Mean-Variance Portfolio Selection
Accounting for Financial Bubbles

A Mean-Field Type Approach

MARCUS HÄGGBOM

SHAYAN NAFAR
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SHAYAN NAFAR

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Supervisor at Nordea Bank Abp: Henrik Segerberg
Supervisor at KTH: Boualem Djehiche
Examiner at KTH: Boualem Djehiche
Abstract

The phenomenon of financial bubbles is known to have impacted various markets since the seventeenth century. Such bubbles are known to form when the market drastically overvalues the price of an asset, causing its market value to increase hyperbolically, only to suddenly collapse once the untenable perceived future prospects of the asset are realized. Hence, it remains crucial for investors to be able to sell off assets residing within a bubble before they burst and their value is significantly diminished. Thus, portfolio optimization methods capable of accounting for financial bubbles in stock dynamics is a field of great value and interest for market participants. Portfolio optimization with respect to the mean-field is a relatively novel approach to accounting for the bubble-phenomenon. Hence, this paper investigates a previously unattempted method of portfolio optimization, providing a mean-field solution to the mean-variance trade-off problem, as well as providing new definitions of stock dynamics capable of diverting investors from bubbles.

Keywords  Financial bubbles · Portfolio optimization · Mean-variance trade-off · Mean-field type optimal control · Fundamental value · Mean reversion
Portföljoptimering av medelfältstyp med hänsyn till finansiella bubblor

Sammanfattning


Nyckelord  Finansiella bubblor · Portföljoptimering · Optimal kontroll av medelfältstyp · Fundamentalt värde · Medelreversion
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Chapter 1

Introduction

Financial, or economic, bubbles have been a reoccurring phenomenon for centuries. These bubbles occur when the market values an investment instrument, such as a common stock or real estate, higher than what it is actually worth. This ‘actual’ worth has been attempted to be described through what is known as the firm-foundation theory, which argues that all investment instruments have a value that they are firmly tied to. Hence, market prices are claimed to oscillate around this so called fundamental, or intrinsic, value. Consequently, according to the firm-foundation theory, financial bubbles build up during periods where the market price of an asset keeps increasing relative to the fundamental value to which it is firmly anchored. Another theory, called the caste-in-the-air theory, strongly opposes the idea of a fundamental value and takes a more psychological approach to explaining how market prices build up unsustainably only to suddenly collapse. Formulated by the renowned economist John Maynard Keynes, the theory claims that market participants do not tend to estimate fundamental values and instead analyze the crowd psychology of other investors. It is argued that in periods during which the market has an optimistic view on future returns, the investors tend to “build castles in the air”. Nevertheless, castles that are built in the air are bound to collapse, and with it the market’s optimistic speculations on future growth and the price of the asset, exactly like what occurs when a financial bubble bursts.

The first ever recorded example of a financial bubble is believed to be the Dutch tulipmania, also known as the tulip-bulb craze in the early seventeenth century. In the late sixteenth century, a botany professor introduced a set of foreign plants, namely tulips, to Dutch society. Although the Dutch where highly impressed by the plant, the price set by the professor was considered
unacceptably high and did not attract buyers. However, after a burglar managed to steal the bulbs from the professor’s house, the bulbs entered the market at a lower, yet highly profitable price. Over time, the popularity of the tulip-bulb continued to grow steadily, until the emergence of a virus causing color breaking in some flowers. This resulted in the petals of these flowers developing contrasting colors, further adding to their beauty and appeal. Popular taste caused the market to quickly increase the value of these flowers, as people became more willing to pay higher prices in order to own these virus infected plants. Subsequently, a tulipmania grew and the higher the price became, the more people considered purchasing tulip-bulbs an intelligent investment. As the profits were stacking up for those owning the infected bulbs, all the more people were tempted to join in on the frenzy, causing the bulb-prices to reach astronomical levels. However, as prices got extraordinarily high, some decided to begin selling their bulbs and others soon followed, triggering a chain reaction where a sudden change in demand caused the prices of the bulbs to suddenly decrease and move in a direction not seen in a long time. As the price decreased, dealers went bankrupt and trade contracts were not honored, quickly leading to a market-wide panic and the collapse of the tulip-bulb prices until they were not worth more than a common onion.

Another well known example of a financial bubble is the South Sea bubble of the early eighteenth century, whilst other more recent examples of financial bubble collapses are the dot-com bubble (early 2000’s) and the U.S. housing bubble (2007-2008). Historically, bubbles are formed when the market has unsustainable positive views about the future of an asset. Whether these views truly are grossly exaggerated to a point of imminent collapse is not always evident, making financial bubbles difficult to identify in real-time. Therefore, bubbles are often identified once they have collapsed, i.e. when there is a sudden drop or correction in the asset’s price.

Financial bubbles pose a great risk to capital markets operators who have invested in assets that may reside within such a bubble. Hence, being able to assess and manage the risks of potential bubble formations provides an essential advantage to any investor who seeks to shield themselves from sudden damaging losses associated with bubble bursts. To address the issue of incorporating bubble dynamics in portfolio optimization, stochastic optimal control of mean-field type can be used. Nevertheless, the mean-field framework is not commonly used in portfolio optimization, especially not when simultaneously seeking to account for bubble processes in the asset price dynamics. Hence, this paper presents a novel model frame for portfolio optimization in the presence of financial bubbles. Additionally, new mathe-
matical formulations for the incorporation of bubbles in stock dynamics are defined and their feasibility is discussed.

1.1 Project Description

This thesis attempts to solve two problems. The first consists of finding an appropriate model for stock dynamics that accounts for the existence of bubbles. The second problem is to optimize a portfolio given these dynamics. The objective studied in the optimization problem is the well-known mean-variance trade-off problem.

The overall aim is to formulate a novel mean-field approach to the mean variance trade-off problem in portfolio optimization, creating a solution method which allows for parameters to be stochastic. This allows one to find optimal solutions for more general stock dynamics.

In order to limit the scope, the work is on a theoretical level. Neither the optimization method nor the proposed stock models are implemented using historical data.

1.2 Outline

This thesis is divided into six chapters. In Chapter 2 relevant mathematical and economic theory is presented and a number of essential as well as state of the art methods are considered with regard to the inclusion of bubbles in stock dynamics, where the main focus is directed towards the works of R.C. Merton, P. Protter and D. Sornette. Chapter 3 presents the mean-variance problem of portfolio optimization in a mean-field setting and ultimately arrives at a solution. Having solved the mean-variance problem, Chapter 4 proposes a number of new stock models designed to not blindly accept the market price during periods of optimism where stocks become overpriced relative to their fundamental value. Instead, these new models prefer taking positions such that the portfolio remains closer to its intrinsic value rather than to the market value. This effect is attained through the use of mean-reverting processes or risk premium. The models all have in common the adoption of the expected value as a proxy for the fundamental value; a motivation for this proposition is also presented. In Chapter 5 the scope and limitations of the proposed models are analyzed, discussed and possible future improvements are suggested. Chapter 6 finally summarizes and concludes the results arrived at in this thesis.
Chapter 2

Theory

2.1 Stock and Bubble Modeling

In order to describe the behavior of financial bubbles, numerous mathematical models have been proposed, and depending on the author’s view on how financial markets operate and react to price changes, various approaches have been attempted. In this section, a number of methods for modeling economic bubbles are presented.

We begin with more classical work regarding general stock modeling, by R.C. Merton and others and then move on to more contemporary, state of the art methods specific to bubbles, proposed by P. Protter and D. Sornette.

2.1.1 Stock Models

The idea of modeling stocks by using a random walk dates back to the beginning of the 20th century. More specifically, in his Ph.D. dissertation in 1900, Louis Bachelier [3] used a Brownian motion to model stock prices. It was later refined by Osborne [31] into the predominant Geometric Brownian Motion (GBM), on which much of contemporary financial theory is built. However, the GBM as a stock model is not without critique. Even before the pioneering work of Black and Scholes [7] in the early 1970’s on option pricing, which was built on the assumption that the underlying stocks were log-normal [1], some research had pointed out flaws with the normality prop-

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[1] It should be pointed out that Merton [29] soon after expanded the theory to allow for weaker assumptions on the stock dynamics.
Since the empirical distributions of price returns seemed leptokurtic (having a more peak-like bulk and longer or heavier tails than a normal distribution), Mandelbrot [25] rejected the normal distribution, suggesting instead that the returns could be modeled by a family of probability laws called stable Paretian. This is a general family of laws, including Gaussian (Mandelbrot was interested in the non-Gaussian cases), satisfying the property

\[ aU_1 + bU_2 \overset{d}{=} cU, \]

where \( a, b, c > 0 \), \( U_1 \) and \( U_2 \) are independent copies of \( U \), and \( \overset{d}{=} \) denotes equality in distribution. The normal distribution is the only distribution with finite second moment that satisfies this property. The Cauchy and Lévy distributions also have this property; other distributions lack analytic density functions and are solely identified by their characteristic functions. The reader is referred to the original source [25] for more details, or to the work of Fama for an overview [14, 16] and application in portfolio analysis [15]. In essence, the appeal of the stable Paretian is that the property extends to a sum of several independent copies, allowing for invariance under addition. It can be seen, Fama [16] explains, as a generalization of the method of using the normal distribution, allowing for infinite second moments.

Whether the empirical distributions actually have infinite second moments is not de facto. On the contrary, Godfrey et al. [18] found at the time no empirical support of Mandelbrot’s and Fama’s hypothesis. In more recent years, [20] comes to the conclusion that stock returns have finite first and second moments, but infinite higher moments, while [34] presents empirical evidence supporting finite-variance models over stable Pareto. The latter paper lifts out two models that have particularly good descriptive powers. The first one is a mixture of normal distributions, suggested by Kon [22]. The model proposes that the returns are randomly drawn with some probability from one of \( N \) normal distributions, each distribution having an individual variance. A less general version of this is a mixture of two normal distributions used by Christie [10], where one has higher variance, encapsulating ‘information events’ (such as particularly good/bad news), and the other has lower variance and represents ‘non-information events’.

The second model which [34] described as having good descriptive powers is the mixed jump-diffusion model proposed by Merton [30].

\[ ^2 \text{See for instance [25] for a brief collection of early work on empirical distributions of returns.} \]
Merton’s Mixed Jump-Diffusion

This section presents the material from Merton’s 1976 paper, *Option pricing when underlying stock returns are discontinuous*. Merton’s approach is also related to new information reaching the market causing extraordinary changes in stock prices. It builds on a premise that the changes in stock prices are assumed to be of two kinds. The first kind is ‘normal’ fluctuations (similar to non-information events), when the stock price moves simply because of perturbations in supply and demand in a liquid market. The second kind of fluctuations are the ‘abnormal’ ones, when the stock drops or gains a significantly large percentage due to important new information.

Merton models the first type of price behavior with the classic GBM with constant coefficients. For the second type, a jump process is suggested. It is justified, Merton argues, by the observation that the critical information arrives in a discrete, and not continuous manner. The model of stock price is a mixture of these two, so that the price follows a GBM, and is at random times subjected to jumps driven by a compound Poisson process.

**Definition 1.** (Poisson point process) A process \( \{N(t)\}_{t \geq 0} \) with intensity \( \nu > 0 \) is a Poisson point process if it satisfies the following:

- \( N(0) = 0 \),
- \( N \) has independent increments,
- for \( 0 \leq s \leq t \), \( N(t) - N(s) \sim \text{Pois}(\nu(t - s)) \), i.e. the increment over a ‘time’ period \( t - s \) is Poisson-distributed\(^3\) with mean \( \nu(t - s) \).

**Definition 2.** (Compound Poisson process) Let \( \{Y_n\}_{n=1,2,3,...} \) be a set of independent and identically distributed random variables, and let \( \{N(t)\}_{t \geq 0} \) be a Poisson point process with intensity \( \nu \), independent of \( Y_n \). A compound Poisson process, with intensity \( \nu \) and with jump size determined by the distribution of \( Y_n \), is a process \( \{J(t)\}_{t \geq 0} \) such that

\[
J(t) = \begin{cases} 
0, & \text{if } N(t) = 0, \\
\sum_{n=1}^{N(t)} (Y_n - 1), & \text{if } N(t) \geq 1.
\end{cases}
\]

The compound Poisson process is a process which subjects the stock price to large changes at random times, events, driven by the Poisson point process.

\(^3\)Recall that a Poisson-distributed variable \( X \) with parameter \( \nu \) (were \( \nu = \mathbb{E}[X] \)) is a discrete random variable with probability mass function \( P(X = k) = \frac{\nu^k e^{-\nu}}{k!} \).
Since the increments of the Poisson point process are Poisson-distributed, the probability of number of events happening in a time span $h$ can be studied; we have

$$P[N(t + h) - N(t) = 0] = e^{-\nu h} = 1 - \nu h + o(h),$$
$$P[N(t + h) - N(t) = 1] = \nu he^{-\nu h} = \nu h + o(h),$$
$$P[N(t + h) - N(t) = 2] = \frac{1}{2}(\nu h)^2 e^{-\nu h} = o(h),$$

for $h \to 0$, with the notation that $\psi(h) = o(h)$ if $\lim_{h \to 0} \frac{\psi(h)}{h} = 0$. We see that the process can be interpreted as modeling arrival of events, where the probability of an event occurring within the next infinitesimal time frame is proportional to the length of the time frame, as well as to the intensity $\nu$. Thus, $\nu$ is the mean number of arrivals per time unit.

The jump-diffusion in the univariate case is posed as

$$dS_t = S_{t-} \alpha \, dt + S_{t-} \sigma \, dW_t + S_{t-} (dJ_t - \nu \kappa \, dt)$$
$$= S_{t-} (\alpha - \nu \kappa) \, dt + S_{t-} \sigma \, dW_t + S_{t-} dJ_t,$$  \hspace{1cm} (2.1)

where $\kappa := \mathbb{E}[Y - 1]$ “compensates” for $J$ in a way that $\mathbb{E}[dJ_t - \nu \kappa \, dt] = 0$. The Wiener process and the compound Poisson process are independent. Note that in the case where an event occurs, the jump size is determined by the left-limiting price $S_{t-}$, and $Y$ determines the fraction of the price that is left after the jump, i.e. $S_t = Y S_{t-}$ conditioned on the event occurring. It is therefore important that $Y \geq 0$; in fact Merton regards the special case when $Y$ are log-normal.

The SDE (2.1) given an initial condition $S(0) = S_0$ has the solution

$$S_t = S_0 \exp \left\{ \left( \alpha - \frac{\sigma^2}{2} - \nu \kappa \right) t + \sigma W_t \right\} \mathcal{Y}(N(t)),$$

if all parameters are deterministic and constant, where $\mathcal{Y}(0) = 1$ and $\mathcal{Y}(N) = \prod_{n=1}^{N} Y_n$ for $N \geq 1$.

Following Merton’s publication, a variety of distributions have been proposed for the jump size $Y$. One such model is the asymmetric double exponential [23], in which the jump can be either positive or negative with exponential distribution with different expected values for up and down, and where the sign of the jump is determined by a coin-toss of some determined fairness. Another, more general example is the affine jump-diffusion model presented in [12].
2.1.2 Financial Bubbles in a Stochastic Setting

In Protter’s publication from 2012, *A Mathematical Theory of Financial Bubbles* [32], the following definition of a financial bubble is formulated for the case of an individual stock:

\[ \beta_t = S_t - S^*_t, \]  

(2.2)

where \( \beta_t \) is the difference between the market price of the stock, \( S_t \), and the fundamental price of the stock at time \( t \), \( S^*_t \). The process \( \beta_t \) is referred to as a bubble. The intuition behind this formulation is that if the price of a stock is subject to the formation of a bubble, then the price of the stock is too high relative to what one ought to pay for the stock, i.e. its fundamental price. Nevertheless, the correct, fundamental price of the stock is not intuitively obvious. First, one must assume that a stock can be traded at inflated prices, then a method for determining the actual worth of the stock is needed.

Protter examines and develops a formula for the fundamental price in a complete market, arriving at the following formula for the wealth process, \( X_t \):

\[ X_t = 1_{\{t < \tau\}} S_t + B_t \int_0^{t \wedge \tau} \frac{1}{B_u} dD_u + \frac{B_t}{B_\tau} \Delta 1_{\{\tau \leq t\}}, \]  

(2.3)

where \( 1 \) is the indicator function, and where

\[ B_t = \exp \left\{ \int_0^t r_u du \right\}, \]

is the value of a money market account and \( r \) is the spot interest rate. Moreover, \( \tau \) denotes the lifetime of the stock and thus acts as a stopping time. \( D \) denotes the dividend process of the stock and \( \Delta \geq 0 \) is the liquidation value of the asset at time \( \tau \).

Using conditional expectation in (2.3) and rearranging the terms, one obtains the following expression for the fundamental value:

\[ S^*_t = E_Q \left[ \int_t^{\tau \wedge T} \frac{1}{B_u} dD_u + \frac{\Delta}{B_\tau} 1_{\{\tau \leq T\}} \right] \mathcal{F}_t B_t. \]  

(2.4)

Interestingly, Protter also claims that \( S_t \geq S^*_t \iff \beta_t \geq 0 \) for all \( t \), a relation that is said to hold for zero and non-zero constant spot rates and
dividend paying as well as non-dividend paying stocks. In [32], a proof is provided for zero spot rate, non-dividend paying stocks, emphasizing that this is only done for simplicity and an analogous outcome holds for non-zero spot rates and dividend paying stocks. The proof leads to the formulation of the following theorem from [32]:

**Theorem 1.** Let $S$ be the non-negative price process of a stock and assume $S$ pays no dividends. Moreover assume the spot interest rate is constant and equal to zero. Let $Q$ be a risk neutral measure under which $S$ is a local martingale (and hence a super-martingale). Let $S^*$ be the fundamental value of the stock calculated under $Q$, and let $\beta_t = S_t - S_t^*$. Then $\beta_t \geq 0$.

In the case of complete markets, the model presented in [32] requires bubbles to exist since the beginning of the modeling period or they can never exist, and that if a bubble collapses before the asset’s maturity, it cannot enter a build-up phase again. Protter claims that this is a weak spot of the theory. Due to this reason, he states that using the models presented in [32], complete market models are not well compatible with the study of bubbles. Nevertheless, he manages to resolve this issue within the context of incomplete markets, where the birth of a bubble is allowed to occur after the model starts.

In order to allow for a bubble to begin to form during the modeling period in an incomplete market, the concept of regime shifting processes are introduced. This is done by letting $(\sigma_t)_{t \geq 0}$ represent an increasing sequence of random times with $\sigma_0 = 0$. It is these random stopping times that represent the, assumed to be unpredictable, times of regime shifts in the economy.

In order to formulate the fundamental price of a risky asset in the setting of an incomplete market, a risk neutral measure must be selected. Protter does this using Theorem 13 in [32]. Due to that the measure specified in the theorem can alter upon regime shifts, so may also the fundamental value of the asset. Thus, it is argued that since the choice of risk neutral measure impacts the fundamental value, which in turn can change when there is a regime shift, the birth of price bubbles can be had.

Once again defining an asset price bubble, $\beta_t$ as

$$\beta_t = S_t - S_t^*, \quad (2.5)$$

Protter notes that within a fixed regime, the theory simplifies to the aforementioned complete market case having one risk neutral measure. This in turn leads to the following theorem from [32]:
Theorem 2. Within a fixed regime, \( S \) admits a unique (up to an evanescent set) decomposition
\[
S = S^* + \beta = S^* + (\beta^1 + \beta^2 + \beta^3),
\]
where \( \beta = (\beta_t)_{t \geq 0} \) is a càdlàg local martingale and

1. \( \beta^1 \) is a càdlàg non-negative uniformly integrable martingale with \( \beta^1_t \to X_\infty \) almost surely,
2. \( \beta^2 \) is a càdlàg non-negative uniformly integrable martingale with \( \beta^2_t \to 0 \) almost surely,
3. \( \beta^3 \) is a càdlàg non-negative super-martingale (and a strict local martingale) such that \( \mathbb{E}[\beta^3_T] \to 0 \) and \( \beta^3_t \to 0 \) almost surely. That is, \( \beta^3 \) is a potential.

Furthermore, \((S^* + \beta^1 + \beta^2)\) is the greatest submartingale bounded above by \( W \).

Note in the theorem above, \( \beta^1, \beta^2, \beta^3 \) are three different types of bubbles. This formulation leads up to the following corollary in [32]:

Corollary 1. Within a fixed regime, any asset price bubble \( \beta \) has the following properties:

1. \( \beta \geq 0 \),
2. \( \beta_\tau 1_{\{\tau < \infty\}} = 0 \),
3. if \( \beta_t = 0 \) then \( \beta_u = 0 \) for all \( u \geq t \), and
4. \( S_t = \mathbb{E}_Q[S_T | \mathcal{F}_t] + \beta^3_t - \mathbb{E}_Q[\beta^3_T | \mathcal{F}_t] \) for any \( t \leq T \leq \tau \).

Protter then argues that bubbles must be nonnegative, regardless of the regime being fixed or not. Hence, \( \beta_t \geq 0 \ \forall t \).

An entire section in [32] is dedicated to addressing the issue of the local martingale approach to bubbles, where it is acknowledged that there are separate types of criticism directed at the approach. For instance, there are two foundational criticisms of the proposed model. Protter mentions that modeling in continuous time is, “especially by economists”, seen as superfluous compared to discrete time models. This being due to that it complicates ideas being comparatively straightforward in a discrete time setting and that discrete time models can be used to approximate models in continuous time, thereby rendering the two approaches equivalent in economic theory.

Protter defends the use of continuous time models by arguing that if seeking to model for instance tick data, where information does not arrive in
uniformly spaced time increments, it is preferable to consider tick data as a frequently sampled collection of observations from an underlying continuous process. Nevertheless, he claims that even this view may be too primitive as there is a presence of microstructure noise and/or rounding errors.

The second foundational criticism directed towards the presented model is the concept of bubbles being described through the difference between a strict local martingale and a true martingale. This is due to that it is widely believed that there are no strict local martingales in discrete time and that all local martingales are true martingales. Protter asserts that he disagrees with this reasoning, presenting a number of arguments against it and referring to [21] for a more detailed discussion on the matter.

\section*{2.1.3 The Log-Periodic Power Law}

In the paper \textit{Financial bubbles: mechanisms and diagnostics} [33], D. Sornette and P. Cauwels argue that financial bubbles are a subject of debate and controversy that are not well understood or well defined. Thus they seek to provide a more specific characterization of financial bubbles.

During the formation of a bubble, the market prices diverge from the fundamental value. In order to address cases where the fundamental price is significantly uncertain as a result of future gains being heavily uncertain, the concept of “irrational exuberance” is considered in [33]. The main takeaway is that bubbles can be seen as a deviation of the market price of an asset from its fundamental value. So, absent a well-anchored fundamental price, the market can more easily disconnect from a reasonable price trajectory, follow herd behavior and push a price along a path of untenable growth. The key aspect here is that of the so called ‘herd behavior’, which is connected to positive feedback mechanisms. For instance, if there are good news about excellent future growth for an asset, there tends to be an initial influx of capital causing the asset price to increase. The subsequent growth and outlook of repeatedly increasing returns attracts additional investors. As the price proceeds to increase, increase in demand will follow, which in itself further increases the price. Thus, there is a (positive) feedback mechanism driving the market price away from equilibrium. So during the building up of a bubble, the market behavior is characterized by a different regime, where no significant attention is paid to the fundamental value. It is then argued that at some time investors tend to question the sustainability of the extraordinary growth. This may trigger sudden, unexpected selling of the overpriced asset which in turn may cause the sudden collapse of the bubble, simply as the market’s trust for the potential of the asset vanishes. Hence,
positive feedback is in \cite{33} considered to be the key factor causing the build up of a financial bubble as it breaks down the equilibrium of supply and demand. Sornette and Cauwels thus attempt to incorporate the positive feedback mechanism into their mathematical model accounting for financial bubbles.

This is done by claiming that under the presence of positive feedback, the rate of return changes from being constant to growing itself, causing the price to experience a “faster-than-exponential” hyperbolic course until a point is reached where the growth rate becomes critically large and the model suddenly breaks down. So if one can verify mathematically that the price of an asset follows a hyperbolic course instead of an exponential trajectory, it is possible to determine whether a positive feedback mechanism, characteristic for a bubble build-up phase, is occurring or not.

The development of the price of Bitcoin is used in \cite{33} to illustrate that the price is not well approximated by an exponential function whilst a hyperbolic function provides a better fit. Bitcoin famously burst and resulted in a significant correction in late 2013. It is claimed that the reason for this discrepancy stems from that the growth rate of the price had not remained constant, but was growing itself. Moreover, option hedging and algorithm trading are presented as causes of positive feedback.

Next, Sornette and Cauwels discuss the concept of a singularity, which they define as the point in time where the asset price peaks before it bursts. Instead of taking the common approach of designing a model ensuring the existence of a solution at any point in time, it is said that the financial market may be in different phases having their own distinct dynamics depending on if the pricing mechanism follows a sustainable process or not. It is this non-existence of a solution that plays an essential role in their proposed methodology for foretelling the collapse of a bubble. Furthermore, it is pointed out that instead of “making naive forecasts based on extrapolated trends”, their non-linear method enables the identification of different market regimes, mainly because the model breaks down at the end of an unsustainable process.

A component of central interest in the model proposed in \cite{33} concerns the patterns of oscillations that may be observed in an asset price. These are claimed to arise due to “specific dynamic and structural features of the market”. These oscillations are then sought to be combined with a hyperbolic power law model, resulting in the so called log-periodic power law (LPPL) model. Furthermore, it is argued that traders and investors adhere to a natural hierarchy causing imitation and herd-like behavior, which in turn
impacts the coordination of buy and sell orders. Hence, when strong herding is present the market’s pricing mechanism will be affected by this behavior. This idea can be translated to mathematics as a specific pattern in the price, reflecting the social hierarchy of traders. The underlying symmetry or pattern, is referred to as discrete scale invariance. The result from this is in [33] then considered to, during bubble phases, manifest itself in the form of “ever increasing oscillations with decreasing amplitude” in the price. This phenomenon is in turn referred to as log-periodicity.

Diagnosing Bubbles

The bubble model presented in [33] combines, as previously hinted, the hyperbolic power law with oscillations, thus obtaining the log-periodic power law. The proposed equation for the model is composed of three components, displayed below:

$$\log S_t = A + B(t_c - t)^m + C(t_c - t)^m \cdot \cos \left( \omega \log [t_c - t] - \phi \right), \quad (2.8)$$

for $t < t_c$. The leftmost boxed term describes the smooth hyperbolic power law. The mid and rightmost boxed components combine to form the log-periodic oscillations. More specifically, $t_c$ is the critical time around which the collapse of the bubble/regime change will occur. The term $A$ is constant and represents the expected value of the log-price when the price peaks at $t_c$ before collapsing. $B$ and $C$ determine the amplitude of the power law acceleration and of the log-periodic oscillations respectively. The exponent parameter $m$ is said to reside between 0 and 1 for a bubble and quantifies the degree of faster-than-exponential growth. Moreover, $\omega$ denotes the log-periodic angular frequency. Lastly, $\phi$ is a phase quantifying the time scale of the oscillations. The following essential information is also given in [33]:

1. For $0 < m < 1$, the first component (box) of the equation takes care of the positive feedback mechanism, when price growth become unsustainable, and at $t_c$ the growth rate becomes infinite;

2. The second part (box) of the formula causes the amplitude of the oscillations to drop to zero at the critical time $t_c$;

3. Part (box) three models the frequency of the oscillations, which becomes infinite at $t_c$.

Sornette and Cauwels thus argue that when looking for bubbles, if one can find the pattern of the LPPL with sufficiently strong statistical confidence,
then there is a clear indication of a bubble process. The model has proven to be effective when fitting it to S&P 500 data. Moreover, it is acknowledged by the authors that model calibration has proven to be difficult for this model.

It must be emphasized that Sornette and Cauwels mention in [33] that a single critical time is not used, but a collection of such times are used for calibration. Also, the proposed model indicates the risk of a crash occurring at a suggested critical time and bubbles are looked for in price data using different window sizes. This has allowed statistical analysis to be carried out on different LPPL fits in order to determine where the asset price is most likely to collapse.

2.2 The Fundamental Value

The fundamental or intrinsic value of a financial asset is supposed to represent its ‘actual’ or ‘true’ value. The fundamental value is therefore not necessarily equal to the market price of the asset. The market price reflects how the investors perceive the potential risks and returns associated with, for instance, a stock and thus reflects how the investors perceive its value. However, this perceived value does not have to reflect the actual value of the stock. In such a case, the asset is overvalued, meaning that the market price is greater than the fundamental price of the asset, and market equilibrium is not reached until the fundamental price equals the market price.

Consequently, the fundamental value is of great importance for investors seeking to account for whether an asset is over- or undervalued in their decision-making process. Nevertheless, there is no single correct definition of the fundamental price and different investors use different methods to determine it. Some common estimation methods for stocks are the dividend discount method and the corporate valuation method.

The dividend discount model calculates the fundamental value at time $t$, denoted $V_t$, as

$$V_t = E \left[ \sum_{s \geq t} e^{-r(s-t)} D_s \middle| \mathcal{F}_t \right],$$

where $D_s$ denotes the (future) dividend at time $s$ and $r$ is the continuously compounded discount rate. This method can be compared to the approach of determining the fundamental value presented in Section 2.1.2. Note that the dividends due at each time $s$ are unknown to the investor, and the same applies to the discount rate. Hence, determining the fundamental value using
all future dividends and discount rates quickly proves to be a non-trivial problem.

The corporate valuation method is used for companies that do not pay dividends to their shareholders. This method estimates the company value, \( V_{co} \), as

\[
V_{co} = \sum_{t=1}^{\infty} \frac{FCF_t}{(1 + WACC)^t},
\]

where \( FCF_t \) is the free cash flow at time \( t \) and \( WACC \) denotes the weighted average cost of capital.

### 2.3 Mean Reversion

The concept of business cycles is a well known phenomenon describing the fluctuations of the GDP around the trend in its long term growth [24]. In other words, markets tend to deviate from an average growth rate only to move towards it again. This assumption can also be made for financial instruments in general and not just the market as a whole.

A general approach to mean reversion is usually taken through the Ornstein-Uhlenbeck (OU) process, which in one dimension is determined by the dynamics

\[
dX_t = -\theta(X_t - \mu) \, dt + \sigma \, dW_t
\]

and the initial value \( X_0 = x_0 \), where \( \theta > 0, \mu \in \mathbb{R}, \sigma > 0 \) and \( W_t \) is a one-dimensional Wiener process. The solution to (2.9) is given by

\[
X_t = \mu + (x_0 - \mu)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} \, dW_s.
\]

Taking the expectation on both sides, we have

\[
E[X_t] = \mu + (x_0 - \mu)e^{-\theta t}.
\]

We see that \( \theta \) indicates how ‘strongly’ the system reacts to perturbations, meaning how fast the process is reverted to \( \mu \). The larger \( \theta \) is, the quicker the reversion. Note that \( X_t \) does not revert to its own mean, i.e. \( E[X_t \mid X_0 = x_0] \), but rather it reverts towards \( \mu \), which is called the asymptotic mean. In the case presented above, \( \mu \) is easily found through (2.11) as

\[
\mu_t = \frac{E[X_t \mid X_0 = x_0] - x_0 e^{-\theta t}}{1 - e^{-\theta t}}.
\]
Nevertheless, determining the asymptotic mean is generally not self-evident. First, one needs to determine what value or measure the price of a financial asset tends to revert towards. Is it for example its expected value or its fundamental value? One would then be required to determine how the value is to be defined and computed. This issue of modeling the asymptotic mean gets more complicated when working with more complex models than the one introduced in this section.

Mean reversion has an interesting use when modeling financial bubbles. During the build up phase of a bubble the asset price strongly differs from the asymptotic mean at some point in time, e.g. right before it collapses, and there is then a rapid reversion towards a new, non-hyped price better reflecting the actual worth of the asset. Such behavior can be modeled by letting for instance $\theta$ be stochastic so that it can increase in size when the asset price strongly deviates from the asymptotic mean. The increase in $\theta$, if large enough, will result in a rapid decay in the asset’s price towards the asymptotic mean, thereby enabling the model to account for financial bubbles.

2.4 SDE’s of Mean-Field Type

In [1], a controlled SDE of mean-field type (MFT) is defined as

\[
\begin{aligned}
dx_t &= b(t, x_t, E[\psi(x_t)], u_t) \, dt + \sigma(t, x_t, E[\phi(x_t)], u_t) \, dW_t, \\
x_0 &= \xi,
\end{aligned}
\]

where $b, \sigma, \psi$ and $\phi$ are some functions and $W_t$ is a Brownian motion. Moreover, $u_t$ is the control, which is allowed to take values in the action space $U$ for every $t$. Note that the coefficients of the SDE depend on the law of the SDE, the state and the control. Now if one considers an interacting particle system, the mean-field SDE can, when $n \to \infty$, be formulated as

\[
\begin{aligned}
dx_{i,n}^t &= b\left(t, x_{i,n}^t, \frac{1}{n} \sum_{j=1}^{n} \psi(x_{j,n}^t), u_t\right) \, dt + \sigma\left(t, x_{i,n}^t, \frac{1}{n} \sum_{j=1}^{n} \phi(x_{j,n}^t), u_t\right) \, dW_{i,t},
\end{aligned}
\]

where $\{W^i\}_{i=1,\ldots,n}$ are independent. An optimal control problem of MFT then usually consist of minimizing or maximizing a value function of the form

\[
J(u) = E\left(\int_0^T h(t, x_t, E[\varphi(x_t)], u_t) \, dt + g(x_T, E[\chi(x_T)])\right),
\]
where \(h, g, \phi\) and \(\chi\) are given functions. Such a value function is of mean-field type as \(h\) and \(g\) depend on the law.

Note that the main idea of mean-field theory is to account for the mean interaction in the considered interacting particle system, thereby reducing a so-called “multi-body problem” to an effective “single-body problem”, enabling an understanding of the average behavior of a complex system.

### 2.5 The Girsanov Theorem

The Girsanov theorem is used when one seeks to change an original probability measure to another, equivalent, measure. It is commonly used in arbitrage pricing and risk neutral valuation as one seeks to instead of describing the dynamics of stochastic processes under the ‘objective’ probability measure, \(P\), describe them under the risk neutral measure, \(Q\). This change of measure swiftly enables risk neutral valuation, making it highly useful when pricing financial derivatives.

The following formulation of the Girsanov theorem is cited from [17]:

**Theorem 3. (Girsanov)** Assume that the probability measures \(P\) and \(Q\) are mutually absolutely continuous on \(\mathcal{F}_\infty\). Let \((D_t)_{t \geq 0}\) be the martingale with càdlàg sample paths such that, for every \(t \geq 0\),

\[
D_t = \frac{dQ}{dP} \bigg|_{\mathcal{F}_t}.
\]

Assume that \(D\) has continuous sample paths, and let \(L\) be the unique continuous local martingale such that

\[
D_t = \mathcal{E}(L)_t := \exp \left( L_t - \frac{1}{2} \langle L, L \rangle_t \right) .
\]

Then, if \(M\) is a continuous local martingale under \(P\), the process

\[
\tilde{M} = M - \langle M, L \rangle
\]

is a continuous local martingale under \(Q\).

Here and throughout, \(\langle \cdot, \cdot \rangle\) denotes the quadratic variation.\(^5\)

---

\(^4\)Two measures \(\mu_1\) and \(\mu_2\) are said to be mutually absolutely continuous or equivalent if \(\mu_1(A) = 0 \Leftrightarrow \mu_2(A) = 0\), for any \(A \in \mathcal{F}\).

\(^5\)With an increasing subdivision \(0 = t_0 < t_1 < \cdots < t_{p_n} = t\) of \([0,t]\) with mesh tending to zero (i.e. \(\lim_{n \to \infty} \sup_{k \in \{1, \ldots, p_n\}} \{t_k - t_{k-1}\} = 0\)), the quadratic variation \(\langle M, M \rangle_t\) of a martingale \(M\) is defined as \(\lim_{n \to \infty} \sum_{k=1}^{p_n} (M_{t_k} - M_{t_{k-1}})^2\) in probability [17, p. 79].
To verify whether $D_t$ is a martingale, one can use the Novikov condition which states that if

$$\mathbb{E}\left[\exp\left(\frac{1}{2} \langle L, L \rangle_T\right)\right] < \infty,$$

then $D_t$ is a martingale and $\mathbb{E}^P[D_T] = 1$.

Now let $W^P$ be an $\mathcal{F}_t$-Brownian motion and $L_t$ be the continuous local martingale given by

$$L_t = \int_0^t \varphi(s, W^P_s) \, dW^P_s,$$

which by (2.12) gives

$$D_t = \mathcal{E}(L)_t = \exp\left(\int_0^t \varphi(s, W^P_s) \, dW^P_s - \frac{1}{2} \int_0^t \varphi^2(s, W^P_s) \, ds\right).$$

Note that $\varphi$ has to be a bounded measurable function on $\mathbb{R}_+ \times \mathbb{R}$. Now for a change of notation, let $\tilde{M} = W^Q_t$ and $M = W^P_t$, where $W^P_t$ is a standard Wiener process under the $P$-measure and $W^Q_t$ is a Wiener process under the $Q$-measure. Then, by (2.13), we have

$$W^Q_t = W^P_t - \left\langle W^P_t, \int_0^t \varphi_s \, dW^P_s \right\rangle = W^P_t - \int_0^t \varphi(s, W^P_s) \, ds,$$

(2.14)

where $W^Q_t$ is an $\mathcal{F}_t$-Brownian motion under $Q$.

As an example, let us consider the standard Black-Scholes model

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW^P_t,$$

$$dB_t = r B_t \, dt,$$

where $\alpha, \sigma$ and $r$ are constants and $S_t$ and $B_t$ are the price of a risky and a risk-free asset, respectively. It is seen from (2.14) that

$$dW^Q_t = dW^P_t - \varphi(t, W^P_t) \, dt.$$

Consequently, we have

$$dS_t = \alpha S_t \, dt + \sigma S_t \left[ dW^Q_t + \varphi(t, W^P_t) \, dt \right] = \left[ \alpha + \sigma \varphi(t, W^P_t) \right] S_t \, dt + \sigma S_t \, dW^Q_t.$$
In order for $Q$ to be a martingale measure, the local rate of return under $Q$ must be equal to the short rate, $r$. Hence, it must hold that 

$$\alpha + \sigma \varphi(t, W^P_t) = r$$

which can be formulated as 

$$\varphi(t, W^P_t) = -\frac{\alpha - r}{\sigma} = -\lambda.$$ 

where $\lambda$ is called the market price of risk as it gives the risk adjusted excess return of an asset, or in other words, its risk premium per unit volatility. The risk premium, $\alpha - r$, is the difference between the return an asset gives, $\alpha$, and the market’s risk-free rate of return, $r$. Thus, the risk premium is the additional return that an investor is expected to receive for exposing themselves to the risk associated with investing in the asset.

### 2.6 Portfolio Optimization

The goal of investors is to over time increase the value of their assets. In other words, an investor seeks to maximize the expected future value of their portfolio. Nevertheless, the greater the growth potential of a financial asset is, the greater the risk it is associated with becomes. Hence, investors are faced with a trade-off between maximizing future portfolio values and minimizing, or at least restricting, the risk they are exposed to. The field of portfolio optimization presents investment principles that address this issue.

There are two ways of approaching portfolio optimization with respect to time. One may either consider a time-continuous approach or a discrete approach. The latter is described in a single-period setting in [19], where the considered time horizon consists of the two times $t_0$, which denotes the present, and $t_1$, which denotes the end of the time horizon during which the value of one’s portfolio is to be optimized. An essential aspect of portfolio optimization is whether there exists a risk-free asset or not. A risk-free asset guarantees a certain return during the investment period and can for instance be a zero-coupon bond.

In his 1969 paper, Merton [27] applied the theory of stochastic optimal control to portfolio selection, moving portfolio optimization from the discrete time domain into the continuous setting. Instead of the returns having a certain distribution over a discrete time interval, the stocks are instead assumed to follow a continuous dynamic, with the stochastic part being modeled by a Brownian motion (and later in [28] also a Poisson jump process).
In the version of Merton’s portfolio optimization procedure with no risk-free alternative, consider a market with $d$ risky assets constituting a vector of asset prices $S(t) = (S^1(t), ..., S^d(t))^*$, where “$^*$” denotes the transpose. Assume the $S$-dynamics are given by

$$
\begin{align*}
\frac{dS(t)}{S(t)} &= D[S(t)]\alpha(t, S(t)) \, dt + D[S(t)]\sigma(t, S(t)) \, dW(t), \\
S(0) &= S_0,
\end{align*}
$$

where $D[S] = \text{diag}[S^1, ..., S^d]$, $\alpha_t \in \mathbb{R}^d$, $\sigma_t \in \mathbb{R}^{d \times d}$ and $W \in \mathbb{R}^d$ is a standard Wiener process. Let $w_t$ denote relative portfolio weights at time $t$. Note that by using relative weights, the constraint $1^*w = 1$ must hold, where $1 \in \mathbb{R}^d$ is a vector with all elements as ones. Confining ourselves to portfolios $w$ that are self-financed, the dynamics of the portfolio value $X_w$, given the initial wealth $x_0$, can be written as

$$
\begin{align*}
\frac{dX_w}{X_w} &= X_t^w w_t^* \alpha_t \, dt + X_t^w w_t^* \sigma_t \, dW_t, \\
X_0^w &= x_0,
\end{align*}
$$

which can be generalized as

$$
\begin{align*}
\frac{dX_w}{X_w} &= b(t, X_t^w, w_t) \, dt + \varphi^*(t, X_t^w, w_t) \, dW_t, \\
X_0^w &= x_0.
\end{align*}
$$

Fix a time $T > 0$ and assume that the objective function is given on the form

$$
J(w) = \mathbb{E}\left[\int_0^T f(t, X_t^w, w_t) \, dt + h(X_T^w)\right],
$$

where $f$ is an instantaneous negative utility function and $h$ determines the negative utility of the terminal wealth. Let $U[0, T]$ be the set of admissible controls.\(^6\) The optimization problem consists of finding a control law $w \in U[0, T]$ which minimizes the total negative utility $J$ given the dynamics (2.17). This is the classical formulation of a stochastic optimal control problem, to which theory of dynamic programming can be applied.\(^7\)

Assume the maps $b(t, x, w)$, $\varphi(t, x, w)$, $f(t, x, w)$ and $h(x)$ are uniformly continuous, bounded in $x = 0$ and Lipschitz-continuous in $x$. Then denote

$$
J(s, y, w) = \mathbb{E}\left[\int_s^T f(t, X_t^w, w_t) \, dt + h(X_T^w)\right]
$$

\(^6\) The precise meaning of the term “admissible” is presented in Section 2.7.2.

\(^7\) See e.g. [5] for an introduction or [35] for a more rigorous inquiry.
and the state equation (which is the same as (2.17) but with a different initial condition)

\[
\begin{aligned}
\frac{dX^w_t}{dt} &= b(t, X^w_t, w_t) \, dt + \varphi^*(t, X^w_t, w_t) \, dW_t, \\
X^w_s &= y,
\end{aligned}
\]

for any \( s \in [0, T] \). Now, we define the (optimal) value function

\[ V(s, y) = \inf_{w \in \mathcal{U}_{[0,T]}} J(s, y, w). \]

By definition, the terminal condition \( V(T, y) = h(y) \) holds.

Assume that \( V \in C^{1,2}([0, T], \mathbb{R}) \). Then \( V \) is a solution to the Hamilton-Jacobi-Bellman (HJB) equation, i.e. \( V \) is a solution to the PDE

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \inf_{w \in \mathcal{U}} \left\{ f(t, x, w) + b(t, x, w)V_x + \frac{1}{2} (\varphi^*(t, x, w)V_{xx}) \right\} &= 0, \\
V(T, x) &= h(x),
\end{aligned}
\]

where subscripts denote derivatives. The infimum in the equation is attained by an optimal control \( \hat{w} \). To obtain the optimal control, the HJB equation provides a verification theorem which roughly states that if an admissible control attains this infimum for some function \( G \) which satisfies the HJB equation, then this control is an optimal control and \( G \) is the value function.

An example of a problem of this sort which is similar to that in [5, p. 302–304] is the special case when \( \alpha \) and \( \sigma \) in (2.16) are constant and deterministic, and \( \sigma \sigma^* = \Sigma \) is positive definite. Let \( h = 0 \). To avoid negative wealth, the stopping time

\[
\tau = \inf\{t > 0 \mid X_t = 0\} \wedge T
\]

is introduced, where \( a \wedge b = \min\{a, b\} \). The optimization problem is then to maximize the objective

\[
E \left[ \int_0^\tau f(t, x_t, w_t) \, dt \right]
\]

given the dynamics in (2.16), whilst satisfying the constraint \( 1^*w = 1 \). Note that the theory above assumes the upper integration limit is deterministic.

\footnote{Note that in this problem, the objective is to be maximized and not minimized like the theory above. This is not an issue, but merely a matter of sign change.}
and not a stopping time like in this example. However, as pointed out in [5, p. 298], the theory still applies. The resulting HJB equation reads

\[
\begin{align*}
V_t + \sup_{w^{*}=1} \left\{ f(t, x, w) + x w^{*} \alpha V_x + \frac{1}{2} x^2 w^{*} \Sigma w V_{xx} \right\} &= 0, \\
V(T, x) &= 0.
\end{align*}
\]

If the instantaneous utility does not explicitly depend on \( w \) so that we have \( \frac{\partial f}{\partial w_i} = 0, i = 1, \ldots, d \), and we assume there is an interior solution, the optimal control is determined as

\[
\hat{w} = \frac{1}{1^* \Sigma^{-1} 1} \Sigma^{-1} 1 + \frac{V_x}{x V_{xx}} \Sigma^{-1} \left( \frac{1^* \Sigma^{-1} \alpha}{1^* \Sigma^{-1} 1} 1 - \alpha \right).
\]

For a corresponding problem (still with \( f = f(t, x) \)) when there also exist a risk-free asset with constant and deterministic rate \( r \) to invest in, the HJB becomes

\[
\begin{align*}
V_t + \sup_{w \in \mathbb{R}^d} \left\{ f(t, x) + x [w^*(\alpha - 1r) + r] V_x + \frac{1}{2} x^2 w^* \Sigma w V_{xx} \right\} &= 0, \\
V(T, x) &= 0,
\end{align*}
\]

and the optimal relative portfolio is

\[
\hat{w} = -\frac{V_x}{x V_{xx}} \Sigma^{-1} (\alpha - 1r)
\]

for the risky assets and the rest \( \hat{w}_0 = 1 - 1^* \hat{w} \) is invested in the risk-free asset.

### 2.7 Pontryagin’s Maximum Principle

There are two main ways of solving an optimal control problem; by dynamic programming and the HJB equation presented in the previous section, or with Pontryagin’s maximum principle. Dynamic programming relies on Bellman’s optimality principle, which states that the optimal control is independent of the given initial state. The problem is therefore solved backwards (see e.g. [8, p. 93]), optimizing \( u \) in \( t \), given that \( \hat{u} \) is the optimal control in \([t + \Delta t, T]\). If Bellman’s optimality principle holds, the problem is time consistent [6] and can therefore be solved directly (assuming enough regularity on dynamics and objective) by solving the HJB equation and optimizing the control.

If the problem is time inconsistent, however, the straightforward application of the HJB equation does not hold. An example of when a stochastic optimal
control problem is time inconsistent is when the objective contains non-linear functions of the expected value of the state variable. From portfolio theory we recognize the mean-variance problem as such a case, when the objective is to optimize a trade-off between risk (modeled by variance) and reward (modeled by expected value). Since variance contains the term $E[X]^2$ which is non-linear in the expected value, the mean-variance problem (in a continuous setting) is time inconsistent.\

There are numerous ways of solving the mean-variance optimal control problem. In [4], the problem is modified using recursion so that the modified problem becomes time consistent, whereupon dynamic programming can be used. In [36], the problem is identified as being a linear-quadratic (LQ) problem for which there is extensive theory. The stochastic LQ problem is a special case of stochastic optimal control and can be solved with the stochastic maximum principle (see [9]), a stochastic expansion of Pontryagin’s maximum principle. A third approach on solving the mean-variance problem, which will be adapted in this work, is by regarding the problem as a stochastic optimal control problem of mean-field type as is done in [1]. The theory used to solve such a problem is a general form of the stochastic maximum principle which allows for objective and dynamics that are of mean-field type.

2.7.1 The Deterministic Case

This section is largely based on lecture notes by Djehiche on Pontryagin’s stochastic maximum principle [11]. Below we demonstrate a heuristic derivation of Pontryagin’s principle by observing the optimal control problem

$$\text{minimize } J(u) = \int_0^T f(x(t), u(t)) \, dt + h(x(T))$$

w.r.t. $u : [0, T] \to \mathcal{U}$, s.t.

$$\begin{cases} x'(t) = b(x(t), u(t)), & 0 < t \leq T, \\ x(0) = x_0, \end{cases}$$

with $\mathcal{U}$ being a given non-empty set of controls and $x'$ denoting the $t$-derivative of $x$.

The strategy used in [11] for finding the local minimum of the value function $J$ subject to the given constraints is the Lagrange multiplier method. Solving (2.21) and (2.22) then becomes equivalent to minimizing the Lagrangian
functional
\[ \mathcal{L}(x,u,p) := \int_0^T \left[ f(x_t,u_t) + p_t \cdot (x'_t - b(x_t,u_t)) \right] \, dt + h(x_T). \] (2.23)

It is emphasized that “·” represents the scalar product. To simplify the notation, the Hamiltonian
\[ H(x,u,p) := p \cdot b(x,u) - f(x,u) \]
is introduced, for which we use the notation \( H_t = H(x_t,u_t,p_t) \) and \( \hat{H}_t = H(\hat{x}_t,\hat{u}_t,p_t) \), where we assume that \((\hat{x},\hat{u},p)\) minimizes \( \mathcal{L} \). Thus (2.23) can be reformulated as
\[ \mathcal{L}(x,u,p) = \int_0^T [-H_t + p_t \cdot x'_t] \, dt + h(x_T). \]

A perturbation argument followed by an integration by parts results in the expression
\[ 0 = \int_0^T \left[ \left( -\nabla_x \hat{H}_t - p'_t \right) \cdot \delta x_t - \nabla_x \hat{H}_t \cdot \delta u_t + \left( -\nabla_p \hat{H}_t + \hat{x}' \right) \cdot \delta p_t \right] \, dt + \left( \nabla_x h(\hat{x}_T) + p_T \right) \cdot \delta x_T, \] (2.24)
where \((\delta x, \delta u, \delta p)\) are the perturbations. This suggests that in order to find \((\hat{x},\hat{u},p)\) we can solve the system
\[
\begin{cases}
\hat{x}_0 = x_0, \\
\hat{x}'_t = \nabla_p H(\hat{x}_t,\hat{u}_t,p_t), \quad 0 < t \leq T, \\
\hat{p}'_t = -\nabla_x H(\hat{x}_t,\hat{u}_t,p_t), \quad 0 \leq t < T, \\
p_T = -\nabla_x h(\hat{x}_T).
\end{cases}
\] (2.25)

This system is the Hamiltonian system associated with the optimal control problem (2.21) and (2.22). Observe that the first two rows are the same as (2.22). The third and fourth rows constitutes the so called adjoint equation. Furthermore, we also have from (2.24) that
\[ \nabla_u H(\hat{x}_t,\hat{u}_t,p_t) = 0, \quad 0 \leq t \leq T, \] (2.26)
meaning that the control \( \hat{u} \) that optimizes the control problem also optimizes the function \( u \mapsto H(\hat{x}_t,\cdot,p_t) \). This is Pontryagin’s maximum principle.
2.7.2 The Stochastic and Mean-Field Case

The theory above is in [11] extended to a stochastic framework where the state variable follows a mean-field type SDE and the objective is of mean-field type. The theory and results from therein are presented in this section.

We work on a given filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) which is complete and right-continuous\(^9\), on which a given \(d\)-dimensional Wiener process \(W = \{W_t\}_{t \geq 0}\) is defined. The filtration is set to be the natural (augmented) filtration generated by \(W\).

For the finite time horizon problem, we set \(T > 0\) as a fixed time horizon.

An admissible control \(u\) is a control that is adapted\(^10\) to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) and takes values in a separable metric space \((U, \delta)\), where \(U \subseteq \mathbb{R}^d\). The set of all admissible controls \(u\) is denoted by \(U[0,T]\). The optimization problem consists of finding an optimal control \(\hat{u} \in U[0,T]\) such that \(J(\hat{u}) \leq J(u)\), \(\forall u \in U[0,T]\), where the objective is formulated as

\[
J(u) = \mathbb{E}\left[\int_0^T f(t, x(t), E[x(t)], u(t)) \, dt + h(x(T), E[x(T)])\right],
\]

(2.27)

where both \(f\) and \(h\) take real scalar values. The state variable \(x \in \mathbb{R}^n\) abide by the dynamics

\[
\begin{align*}
\text{d}x(t) &= b(t, x(t), E[x(t)], u(t)) \, dt + \varphi(t, x(t), E[x(t)], u(t)) \, dW_t, \\
x(0) &= x_0,
\end{align*}
\]

(2.28)

where \(b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n\) and \(\varphi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}\). We denote \(\hat{x} := x^{\hat{u}}\) as a solution to the equation (2.28) when an optimal control \(\hat{u}\) is used.

To obtain a maximum principle similar to that of the deterministic case, the following two assumptions are imposed.

\(^9\)A filtered probability space is said to be complete if \(\mathcal{F}_0\) contains all subsets of every null set, i.e. if \(\mathcal{N} \subset \mathcal{F}_0\), where \(\mathcal{N}\) is the set of all \(A \subset A' \in \mathcal{F}_{\infty}\) such that \(\mathbb{P}(A') = 0\). Right-continuity is satisfied if \(\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s\) for all \(t \geq 0\) [17].

\(^{10}\)A random process \(\{X_t\}_{t \geq 0}\) is adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) if \(X_t\) is \(\mathcal{F}_t\)-measurable for all \(t \geq 0\) [17].
(A1) All maps $\psi \in \{b, \varphi, f, h\}$ are measurable and satisfy

$$|\psi(t, x_1, \mu_1, u) - \psi(t, x_2, \mu_2, u)| \leq L(|x_1 - x_2| + |\mu_1 - \mu_2|),$$

for all $t \in [0, T], x_1, x_2, \mu_1, \mu_2 \in \mathbb{R}^n, u \in U,$

$$|\psi(t, x, \mu, u)| \leq L(1 + |x| + |\mu|),$$

for all $t \in [0, T], x, \mu \in \mathbb{R}^n, u \in U,$

for some real constant $L > 0$.

(A2) The derivatives $w \in \{x, \mu, xx\}$ of all maps $\psi \in \{b, \varphi, f, h\}$ satisfy

$$|\psi_w(t, x_1, \mu_1, u) - \psi_w(t, x_2, \mu_2, u)| \leq L(|x_1 - x_2| + |\mu_1 - \mu_2|),$$

for all $t \in [0, T], x_1, x_2, \mu_1, \mu_2 \in \mathbb{R}^n, u \in U,$

for some real constant $L > 0$.

For $h$, the arguments $t$ and $u$ are of course omitted from the expressions above, having instead $h = h(x, \mu)$.

The Hamiltonian is similar to the deterministic case, but with an additional term induced by a Lagrange multiplier $q$ which is needed since the dynamics are stochastic; we define

$$H(t, x, \mu, u, p, q) := p \cdot b(t, x, \mu, u) + \text{tr}[q \varphi(t, x, \mu, u)] - f(t, x, \mu, u), \quad (2.29)$$

for $(t, x, \mu, u, p, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{d \times n}$. Next, we take a look at the adjoint equations. In the general stochastic case, it is necessary to include second-order terms in the Taylor expansion. This results in two adjoint equations. The first, from the first-order term is

$$\begin{cases}
dp(t) = -\{H_x(t) + \mathbb{E}[H_{\mu}(t)]\} dt + q(t) dW(t), \\
p(T) = -h_x(T) - \mathbb{E}[h_{\mu}(T)].
\end{cases} \quad (2.30)$$

Under the assumptions (A1) and (A2), this backward mean-field type SDE adapts a unique and adapted solution.
The second order adjoint equation is

\[
\begin{aligned}
dP(t) &= -\left( b^*_x(t)P(t) + P(t)b_x(t) + \sum_{j=1}^d \varphi^j_x(t)^{\ast} P(t) \varphi^j_x(t) \right. \\
&\quad + \sum_{j=1}^d \left( \varphi^j_x(t)^{\ast} Q_j(t) + Q_j(t) \varphi^j_x(t) + H_{xx}(t) \right) dt \\
&\quad + \sum_{j=1}^d Q_j(t) \, dW^j(t), \\
\left. P(T) \right) &= -h_{xx}(\hat{x}(T)),
\end{aligned}
\] (2.31)

where \( \varphi^j \) is the \( j \)-th column of \( \varphi \). Note that this is a backward SDE which is not of mean-field type. It also admits a unique adapted solution when assumptions [A1] and [A2] hold.

Defining

\[
\delta \psi(t) := \psi(t, x, \mu, u) - \psi(t, x, \mu, \hat{u})
\] (2.32)

for all maps \( \psi \in \{ b, \varphi, f \} \) (recall that \( \hat{u} \) is an optimal control whereas \( u \) is any admissible control), and

\[
\delta H(t) := H(t, x, \mu, u, p(t), q(t)) - H(t, x, \mu, \hat{u}, p(t), q(t)) \\
= p(t) \cdot \delta b(t) + \text{tr}\{ \delta \varphi(t) \} - \delta f(t),
\] (2.33)

we are now equipped to state the stochastic maximum principle.

**Theorem 4.** (Stochastic Maximum Principle) Let assumptions [A1] and [A2] hold. If \((\hat{x}, \hat{u})\) is an optimal solution of the optimal control problem, then there are unique pairs of processes \((p, q)\) and \((P, Q)\) adapted to \( \{ \mathcal{F}_t \}_{t \geq 0} \), satisfying (2.30) and (2.31), respectively, such that

\[
\delta H(t) + \frac{1}{2} \text{tr}\{ \delta \varphi^\ast(t) P(t) \delta \varphi(t) \} \leq 0, \quad \forall u \in U, \; \text{a.e.} \; t \in [0, T], \; \mathbb{P}\text{-a.s.}
\]

The proof of the theorem can be found in [11].

In the special case where \( U \) is convex and all maps are continuously differentiable w.r.t. \( u \), the second order adjoint equation is not needed to solve the optimal control problem (which entails that the assumption [A2] can be weakened so that the maps must not necessarily have second-order derivatives in \( x \) that are Lipschitz).
Chapter 3

The Mean-Variance Problem

3.1 The Case Without a Risk-Free Asset

We work on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) which is complete and right-continuous. Set \(T > 0\) as a fixed time horizon.

For \(i = 1, \ldots, d\), the \(i\):th stock is assumed to have dynamics on the general form

\[
dS_i(t) = S_i(t)\alpha_i(t, S_i(t), E[S_i(t)]) \, dt + S_i(t)\sigma_i \cdot \left(t, \{S_j(t)\}_{j=1,\ldots,d}, \{E[S_j(t)]\}_{j=1,\ldots,d}\right) \, dW(t).
\]

\((3.1)\)

\(W(t) \in \mathbb{R}^d \ \forall t \geq 0\), is a \(d\)-dimensional Brownian motion on the probability space, and we assume the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural (augmented) filtration of \(W\).

\(\alpha_i\) is the \(i\):th element of the vector \(\alpha \in \mathbb{R}^d\).

\(\sigma_i \cdot = [\sigma_{i1}, \ldots, \sigma_{id}] \in \mathbb{R}^{1 \times d}\) is the \(i\):th row of the matrix \(\sigma \in \mathbb{R}^{d \times d}\) which satisfies \(\sigma \sigma^* = \Sigma\). Thus,

\[
\sigma_i \cdot \, dW(t) = \sum_{j=1}^{d} \sigma_{ij} \, dW_j(t).
\]

Below, we will also use \(\sigma_{i \cdot} = [\sigma_{i1}, \ldots, \sigma_{id}]^* \in \mathbb{R}^{d \times 1}\) to denote the \(i\):th column of \(\sigma\).

It is assumed that \(\Sigma\) is symmetric, real and positive-definite. This assures that \(\Sigma\) and \(\sigma\) are invertible.
Here and throughout the paper, the arguments of the functions $\alpha$, $\Sigma$ and $\sigma$ will be suppressed, using notation like $\alpha_i(t) = \alpha_i(t, S_i(t), E[S_i(t)])$ and $\sigma_t = \sigma(t, \{S_j(t)\}_{j=1}^{d}, \{E[S_j(t)]\}_{j=1}^{d})$.

Let $h_i(t)$ be the number of shares of stock $S_i$ that a portfolio contains. Define $u_i(t)$ as the monetary value of those shares, i.e.

$$u_i(t) := h_i(t)S_i(t),$$

and let $x^u(t)$ be the price process of the corresponding portfolio $u(t) = [u_1(t), \ldots, u_d(t)]^* \in \mathbb{R}^d$, such that

$$x^u(t) = \sum_{i=1}^{d} u_i(t)S_i(t).$$

Here, and throughout the rest of this section, the summation $\sum_{i=1}^{d}$ is abbreviated as $\sum_i$. Assume $u$ describes a self-financing portfolio, which would imply

$$dx^u(t) = \sum_i h_i(t) dS_i(t) = \sum_i u_i(t) \frac{dS_i(t)}{S_i(t)} = \sum_i u_i(t)[\alpha_i(t) dt + \sigma_i(t) dW(t)] = \sum_i u_i(t)\alpha_i(t) dt + \sum_i u_i(t)\sigma_i(t) dW(t) = u^*(t)\alpha(t) dt + u^*(t)\sigma(t) dW(t).$$

Thus, we have the portfolio dynamics

$$\left\{ \begin{array}{l}
    dx^u_t = u^*_t \alpha_t dt + u^*_t \sigma_t dW_t, \\
    x^u_0 = x_0,
\end{array} \right. \tag{3.3}$$

for some initial wealth $x_0$. Observe that $\alpha$ and $\sigma$ are determined by the underlying stock dynamics which are not affected by the control $u$. Therefore, we can regard these functions as adapted stochastic processes, i.e. $\alpha : [0, \infty) \times \Omega \to \mathbb{R}^d$ and $\sigma : [0, \infty) \times \Omega \to \mathbb{R}^{d \times d}$.

The objective studied here will be the mean-variance trade-off; we search for the optimal control $\hat{u} \in U[0, T]$, where $U[0, T]$ is the set of admissible controls, such that

$$\mathcal{J}(\hat{u}) = \min_{u \in U[0, T]} \mathcal{J}(u)$$
for the objective function
\[ J(u) = \gamma \frac{1}{2} \text{Var}[x^u(T)] - E[x^u(T)], \tag{3.4} \]
where \( \gamma > 0 \). Note that we can write the objective on the form (2.27) by letting \( f = 0 \) and
\[ h(x, \mu) = \frac{\gamma}{2} (x^2 - \mu^2) - x. \tag{3.5} \]
We assume that \( U \) is convex. To solve the optimization problem, we must therefore solve the backward-forward system of the state dynamics and the first-order adjoint equation.

The Hamiltonian is defined as
\[ H(x, u, p, q) := u^* \alpha p + \text{tr}(qu^* \sigma) \tag{3.6} \]
for \((x, u, p, q) \in \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}^d\). Note that \( \alpha \) and \( \sigma \) are still functions of \( t \in [0, T] \) and \( \omega \in \Omega \). Taking a closer look at the second term and using \( e_i \) as the notation for the \( i \):th unit vector, we have
\[(qu^* \sigma)_{ij} = e_i^* qu^* \sigma e_j = q_i u^* \sigma_{ij} = q_i \sum_k u_k \sigma_{kj},\]
\[ \Rightarrow \text{tr}(qu^* \sigma) = \sum_i (qu^* \sigma)_{ii} = \sum_i q_i \sum_k u_k \sigma_{ki} \]
\[ = \sum_k \sum_i u_k \sigma_{ki} q_i = u^* \sigma q. \]
Thus, we can simplify the Hamiltonian slightly to
\[ H(x, u, p, q) = u^* \alpha p + u^* \sigma q. \tag{3.7} \]
Noting that the Hamiltonian is constant in \( x \) and using the respective derivatives of (3.5), the adjoint equation (2.30) becomes
\[
\begin{aligned}
\begin{cases}
dp_t = q_i^* \mathrm{d}W_t, \\
p_T = 1 - \gamma (x_T - \mu_T),
\end{cases}
\end{aligned}
\tag{3.8}
\]
where \( \mu_t := E[x_t] \). Looking at the Hamiltonian further, we have
\[
\frac{\partial H}{\partial u_i} = \frac{\partial}{\partial u_i} \left( \sum_j u_j \alpha_j p + \sum_j \sum_k u_j \sigma_{jk} q_k \right) \]
\[ = \alpha_i p + \sum_k \sigma_{ik} q_k = \alpha_i p + \sigma_{i*} q, \]

which means
\[ \nabla_u H = \alpha p + \sigma q. \]  
(3.9)

To minimize the Hamiltonian, we let
\[ \nabla_u H = 0, \]
\[ \iff \alpha p + \sigma q = 0, \]
\[ \iff q = -p \sigma^{-1} \alpha. \]  
(3.10)

Observe that the Hamiltonian is zero everywhere when this holds, which implies lack of uniqueness.

Inserting (3.10) in (3.8) results in the adjoint system
\[
\begin{cases}
    dp_t = -p_t (\sigma_t^{-1} \alpha_t)^* dW_t, \\
    p_T = 1 - \gamma (x_T - \mu_T).
\end{cases}
\]  
(3.11)

We see that \( p \) is a martingale if Novikov’s condition (see e.g. \([17, p. 137]\)) is satisfied, that is if
\[
E \left[ \exp \left\{ \frac{1}{2} \int_0^T \| \sigma_t^{-1} \alpha_t \|^2 dt \right\} \right] < \infty,
\]
\[ \iff E \left[ \exp \left\{ \frac{1}{2} \int_0^T \alpha_t^* \Sigma_t^{-1} \alpha_t dt \right\} \right] < \infty. \]  
(3.12)

Assume that this condition holds so that \( p \) is a martingale. Then we have
\[
p_0 = E[p_T \mid \mathcal{F}_0] = E[1 - \gamma (x_T - \mu_T) \mid \mathcal{F}_0]
= 1 - \gamma (E[x_T \mid \mathcal{F}_0] - \mu_T).
\]  
(3.13)

Since \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \) then
\[
E[x_T \mid \mathcal{F}_0] = E[x_T \mid \{ \emptyset, \Omega \}] = E[x_T \mid \Omega] = E[x_T] = \mu_T.
\]

Inserting this equality in (3.13) yields
\[ p_0 = 1. \]  
(3.14)

Now to the adjoint equation (3.11), we see that it is a linear SDE, which means we can solve \( p_t \) directly as
\[ p_t = p_T \frac{Z_t}{Z_T}, \]
where we have introduced the function
\[ Z_t = \exp \left\{ -\frac{1}{2} \int_0^t \alpha_s^* \Sigma_s^{-1} \alpha_s \, ds - \int_0^t \left( \sigma_s^{-1} \alpha_s \right)^* dW_s \right\}. \tag{3.15} \]

Thus,
\[ p_0 = p_T \frac{Z_0}{Z_T} = p_T \frac{1}{Z_T}. \]

Hence,
\[ p_T \frac{1}{Z_T} = 1, \]

where we have used (3.14), which implies
\[ p_t = Z_t. \tag{3.16} \]

To retrieve the control, we use the fact that \( p_t \) is a martingale again, expressing \( p_t \) as
\[ p_t = \mathbb{E} [p_T | \mathcal{F}_t] = \mathbb{E} [x_t + \int_t^T u_s^* \alpha_s \, ds + \int_t^T u_s^* \sigma_s \, dW_s | \mathcal{F}_t] = 1 + \gamma \mu_t - \gamma \mathbb{E} [x_t | \mathcal{F}_t] \]
\[ = 1 + \gamma \mu_T - \gamma \int_t^T \mathbb{E} [u_s^* \alpha_s | \mathcal{F}_t] \, ds, \]

where we use the dynamics of \( x_t \) from (3.3). This tells us that \( p \) and \( x \) have proportional quadratic variation. In particular,
\[ d\langle p \rangle_t = \gamma^2 \, d\langle x \rangle_t. \tag{3.17} \]

From (3.11) and (3.3), respectively, we can write (3.17) explicitly as
\[ (p_t \sigma_t^{-1} \alpha_t)^* \left( p_t \sigma_t^{-1} \alpha_t \right) \, dt = \gamma^2 (u_t^* \sigma_t) (u_t^* \sigma_t)^* \, dt, \]
\[ \Leftrightarrow p_t^2 \alpha_t^* (\sigma_t^*)^{-1} \sigma_t^{-1} \alpha_t = \gamma^2 u_t^* \sigma_t \sigma_t^* u_t, \]
\[ \Leftrightarrow p_t^2 \alpha_t^* \Sigma_t^{-1} \alpha_t = \gamma^2 u_t^* \Sigma_t u_t. \tag{3.18} \]

Inserting (3.16), we see that we can choose our optimal control as any \( u \in \mathcal{U}[0, T] \) satisfying
\[ u_t^* \Sigma_t u_t = \frac{Z_t^2}{\gamma^2 \alpha_t^* \Sigma_t^{-1} \alpha_t}. \tag{3.19} \]
In this case when there is no risk-free asset, we have defined $u$ so that $\mathbf{1}^* u_t = x_t$. Thus, the solution set is the intersection between this hyperplane and the hyperellipsoid (3.19). This solution set could be limited further by other constraints, for example the no-short constraint $u \geq 0$. Note that imposing these constraints could result in a set of optimal controls which is empty.

### 3.2 The Case With a Risk-Free Asset

This section is very similar to the previous. Except for the $d$ available stocks with the same dynamics above, we have the option of investing in a risk-free asset, $S_0$, following the dynamics

$$dS_0(t) = r(t)S_0(t)\,dt,$$  \hfill (3.20)

where $r$ is deterministic.

We define the control variable $u$ as before according to (3.2), that is $u_i$ is the monetary value of the position in the asset $S_i$, for $i = 1, \ldots, d$. The difference between the portfolio value and $\sum_{i=1}^d u_i$ is placed in the risk-free asset, which allows a short position.

The dynamics of the portfolio value is then

$$dx^u_t = \left[ r_t(x^u_t + u^* \beta_t) + u^* \sigma \right] dt + u^* \sigma dW_t,$$

where $\beta_t := \alpha_t - r_t \mathbf{1}$. Hence, the dynamics of the controlled portfolio is now

$$\begin{cases}
    dx^u_t = \left[ r_t(x^u_t + u^* \beta_t) + u^* \sigma \right] dt + u^* \sigma dW_t, \\
    x_0^u = x_0,
\end{cases}$$  \hfill (3.21)

which means the Hamiltonian is

$$H(x, u, p, q) := (r_t x + u^* \beta_t)p + u^* \sigma q,$$  \hfill (3.22)

and the adjoint equation is

$$\begin{cases}
    dp_t = -r_t p_t dt + q^*_t dW_t, \\
    p_T = 1 - \gamma(x_T - \mu_T).\end{cases}$$  \hfill (3.23)
We can rewrite the dynamics slightly as
\[
dp_t + r_t p_t \, dt = q_t^* \, dW_t,
\]
\[
\Leftrightarrow e^{-\int_0^t r_s \, ds} \, d \left( e^{\int_0^t r_s \, ds} p_t \right) = q_t^* \, dW_t,
\]
\[
\Leftrightarrow d \left( e^{\int_0^t r_s \, ds} p_t \right) = e^{\int_0^t r_s \, ds} q_t^* \, dW_t.
\]

Defining
\[
\tilde{p}_t := \exp \left\{ \int_0^t r_s \, ds \right\} p_t, \quad \tilde{q}_t := \exp \left\{ \int_0^t r_s \, ds \right\} q_t,
\]
we can rewrite the adjoint equation \((3.23)\) as
\[
\begin{cases}
\, d\tilde{p}_t = \tilde{q}_t^* \, dW_t, \\
\tilde{p}_T = \exp \left\{ \int_0^T r_s \, ds \right\} (1 - \gamma (x_T - \mu_T)).
\end{cases}
\]

The gradient of \( H \) w.r.t. \( u \) is the same as in the previous case:
\[
\nabla_u H = \beta_t p + \sigma_t q,
\]
so the Hamiltonian attains its optimum when
\[
q = -p \sigma_t^{-1} \beta_t,
\]
\[
\Leftrightarrow \tilde{q} = -\tilde{p} \sigma_t^{-1} \beta_t.
\]

Inserting \((3.26)\) in \((3.25)\), the adjoint equation finally becomes
\[
\begin{cases}
\, d\tilde{p}_t = -\tilde{p}_t (\sigma_t^{-1} \beta_t)^* \, dW_t, \\
\tilde{p}_T = \exp \left\{ \int_0^T r_s \, ds \right\} (1 - \gamma (x_T - \mu_T)).
\end{cases}
\]

Similarly to the previous case, the Novikov criterion must hold for \( \tilde{p} \) to be a martingale:
\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T \beta_t^* \Sigma_t^{-1} \beta_t \, dt \right\} \right] < \infty.
\]

We solve the adjoint equation as
\[
\tilde{p}_t = \tilde{p}_T \frac{Z_t}{Z_T},
\]
where $Z$ is the same as above but with $\alpha$ replaced by $\beta$:

$$Z_t = \exp\left\{-\frac{1}{2} \int_0^t \beta_s \Sigma_s^{-1} \beta_s \, ds - \int_0^t (\sigma_s^{-1} \beta_s)^* \, dW_s \right\}. \quad (3.30)$$

Since $\tilde{p}$ is a martingale and the risk-free rate $r_t$ is assumed to be deterministic,

$$\tilde{p}_0 = \mathbb{E}[\tilde{p}_T | \mathcal{F}_0] = \mathbb{E}\left[\exp\left\{\int_0^T r_s \, ds \right\} \left(1 - \gamma(x_T - \mu_T)\right) \bigg| \mathcal{F}_0\right]$$

$$= \exp\left\{\int_0^T r_s \, ds \right\} (1 - \gamma (\mathbb{E}[x_T | \mathcal{F}_0] - \mu_T))$$

$$= \exp\left\{\int_0^T r_s \, ds \right\} (1 - \gamma (\mathbb{E}[x_T | \{\emptyset, \Omega}\] - \mu_T))$$

$$= \exp\left\{\int_0^T r_s \, ds \right\}. $$

Thus,

$$\tilde{p}_t = \exp\left\{\int_0^T r_s \, ds \right\} Z_t, \quad (3.31)$$

which implies that $p_t$ has the solution

$$p_t = \exp\left\{\int_t^T r_s \, ds \right\} Z_t. \quad (3.32)$$

Following the procedure of the previous case, we use the fact that $\tilde{p}$ is a martingale:

$$\tilde{p}_t = \mathbb{E}[\tilde{p}_T | \mathcal{F}_t] = \mathbb{E}\left[\exp\left\{\int_0^T r_s \, ds \right\} \left(1 - \gamma(x_T - \mu_T)\right) \bigg| \mathcal{F}_t\right]$$

$$= \exp\left\{\int_0^T r_s \, ds \right\} (1 + \gamma \mu_T - \gamma \mathbb{E}[x_T | \mathcal{F}_t])$$

$$= \exp\left\{\int_0^T r_s \, ds \right\} \times \left(1 + \gamma \mu_T - \gamma E\left[ x_t + \int_t^T (r_s x_s + u_s^* \beta_s) \, ds + \int_t^T u_s^* \sigma_s \, dW_s \bigg| \mathcal{F}_t \right]\right)$$

$$= \exp\left\{\int_0^T r_s \, ds \right\} \left(1 + \gamma \mu_T - \gamma \int_t^T E[ r_s x_s + u_s^* \beta_s | \mathcal{F}_t ] \, ds \right).$$
Thus,

\[
d\langle \tilde{p} \rangle_t = \exp\left\{ 2 \int_0^T r_s \, ds \right\} \gamma^2 d\langle x \rangle_t,
\]

\[
\Leftrightarrow (\tilde{p}_t \sigma_t^{-1} \beta_t)^* (\tilde{p}_t \sigma_t^{-1} \beta_t) \, dt = \exp\left\{ 2 \int_0^T r_s \, ds \right\} \gamma^2 (u_t^* \sigma_t)(u_t^* \sigma_t)^* \, dt,
\]

which together with (3.31) implies we can choose our optimal control as any admissible control \( u \) satisfying

\[
u_t^* \Sigma_t u_t = \frac{Z^2_t}{\gamma^2} \beta_t^* \Sigma_t^{-1} \beta_t.
\]

(3.33)

Again, we can impose further constraints, e.g. no short positions for which \( u_t \geq 0 \) and \( 1^* u_t \leq x_t \) must also hold.
Chapter 4

Modeling Stock Price and Bubbles

4.1 Choosing the Fundamental Value

As mentioned in Section 2.2 there are numerous ways of defining the fundamental value. We will consider dividend paying stocks and thus utilize the dividend discount method, also presented in Section 2.2. We recall that the fundamental value, \( V_t \), is according to this method given by

\[
V_t = E \left[ \sum_{s \geq t} e^{-r(s-t)} D_s \mid \mathcal{F}_t \right],
\]

(4.1)

where \( D_s \) denotes the (future) dividend at time \( s \) and \( r \) is the continuously compounded discount rate.

The efficient market hypothesis (EMH) suggests that the market price of a stock, \( S_t \), equals the intrinsic value for all \( t \). However, this entails that financial bubbles, which arise due to an extreme over-valuation of a stock relative to its actual value, are not possible. Hence, the collapse of a bubble is not supported by the EMH. Nevertheless, the existence of bubbles is a refutable fact that has been observed time after time on the financial markets. Consequently, claiming that \( S_t = V_t \) at all times is not a viable option.

We assume that investor’s always seek to trade a stock at a price they consider to be its fundamental value. Hence, the current price of a stock is the best estimate the market provides for the fundamental value. This intuition does, unlike the EMF, not claim that the market price equals the fundamental
value at all times but rather assumes that the market reverts the stock’s price back towards the fundamental value whenever new information becomes available.

In accordance with the intuition explained above, the following relationship formulated in [2], between the market price of a stock and its fundamental value was chosen:

\[ S_t = V_t \frac{e^{\Delta_t}}{E[e^{\Delta_t}]}, \quad (4.2) \]

where \( \Delta_t \) is a noise process independent of \( V \) and \( D \) for all \( t \). Moreover, the noise is stationary and mean-reverting around its mean which is given by \( E[\Delta_t] = 0 \).

Note that

\[
E[S_t] = E \left[ V_t \frac{e^{\Delta_t}}{E[e^{\Delta_t}]} \right] = \{ V_t and \Delta_t are independent \} \\
= E[V_t] = E \left[ \sum_{s \geq t} e^{-r_s(s-t)} D_s \bigg| \mathcal{F}_t \right] = \{ notation \} \\
= E \left[ \sum_{s \geq t} e^{-r_s(s-t)} D_s \bigg| \mathcal{F}_t \bigg| \mathcal{F}_0 \right] = \{ the tower property \} \\
= E \left[ \sum_{s \geq t} e^{-r_s(s-t)} D_s \bigg| \mathcal{F}_0 \right] := \hat{V}_t, \quad (4.3)
\]

where we define \( \hat{V}_t \) as the best guess of what the fundamental value will be at the future time \( t \). Consequently, we see that the expected value of the stock’s price at time \( t \) is equal to the estimated fundamental value at time \( t \), given the information at time 0.

A significant weakness of the dividend discount method is that future dividends, \( D \), and the return rate \( r \) in the equations above are unknown and need to be predicted for all times \( s \geq t \). By using \( E[S_t] \) as a proxy for the fundamental value, one does however not need to predict the future dividends nor the discount rates. Since predictions are done with a level of uncertainty, computing \( V_t \) using predicted values for \( D \) and \( r \) does not necessarily give a more accurate fundamental value than if one computes \( \hat{V}_t \) instead. Due to this reason, the expected value of the stock price at time \( t \) could perhaps be employed advantageously and is thus henceforth used as the fundamental value of the stock at time \( t \).
4.2 Proposed Stock Models

4.2.1 Bubbles and Mean Reversion

Contemporary theory on financial bubbles relates two values of a stock, one being the actual market price at which the stock is traded, and the other being some value inherent to the stock itself. Both Protter and Sornette calls this latter value the fundamental value of the stock. Recall that Protter defines a bubble as

$$\beta_t = S_t - V_t,$$

where $V_t$ is the fundamental value defined as

$$V_t = \mathbb{E}_Q \left[ \int_t^{\tau \wedge T} \frac{1}{B_u} dD_u + \frac{\Delta}{B_{\tau}} 1_{\{\tau \leq T\}} \left| \mathcal{F}_t \right| B_t \right],$$

(4.4)

where all terms are defined as in Section 2.1.2. He later goes on to prove that, with his definition of the fundamental value, $\beta_t \geq 0$, i.e. a stock is never undervalued. We find this result peculiar, and reject the claim that the fundamental value must always be at or below trading price. Embracing the notion of the expected value as a proxy for the fundamental value, we study instead the difference $S_t - \mathbb{E}[S_t]$. Since this difference can be negative and we limit us to positive bubbles only, it appears rational to define a bubble instead as

$$\beta_t = (S_t - \mathbb{E}[S_t])_+.$$

To suggest some possible stock dynamics that are based on this idea, we begin with the GBM as a foundation. We assume that a bubble appears not in a systematic way but in a random way. Thus, we assume that the diffusion, and not drift, of the Itô process drives a bubble; a bubble occurs when the stock follows a sample path where many ‘fortunate events’ have occurred frequently in a period. This would represent the stock being heavily overvalued.

Further, we propose that the deflation is somewhat systematic, acting as a gravity on the drift when the stock is overvalued. Modeling this gravity as a mean reversion, the first proposed stock model is

$$dS_t = aS_t dt + \sigma S_t dW_t - \theta (S_t - \mathbb{E}[S_t])_+ dt.$$  \hfill (A)

A weakness with this model is that the mean reversion will be equally strong in proportion to the bubble size (since $\theta$ is constant) for small and large bubbles. If the mean reversion is scaled with the fraction of the trading price and the fundamental price, the mean reversion will behave more like a rubber
band, being more prominent for large bubbles:
\[
dS_t = aS_t \, dt + \sigma S_t \, dW_t - \theta \frac{S_t}{E[S_t]} (S_t - E[S_t])_+ \, dt.
\] (B)

With the first order Taylor approximation \( \log(1+x) \approx x \), we see that this suggested process is similar to the Geometric Ornstein-Uhlenbeck\(^1\) process

\[
dX_t = \sigma X_t \, dW_t + \theta(\mu - \log X_t) X_t \, dt.
\]

If we set \( \mu = \log E[X_t] \), we can rewrite the GOU process as

\[
dX_t = \sigma X_t \, dW_t - \theta \left( \frac{X_t}{E[X_t]} - 1 \right) X_t \, dt
\]

\[
\approx \sigma X_t \, dW_t - \theta \frac{X_t}{E[X_t]} (X_t - E[X_t]) \, dt.
\]

The difference from this equation and the stock model (B) is that the mean reversion here is two-sided and that the second drift term with \( a \) is missing.

Because of the close relation between these two equations, a third proposed model is

\[
dS_t = aS_t \, dt + \sigma S_t \, dW_t - \theta S_t \log \left( \frac{S_t}{E[S_t]} \right) \, dt,
\] (C)

where \( \log_+ x := (\log x)_+ \). In this model, the force of the mean reversion is somewhere in between the models (A) and (B).

A theoretical problem with models (B) and (C) is that they are undefined for \( E[S_t] = 0 \). On a practical note, this will not be an issue if implementing the model, as an investment in a stock with a fundamental value of zero is ill-advised and can be singled out and omitted.

Another issue with all proposed models is that the drift component \( \alpha \) is unbounded from below\(^2\). If these were to be implemented in the optimal control problem, Novikov’s condition must hold, i.e.

\[
E \left[ \exp \left\{ \frac{1}{2} \int_0^T \alpha_t^* \Sigma_t^{-1} \alpha_t \, dt \right\} \right] < \infty.
\]

\(^1\)The Geometric Ornstein-Uhlenbeck has appeared in a financial context before, in the modeling of commodities, like in [26] where the model parameters are estimated from the spot price of gold.

\(^2\)For instance with the model (A), we have \( \alpha_t = a - \theta(S_t - E[S_t])_+ \).
This expected value is difficult to calculate for the suggested versions of $\alpha_t$. A practitioner might consider the (albeit unsatisfying) capped version

$$\alpha^M_t := \max\{\alpha_t, -M\}$$

for some large enough $M > 0$ which evades the problem, as $\alpha^M_t$ is then bounded (from above by $a$ and from below by $-M$), ensuring that Novikov’s condition holds.

A disadvantage that all these models share, is that the bubble collapses very slowly and the mean reversion is always active, even for small deviations from the fundamental value. An alternative could be to have a large mean reversion that happens in an instant, simulating a crash. This can be modeled by a jump process, in line with the work of Merton. A proposed model with fast mean reversion is

$$dS_t = aS_t \, dt + \sigma S_t \, dW_t - (S_t - E[S_t])_+ \, dN_t,$$

where $N$ is a Poisson point process with some intensity $\nu$. In this case, the bursting of the bubble is random (corresponding to an event occurring) and the stock price is immediately corrected to its fundamental value.

### 4.2.2 The Risk Premium Approach

An alternative model proposal for the stock dynamic is to express the drift term through the use of risk premium. We begin by observing the dynamics of a single stock $i = 1, \ldots, d$ under the $P$-measure and the $Q$-measure:

$$dS_i(t) = S_i(t) \alpha_i(t) \, dt + S_i(t) \sigma_i \, dW^P_t,$$

$$dS_i(t) = S_i(t) r(t) \, dt + S_i(t) \sigma_i \, dW^Q_t,$$

where all terms above have the same dimensions as in Chapter 3 and $r$ is a deterministic function representing the risk-free interest rate. The Girsanov theorem gives $dW^P = \phi(t) \, dt + dW^Q$, where $\phi(t) \in \mathbb{R}^d$, and thus (4.5) can be written as

$$dS_i(t) = S_i(t) \alpha_i(t) \, dt + S_i(t) \sigma_i(t) \phi(t) \, dt + S_i(t) \sigma_i \, dW^Q_t.$$

Comparing (4.7) to (4.6) it is seen that

$$\alpha_i(t) + \sigma_i(t) \phi(t) = r(t).$$

Consider the multidimensional case where $S = [S_1(t), \ldots, S_d(t)]$ so that $\alpha \in \mathbb{R}^d$. Then

$$\alpha(t) + \sigma(t) \phi(t) = 1r(t),$$

$$\iff \alpha(t) - 1r(t) = -\sigma(t) \phi(t).$$
Note that the left hand side in (4.8) is the risk premium. In a Black-Scholes setting, the risk premium is assumed to be proportional to the risk as

\[ \alpha - 1r(t) = \sigma \lambda, \]

where \( \lambda \in \mathbb{R}^d \) is the market price of risk. We propose instead an adjustment to this model of risk premium, where the risk premium is adjusted by a factor connected to a potential overvaluation of a stock. In the univariate case this would look like

\[ \alpha(t) - r(t) = \phi(S_t, E[S_t])\sigma \lambda. \] (4.9)

In this model, \( \lambda \) is still a constant determined by the market.

In the multivariate case, let \( \phi^\psi(t) = [\phi^\psi_1(t), ..., \phi^\psi_d(t)]^* \in \mathbb{R}^d \), where \( \phi^\psi_i \) is the adjustment factor for stock \( i \), and let \( \Phi^\psi(t) = D[\phi^\psi(t)] \in \mathbb{R}^{d \times d} \) be the diagonal matrix of \( \phi^\psi \). The proposed overvaluation adjustment factor here will be

\[ \phi^\psi_i(t) = \left( \frac{E[S_i(t)]}{S_i(t)} \wedge \psi \right) \] (4.10)

for \( i = 1, ..., d \), where \( \psi \geq 1 \) is a constant. The purpose of this factor is to add a time-dependent scaling parameter to the market price of risk, \( \lambda \). The intuition behind it is that as the market price \( S_i(t) \) of stock \( i \) grows larger than \( E[S_i(t)] \), i.e. the stock’s fundamental value, the market price of risk should decrease and conversely, if \( S_i(t) < E[S_i(t)] \) then the market price of risk should increase. However, if \( S_i(t) \ll E[S_i(t)] \) then the market price of risk should not be allowed to grow uncontrollably. Therefore, an upper bound \( \psi \) is set as a restriction. For \( \psi > 1 \), the risk premium will (up to the point \( \psi \)) be amplified if the stock is undervalued. This could be tied to the existence of negative bubbles. For \( \psi = 1 \) the risk premium can only be adjusted down, which happens if the stock is overvalued.

For the multivariate version of (4.9) we get

\[ \alpha(t) - 1r(t) = \Phi^\psi(t)\sigma(t)\lambda. \]

Using the relation (4.8) we get

\[ -\sigma(t)\varphi(t) = \Phi^\psi(t)\sigma(t)\lambda, \]

\[ \Leftrightarrow \varphi(t) = -\sigma^{-1}(t)\Phi^\psi(t)\sigma(t)\lambda. \] (4.11)

Note that since \( \Phi^\psi \) is bounded, and \( \sigma \) is assumed to be invertible, the Girsanov kernel \( \varphi \) will be bounded and so the Novikov condition holds.
Inserting (4.11) into (4.8) and rewriting, one obtains

$$\alpha(t) = 1r(t) + \Phi^\psi(t)\sigma(t)\lambda.$$  \hspace{1cm} (4.12)

Hence theoretically,

$$\alpha_{i,\text{max}}(t) = r(t) + \psi\sigma_i(t)\lambda,$$

$$\alpha_{i,\text{min}}(t) = r(t).$$  \hspace{1cm} (4.13)

Note that $\alpha_{i,\text{min}}(t)$ corresponds to the case where the risk premium is zero.
Chapter 5

Discussion

Regarding the use of the expected value of stock prices as their fundamental value, there are certain benefits but also debatable downsides.

One particular advantage is that $E[S_t]$ does not have to be expressed explicitly. Hence, instead of attempting to fit a model of the expected value to past data and extrapolating it to represent future instances, one may express $E[S_t]$ implicitly (as a stochastic variable) and thus obtain mean reversion to the actual expected value at time $t$ and not a predicted value. Predictions of future expected values can strongly vary in accuracy and precision depending on the robustness of the prediction model, e.g., how sensitive it is to changes in the prediction horizon as well as the size, frequency and sample window of the analyzed data. Thus, being able to express the fundamental value implicitly allows one to avoid any uncertainties caused by or related to the use of prediction models.

The use of a stochastic expressions of the fundamental value was for example shown in the work of Protter in Section 2.1.2. However, unlike the approach formulated in said section, the models suggested in this paper do not assume that the market price of the stock has to be greater than or equal to the fundamental value. Instead, as mentioned in Section 4.2 it is assumed that the market price of a stock may be smaller than the fundamental value, meaning that a stock can indeed be undervalued by the market relative to its fundamental value. Moreover, to assure that mean reversion only applies to positive bubbles, i.e., when the market price exceeds the fundamental value, maximum functions have been used.

It is noteworthy that the proposed models are not built to identify or predict financial bubbles during their build up phase in order for the investor to
exit their positions right before the bubble collapses. To do so the proposed models would require predictive abilities. The prediction of high likelihoods of bubble bursts is a complex problem which was regarded as being beyond the scope and limitations of this paper. A predictive model for regimes where bubbles form and build up were presented in Section 2.1.3. Similar models have been rigorously tested and their performances are reported by the Financial Crisis Observatory [13]. Subsequently, the purpose of this paper was never to create a predictive model superior to existing models. The models formulated in this paper instead aim to prevent an investor from entering trades that result in them following the market in the hyperbolic price curve of a positive price bubble. As mentioned in Section 2.1.3 positive feedback mechanisms tend to cause the stock price to rise hyperbolically during a bubble’s build-up phase. Rather than attempting to predict when the bubble bursts in order for the investors to exit their position close to an imminent collapse, the proposed models take a more risk averse approach and suggest positions decreasing the investor’s exposure to the possibility of substantial negative returns during periods where a bubble burst becomes increasingly likely.

The models in Section 4.2.1 have mean-reverting properties. Consequently, when a positive bubble builds up and $S_t$ diverges from $E[S_t]$, the models will suggest taking positions reverting the price towards the fundamental value, preventing an investor from getting caught in a positive feedback mechanism and taking high risks during a regime of extraordinarily increasing returns. This risk-averting property has more of an impact in models (B) and (C), rather than in (A), as they allow mean reversion to increase in magnitude the greater the market price becomes relative to the fundamental value. Since the models strive to redirect the investor towards the fundamental value during a bubble build-up, the possibility of obtaining optimal returns is reduced, but simultaneously prevents having significant positions in a stock upon its collapse, thereby limiting losses. Hence, the models in Section 4.2.1 are meant for investors seeking to maintain positions in overvalued stocks whilst suppressing their value at risk in case of a collapse.

The risk premium based model in Section 4.2.2 lacks mean-reverting properties but also possesses dynamics preventing an investor from following the market during hyperbolic increases in price. The greater the market price is than the fundamental value, the smaller the risk premium becomes, resulting in a diminished drift term in the stock dynamics. Thus, as a bubble increases in size, the model opts for positions where the interest rate falls closer to $r$ rather than the $\alpha$-value of the market, effectively preventing the investor from maximizing investments into a stock under an increasing risk
of residing within a bubble. In case the market price of the stock is under-
valued, i.e. \( S_i(t) < \mathbb{E}[S_i(t)] \), it would be counterintuitive to allow \( \phi_i^\psi(t) \) to
grow unrestrictedly and thus resulting in an overblown, unmaintainable \( \alpha \)
in the drift term. Hence, the use of \( \psi \) prevents the model from suggesting
positions expecting highly unlikely returns. The value of \( \psi \) is nevertheless
required to be carefully chosen and will be a subjective choice of the investor
and a projection of his or her willingness towards risk-taking in the presence
of undervalued stocks.

Although the proposed models are constructed to avoid investments that
significantly divert from a fundamental value, they do not allow for (near)
instant price corrections in the stock price dynamics. Therefore, a mod-
ification worth considering would be the incorporation of jump processes,
for example as presented in Section 2.1.1 and model (D). This would en-
able a model better suited for simulating sudden price corrections. Such
occurrences can be detected in historical price data when news changing the
market’s perception of a stock’s value breaks. A clear example of a sudden
price correction was seen in the stock of Facebook in July 2018 when the
company released an unsatisfactory earnings report. The stock price went
from a closing price of $217.48 on July 25\(^{th}\) to a closing price of $176.27 on
July 26\(^{th}\), loosing just beneath 20% of its market value between two closing
hours. Mean reversion will likely not suffice in accounting for such sudden
corrections on its own. Also, depending on how the fundamental, i.e. ex-
pected value, is measured the magnitude of mean reversion can significantly
vary in the proposed models. The addition of jump processes would thus aid
the proposed models in accounting for occurrences of near instant price
drops. It is however noteworthy that the method used in Chapter 3 will
be further complicated through the addition of a jump parameter and thus
requires adaptation.

The use of \( \mathbb{E}[S_t] \) as a proxy for the fundamental value stems from the assu-
ption that the EMH does not hold at all times \( t \). Nevertheless, not all
investors share this idea and are at odds with the assumptions of funda-
mental analysis. Since such investors make investments based on the EMH,
one cannot expect the entire market to trade stocks at/close to what they
consider to be fundamental prices. Instead, some can tend to consider the
current market price to be a fair measure of the stock’s intrinsic value. Due to
such theoretical disagreements between market participants, it is not evident
how the expected value of a stock is to be defined or measured and it thus
becomes a subjective, philosophical question. However, in order for a model
to have any practical implementation value, it should be supported by em-
pirical evidence. This has not been done for the proposed models and would
be essential in order to legitimize them for practical implementation.

Continuing on the practical aspects of the proposed models, their performance will be highly dependent on parameter estimations of numerous terms. More specifically, \( \sigma, a \) and \( \theta \) need to be estimated when implementing the model. A difficulty with the proposed models is that they are not easily converted into a stationary time series, such as the standard Black-Scholes framework when log-returns are assumed to be IID. The jump model (D) is perhaps the most easily implemented model, since the intensity parameter \( \nu \) could be estimated by looking at frequencies of extreme negative returns for each individual stock. Since parameter estimation falls outside the scope of this paper, no specific methods have been considered. Thus, a suggestion on future work is to construct ample methods for parameter estimation, enabling the model to be implemented and its performance measured.
Chapter 6

Conclusion

The aim of the project was to formulate a solution method for a portfolio optimization problem when the underlying stocks may be residing within a bubble. Using a generalized version of the stochastic maximum principle which applies to optimal control problems of MFT, the mean-variance problem was solved without assumptions on the stock dynamics’ parameters being deterministic. With the control $u_i$, where $u_i$ is the monetary amount invested in stock $i$, the optimal control set is the intersection between the hyperellipsoid (3.19) and the hyperplane $1^*u$ in the case where there is no risk-free asset available. If investing in a risk-free asset is possible but there are no constraints on portfolio weights or the portfolio value, the solution set is the entire hyperellipsoid (3.33). In this case, further constraints can be added for example in order to ensure non-negative portfolio value, which of course limits the solution set to a subset of the hyperellipsoid.

The stock models proposed in Section 4.2.1 all included an asymmetric mean-reverting term of some sort. The mean to which the stock is reverting to is taken to be the expected value of the stock. The expected value is supposed to emulate the fundamental price of the stock. The purpose of the asymmetric mean reversion is to decrease the expected return of the stock if it is overvalued, i.e. if it resides within a bubble. In the case of the continuous models, the optimal portfolio results apply, although the drift must be capped to ensure that Novikov’s condition is satisfied. Since the solution method was not implemented for dynamics containing a jump process, the stock model with jumps cannot be applied given the results presented here. Moreover, the models are derived from intuitive reasoning but are not yet tested on historical data, the reason being difficulty of parameter estimation. Thus, their usefulness remains to be determined.
A different approach to the stock dynamic modeling is to look at the risk premium. The same idea of using the expected value as a proxy for the fundamental value can be applied, assuming that the risk premium is reduced when the stock is overvalued. This is simply extended to allow for under-valued stocks to have higher risk premium. Although this does not model bubble burst or bubble deflation, it does account for fads so that bullish stocks eventually cool off.
Bibliography


