m)]
\] (3.18)

This is exactly the \(n^{\text{th}}\) partial gradient of \(f\).

To generalise our proof of computing the EDD by looking at the hyperdeterminant we would have to relate tangent solutions between similar systems of equations.

Proposition 19. Let \(f_1(x, y)\), \(f_2(x, y)\) be two polynomials and \(F_1, F_2\) be the corresponding multilinear forms. Then the equations describing a point of tangency between their algebraic sets can be formulated as follows
\[
\exists \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad \text{s.t.} \quad \begin{cases} 
    F_1(\chi, \ldots, \chi) = 0 \\
    F_2(\chi, \ldots, \chi) = 0 \\
    s_1 F_1(\chi, \ldots, \chi) + s_2 F_1(\chi, \ldots, \chi) = 0
\end{cases}
\] (3.19)

And if the following is satisfied as well the point is a regular point of tangency
\[
\overline{F}_1(\chi, \ldots, \chi), \overline{F}_2(\chi, \ldots, \chi) \neq 0
\] (3.20)
Proof. By construction the first two equations directly implies that \( x \) is a point on both varieties. The third equation should be equivalent to the gradients of \( f_1 \) and \( f_2 \) being linearly dependent at \( x \). By proposition 18 we have that

\[
\nabla f_1(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_1}{\partial y} \end{bmatrix} = n \begin{bmatrix} F_1(x, \ldots, x, e_1) \\ F_1(x, \ldots, x, e_2) \end{bmatrix}
\]

(3.21)

Likewise for \( f_2 \). The gradient is extremely similar to \( F \). So the only difference is that we added the equation

\[
s_1 F_1(x, \ldots, x, e_3) + s_2 F_2(x, \ldots, x, e_3) = 0
\]

But this does not add any extra information since by the gradients being linearly dependent we already know that

\[
0 = s_1 F_1(x, \ldots, x, e_1) + s_2 F_2(x, \ldots, x, e_1)
\]

\[
x \frac{\partial f_1}{\partial x} + y \frac{\partial f_1}{\partial y} + F_1(x, \ldots, x, e_3) = 0 + 0 + F_1(x, \ldots, x, e_3)
\]

(3.22)

In the same way the third component of the fourth equation does not add any extra information.

If the gradient of \( f_1 \) is 0 then

\[
0 = f_1(x, y) = F_1(x, \ldots, x) =
\]

\[
0 = \frac{x}{n} \frac{\partial f_1}{\partial x} + \frac{y}{n} \frac{\partial f_1}{\partial y} + F_1(x, \ldots, x, e_3) = 0 + 0 + F_1(x, \ldots, x, e_3)
\]

(3.23)

And likewise for \( f_2 \).

\[
\]

Proposition 20. Let \( F_1, F_2 \) be multilinear forms on \((\mathbb{R}^3)^n\). Then create the multilinear form defined by the tensor symmetric in all except the first index

\[
i \in \{1, 2, 3\}^n \quad f_{(1, i)} = (f_1)_i \quad f_{(2, i)} = (f_2)_i
\]

such that the corresponding linear form is

\[
F(s, x^{(1)}, \ldots, x^{(n)}) = s_1 F_1(x^{(1)}, \ldots, x^{(n)}) + s_2 F_2(x^{(1)}, \ldots, x^{(n)})
\]

(3.25)

The hyperdeterminant of \( f_{(a,i)} \) is 0 iff there is a solution with all vector arguments non-zero to

\[
\begin{cases}
F_1(x^{(1)}, \ldots, x^{(n)}) = 0 \\
F_2(x^{(1)}, \ldots, x^{(n)}) = 0 \\
\forall j \quad s_1 \bar{F}_1(x^{(1)}, \ldots, \hat{x}^{(j)}, \ldots, x^{(n)}) + s_2 \bar{F}_2(x^{(1)}, \ldots, \hat{x}^{(j)}, \ldots, x^{(n)}) = 0
\end{cases}
\]

(3.26)

where \( \hat{x}^{(j)} \) signifies that vector not being included since

Proof. This is just a direct application of the definition of the hyperdeterminant, \( \bar{F} \), and the fact from proposition 14 stating that

\[
\frac{\partial F_1}{\partial x_j^{(i)}} = F_1(x^{(1)}, \ldots, x^{(j)} = e_k, \ldots, x^{(n)})
\]

(3.27)
Appendix A

Code

A.1 Computing EDD

Code for calculating the EDD of an explicit ideal. Here we calculate the EDD over \( \mathbb{Q} \). This example is from [2].

\[
R = \mathbb{Q}[x_1,x_2,x_3];
I = \text{ideal}(x_1^5+x_2^5+x_3^5); u = \{5,7,13\};
sing = I + \text{minors}(\text{codim} I,\text{jacobian}(I));
M = (\text{matrix}[\text{apply} (#\text{ gens} R,i->(\text{gens} R)_i-u_i)])||((\text{transpose}(\text{jacobian} I))); J = \text{saturate}(I + \text{minors}(\text{codim} I)+1,M), \text{sing});
dim J, degree J
\]

A.2 The discriminant of the hyperdeterminant is non-constant w.r.t. \((u,v)\)

Under the assumption that the highest degree term w.r.t. \( r^2 \) of the hyperdeterminant does not vanish, we verify with this Macauley2 code that the discriminant of the hyperdeterminant does depend on \((u,v)\).

```
restart
R = \mathbb{Q}[a,b,c,d,e,f, K1, K2, D, ApB, Par][u,v]
-- Assume Par != 0 and
c0 = ApB^2*Par^2 + 4*Par^3
    +18*ApB*K2*D - 18*Par*K1*D
    -18*K1^2*K2*D

c4 = K1^2*K2^2 - 4*D*K1^3
t1 = 256*c0^3*c4^3 -192*c0^2*c1*c3*c4^2 -128*c0^2*c2^2*c4^2;
t2 = 144*c0^2*c2^2*c3^2*c4 -27*c0^2*c3^2*c4 +144*c0*c1^2*c2*c4^2;
t3 = -6*c0*c1^2*c3^2*c4 +80*c0*c1*c2^2*c3^3 +18*c0^2*c1^2*c2^3;
t4 = 16*c0^2*c2^4*c4^4 -4*c0*c2^3*c4^2 + 18*c1^3*c2^2*c3^2;
t5 = -4*c1^3*c3^3 + 4*c1^2*c2^2*c3^2 + c1^2*c2^2*c3^2;
disc = t1+t2+t3+t4+t5
disc2 = \text{factor}(\text{disc})
f0 = disc2#0#0
f1 = disc2#1#0
f2 = disc2#2#0
f3 = disc2#3#0
```
C1 = a*u^2 + 2*c*u*v + b*v^2 + 2*d*u + 2*e*v + f
C2 = -Par*(u^2 + v^2) + (2*b*d - 2*c*e)*u + (-2*c*d + 2*a*e)*v - d^2 - e^2 + ApB*f
Dd = a*b*d + 2*c*e*d - a*e^2 - b*d^2 - f*c^2

-- only f1 and f2 can be 0

f11 = sub(f1, {K1=>C1})
f11 = sub(f11, {K2=>C2})
f11 = sub(f11, {ApB=>(a+b)})
f11 = sub(f11, {Par=>(c^2 - a*b)})
f11 = sub(f11, {D=>Dd})

factor(coefficient(v^4, f11))
-- we must set c=0
f11 = sub(f11, {c=>0})

factor(coefficient(u^2*v^2, f11))
-- Either a*b = 0 or (a-b)^2=0
-- (a-b)^2 != 0 since otherwise we would have a circle!
-- we must set a=b=0
f11 = sub(f11, {/(a*b)})
-- We must set a*e = b*d = 0
-- Contradiction, these conditions forces the conic to be degenerate! Det(M) = Dd = 0
-- It must be the second factor that evaluates to 0

f22 = sub(f2, {K1=>C1})
f22 = sub(f22, {K2=>C2})
f22 = sub(f22, {ApB=>(a+b)})
f22 = sub(f22, {Par=>(c^2 - a*b)})
f22 = sub(f22, {D=>Dd})

factor(coefficient(u^6, f22))
-- we must set b=0
f22 = sub(f22, {b=>0})

factor(coefficient(u^3*v^3, f22))
-- we must set c=0
-- Contradiction, now (c^2-a*b = 0) and this is assumed not to be a parabola

restart
R = QQ[a,b,c,d,e,f, K1, K2, D, ApB, Par][u,v]
c0 = ApB^2*Par^2 + 4*Par^3

-- Start by assuming parabola, i.e. set Par = 0

c3 = sub(2*Par*K2*K1^2 -2*ApB*K1*K2^2 + 4*K2^3 + 12*D*ApB*K1^2 -18*K1*K2*D, {Par=>0})
c4 = sub(K1^2*K2^2 - 4*D*K1*K2, {Par=>0})

disc = c2^2*c3^2 - 4*c1*c3^3 - 4*c2^3*c4 + 27*c1^2*c4^2 + 18*c1*c2*c3*c4
disc2 = factor(disc) -- it has two distinct prime factors
f1 = disc2#0#0
f2 = disc2#1#0

C1 = a*u^2 + 2*c*u*v + b*v^2 + 2*d*u + 2*e*v + f
C2 = sub(-Par*(u^2 + v^2) + (2*b*d - 2*c*e)*u + (-2*c*d + 2*a*e)*v - d^2 - e^2 + ApB*f,
\[
\begin{align*}
Dd &= a*b*d + 2*c*e*d - a*e^2 - b*d^2 - f*c^2 \\

f11 &= sub(f1, \{K1=\text{C1}\}) \\
f11 &= sub(f11, \{K2=\text{C2}\}) \\
f11 &= sub(f11, \{\text{ApB}=\text{(a+b)}\}) \\
f11 &= sub(f11, \{D=\text{Dd}\}) \\

\text{factor(coefficient}(u^2, f11)) \\
\text{factor(coefficient}(v^2, f11)) \\
\text{factor(coefficient}(u*v, f11)) \\
\quad \text{-- either } a=b=c=0 \text{ or } (a+b)=0 \\
\quad \text{-- it must be } a=-b \\
\quad \text{-- Contradiction: } 0 = c^2 - a*b = c^2 + a^2 \quad \rightarrow \quad a = b = c=0 \\
f22 &= sub(f2, \{K1=\text{C1}\}) \\
f22 &= sub(f22, \{K2=\text{C2}\}) \\
f22 &= sub(f22, \{\text{ApB}=\text{(a+b)}\}) \\
f22 &= sub(f22, \{D=\text{Dd}\}) \\

\text{factor(coefficient}(u^3, f22)) \\
\text{factor(coefficient}(v^3, f22)) \\
\quad \text{-- Either } a+b=0 \text{ or } (b*d - c*e) = (c*d - a*e) = 0 \\
\quad \text{-- The first case is impossible assuming parabola. This implies that } Dd = a*c*e - c^2*f \\
f22 &= sub(f22, R/((b*d - c*e), (c*d - a*e))) \\
\text{factor(coefficient}(u*v, f22)) \\
\quad \text{-- Either } c=0 \text{ or } a*e - c*f = 0 \\
\quad \text{-- It must be the second case, hence } Dd = 0. \text{ Contradiction!}
\end{align*}
\]
f11 = sub(f11, {c=>0})
isPrime(f11)

factor(coefficient(u^2, f11))
factor(coefficient(v^2, f11))
factor(coefficient(u*v, f11))
-- either a=0 or 3*a^3*d = 7*d^2 + 3*e^2 = 3*d^2 + 7*e^2
-- it cannot be a=0
-- This implies d^2 = e^2. Now Dd = a^2*d -a*d^2 - a*d^2 = a^2 = ae^2
f11 = sub(f11, R/(e^2-d^2))
factor(coefficient(u*v, f11))
-- Either a=0, e=0 or d=0, contradiction since Dd = ae^2 = ad^2 != 0

f22 = sub(f2, {K1=>C1})
f22 = sub(f22, {K2=>C2})
f22 = sub(f22, {D=>Dd})
f22 = sub(f22, {b=>a})
f22 = sub(f22, {c=>0})

factor(coefficient(u^6, f22))
-- Either a=0 or (a*d - d^2 - e^2)= 0. Contradiction since Dd = 0 in both cases!
Appendix B

Further theory

B.1 \( \mathcal{V}(I : J^\infty) = (\mathcal{V}(I) - \mathcal{V}(J)) \) for algebraically closed fields

This proof is a modification of a theorem in [6]. In that proof the coefficient field \( k \) needs to be an algebraically closed field and \( I \) a radical ideal. We do not need \( I \) to be radical.

By the closure \( \bar{S} \) of a set \( S \subset k^n \) we mean the Zariski topological closure, which is either the intersection of all closed sets containing \( S \) or one can easily prove that is the same as

\[ \bar{S} = \mathcal{V}(I(S)) \]

It is the affine algebraic set corresponding to the largest ideal of polynomials that vanishes on \( S \).

From now on let \( I, J \subset R = k[x_1, \ldots, x_n] \) be ideals. By Hilbert’s basis theorem \( R \) is Noetherian. We can now define the ideal quotient

\[ (I : J) := \{ r \in R \mid rJ \subset I \} \quad \text{(B.1)} \]

that is a well-defined ideal and from this we can define the saturated quotient ideal

\[ (I : J^\infty) := \bigcup_{m \geq 1} (I : J^m) \quad \text{(B.2)} \]

where \( J^m \) is the ideal of elements in \( R \) generated by products of \( m \) elements of \( J \). It is easy to see that for \( R \) Noetherian \( J^m \) is generated by products of the (finitely many) generators of \( J \) with \( m \) factors. The union of an increasing chain of ideals is again an ideal. The set \( (I : J^\infty) \) such a union since it is clear that \( (I : J^n) \subset (I : J^{n+1}) \).

We begin by proving that \( \mathcal{V}(I : J^\infty) \supset \mathcal{V}(I(\mathcal{V}(I) - \mathcal{V}(J))) \). This is done by proving that \( (I : J^\infty) \subset I(\mathcal{V}(I) - \mathcal{V}(J)) \).

Let \( f \in (I : J^\infty) \) and \( x \in \mathcal{V}(I) - \mathcal{V}(J) \) be arbitrary. By definition of \( (I : J^\infty) \) we have that

\[ \exists n \text{ s.t. } \forall g' \in J^n \quad fg' \in I \quad \text{(B.3)} \]

Now \( x \in \mathcal{V}(I) - \mathcal{V}(J) \implies x \in \mathcal{V}(I) \) and we have that

\[ \forall g' \in J^n \quad f(x)g'(x) = 0 \quad \text{(B.4)} \]

Because \( x \in \mathcal{V}(I) - \mathcal{V}(J) \implies x \notin \mathcal{V}(J) \) means that at least one of the elements, \( g \) of \( J \) is not 0 at \( x \). Now we use that every field is an integral domain and therefore we have that \( g(x) \neq 0 \implies g^n(x) \neq 0 \). Then

\[ f(x)g^n(x) = 0 \implies f(x) = 0 \quad \text{(B.5)} \]

We proved that every \( f \in (I : J^\infty) \) vanishes on \( \mathcal{V}(I) - \mathcal{V}(J) \) i.e.

\[ (I : J^\infty) \subset I(\mathcal{V}(I) - \mathcal{V}(J)) \implies \mathcal{V}(I : J^\infty) \supset \mathcal{V}(I(\mathcal{V}(I) - \mathcal{V}(J))) \quad \text{(B.6)} \]
Now for the other inclusion. We do this directly. Assume \( x \in \mathcal{V}(I : J^\infty) \). Let \( h \in I(\mathcal{V}(I) - \mathcal{V}(J)) \) and \( g \in J \) be arbitrary. Clearly \( hg = 0 \) on \( (\mathcal{V}(I) - \mathcal{V}(J)) \cup \mathcal{V}(J) \). Here we have to use the strong nullstellensatz and use that \( k \) is algebraically closed. So \( hg = 0 \) on \( \mathcal{V}(I) \) now implies that

\[
\forall g \in J \quad hg \in I(\mathcal{V}(I)) = \sqrt{I} \implies hJ \subset \sqrt{I} \quad (B.7)
\]

If we let \( J = (g_1, \ldots, g_s) \) then \( hJ = (hg_1, \ldots, hg_s) \). And we have that for every \( i \)

\[
\exists n_i \text{ s.t. } h^{n_i}g_i^{n_i} \in I \quad (B.8)
\]

Then since there are only finitely many generators of \( J \) there is some \( n_{max} \) among these exponents. So \( h^{n_{max}}g_i^{n_{max}} \in I \) for all indices \( i \). Now let \( g' \) be any generator of \( J^s \cdot n_{max} \). We have \( g' = g_1^{a_1} \cdot \ldots \cdot g_s^{a_s} \) with

\[
\sum_{i=1}^{s} a_i = s \cdot n_{max} \quad a_i \geq 0 \quad (B.9)
\]

So all \( a_i \) cannot be less than \( n_{max} \). Assume \( a_1 \geq n_{max} \). Then

\[
h^{n_{max}}g' = g_1^{a_1-n_{max}} \cdot g_s^{a_s} \cdot (h^{n_{max}}g_i^{n_{max}}) \in I \quad (B.10)
\]

So this means exactly that

\[
h^{n_{max}} \in (I : J^s \cdot n_{max}) \subset (I : J^\infty) \quad (B.11)
\]

Now remember we picked \( x \in \mathcal{V}(I : J^\infty) \). That means that

\[h^{n_{max}}(x) = 0 \implies h(x) = 0\]

We just proved that every polynomial in \( I(\mathcal{V}(I) - \mathcal{V}(J)) \) vanishes on \( \mathcal{V}(I : J^\infty) \) implying

\[
\mathcal{V}(I : J^\infty) \subset \mathcal{V}(I(\mathcal{V}(I) - \mathcal{V}(J))) \quad (B.12)
\]

\[\square\]
Bibliography


