Pricing of a balance sheet option limited by a minimum solvency boundary

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Abstract

Pension companies are required by law to remain above a certain solvency level. The main purpose of this thesis is to determine the cost of remaining above a lower solvency level for different pension companies. This will be modelled by an option with a balance sheet as the underlying asset. The balance sheet is assumed to consist of bonds, stocks, liabilities and own funds. Both liabilities and bonds are modelled using forward rates. Data used in this thesis is historical stock prices and forward rates. Several potential models for stock and forward rate processes are considered. Examples of models considered are Bates model, Libor market model and a discrete model based on normal log-normal mixture random variables which have different properties and distributions. The discrete normal log-normal mixture model is concluded to be the model best suited for stocks and bonds, i.e. the assets, and for liabilities.

The price of the balance sheet option is determined using quasi-Monte Carlo simulations. The price is determined in relation to the initial value of the own funds for different portfolios with different initial solvency levels and different lower solvency bounds. The price as a function of the lower solvency bound seems to be an exponential function and varies depending on portfolio, initial solvency level and lower solvency bound. The price converges with sufficient accuracy. It is concluded that the model proves that remaining above a lower solvency level results in a significant cost for the pension company. A further improvement suggested is to validate the constructed model with other models.
Prissättning av en balansräkningsoption med en undre solvensbegränsning

Sammanfattning


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1 Introduction

1.1 Background

Pension companies invest their customers’ premiums to give them a return that is as high as possible once they retire. The pension company’s portfolio modelled as a balance sheet consists of assets, liabilities and own funds, illustrated in Figure 1. The assets could consist of for example stocks, bonds, derivatives and real estate. The liabilities mainly consists of actuarial provisions, i.e. the future pay-out’s to the customers. Own funds consists of the economic net worth, i.e. assets minus liabilities.

![Balance Sheet Illustration](image)

Figure 1: Illustration of a balance sheet.

The level of solvency of a pension company’s balance sheet is determined by the value of the assets divided by the value of the liabilities,

\[
\text{solvency ratio} = \frac{\text{assets}}{\text{liabilities}}. \tag{1}
\]

By law, the solvency ratio of a balance sheet has to be at least 104% \[1\]. However, a solvency larger than 104% is preferred as it implies less exposure and sensitivity towards changes in the market.

When the market moves and the value of the assets decrease for example, the solvency ratio will decrease as well. Before the ratio reaches a level below 104%, the pension company needs to act in order to be able to continue running their business. They are forced to sell an asset that they consider risky, i.e. might depreciate even more and result in a solvency below 104% or an individually chosen buffered minimum solvency level. This means that they will sell at a cheap price compared to the initial value of the asset. Assume that the market has recovered and they are able to and want to buy the asset back. Now, the price has increased and they have to buy it back at a higher price. The loss that this has resulted in can be considered a cost, the cost of staying above a certain solvency level when the market moves. This is what pension companies usually do in reality.

Now, consider that this cost can be modelled, in theory, using an option with the underlying asset being the balance sheet. If a pension company would be able to get the exact amount from their customers that they need to stay above their minimum solvency level when the market moves, they would not have to neither sell cheap nor buy expensive and loose money due to those transactions. In theory, the pension company could create a sort of put option to their customers where the underlying asset is the balance sheet. The idea is that as long as the pension company has a solvency level that is larger than the minimum solvency level,
the option is worth nothing. Once the solvency level of the pension company's balance sheet goes below their minimum solvency boundary, the option will be worth the amount that the pension company requires to get back above their minimum solvency level again. This means that the pension company will not be forced to rebalance the portfolio, i.e. make transactions where they sell cheap and buy expensive, to remain above their minimum solvency level. The cost will instead be covered by the option.

1.2 Purpose

Common practice amongst pension companies is to present the result of their asset portfolio but not always in comparison with their liabilities. This means that a pension company can present a positive result based on the yield of their assets meanwhile their solvency has decreased due to increasing liabilities.

Maintaining a certain solvency level is an aspect that all pension companies must take into consideration when managing their portfolio of investments. The purpose of this thesis is to demonstrate that there is a cost that originates from remaining above a certain solvency level and that the size of this cost is of significance. The aim is also to demonstrate that the change in solvency of a pension company depends on the initial solvency level. Another goal of this thesis is to study how the price of the option changes when the limit for the solvency level varies. This is done by developing a new model that can be used to describe the behaviour of an option with a balance sheet as the underlying asset.

1.3 Research question

The research question that is investigated in this thesis is the following.

"What is the cost of remaining above a certain solvency level for a given portfolio?"

This question is approached by thorough literature studies on possible methods to use and then implementation of the most appropriate method. Also, the selection of models to best describe the movements for stocks, bonds and liabilities are investigated. This thesis is limited to portfolios where the assets consists of stocks and bonds and liabilities consisting of FTA (actuarial provisions). The aim is to find the corresponding cost of hedging a portfolio which can be equated with the price of an option with the balance sheet as the underlying asset.

1.4 Procedure

The option with the balance sheet as the underlying asset is modelled based on stocks, liabilities and bonds. Both bonds and liabilities are governed by forward rates as these affect the discounting of future coupon payments. The option depends on the composition of the balance sheet and which lower solvency level is chosen.

The data used in this thesis is provided by Captor. It consists of multiple stock prices and forward rates (for years 1-10, 15 and 20) from 2009-01-30 to 2019-02-28, extracted monthly. Given these inputs, a model describing stock movements and forward rate movements is created. The model is based on normal log-normal random variables which are mixtures between normal and log-normal random variables. This model is designed to account for fat-tailed distributions, i.e. with non-zero skewness and kurtosis, which is in line with the historical behaviour of the stocks and forward rates. The forward rates are strongly
correlated with each other which is incorporated in the model together with a correlation between the stocks and forward rates. Methods for determining the parameters included in the model and for determining the correlation between the stocks and forward rates from historical data are derived.

The derived model is modelled under the real-world measure whereas when pricing the option, a model under the risk-neutral measure is required. Since the market is incomplete there exists no unique risk-neutral measure. Therefore, one measure amongst several possible is chosen to retrieve a risk-neutral price. The price of the option is obtained using quasi-Monte Carlo simulations and is simulated for three different portfolios. The difference between them is that they consist of bonds with different maturity times and different stocks. A price of the option is simulated for each portfolio with three different initial solvency levels and several different lower solvency bounds. The price is relative to the initial value of the balance sheet, i.e. the initial value of the own funds.
2 Theoretical background

In this chapter, the concepts of skewness and kurtosis are explained in section 2.1. It is followed by an introduction to normal log-normal mixture random variables in section 2.2 that takes skewness and kurtosis into account. This distribution will be used to model forward rates and stocks. The forward rates are adjusted to a regulatory curve according to Swedish Financial Supervisory Authority, SFSA, which is explained in section 2.3. In section 2.4, the Monte Carlo method for option price simulation is described. Bates model for modelling stocks is introduced in section 2.5 and the Libor market model for modelling forward rates is described in section 2.6. These models are researched but not chosen to be the most optimal to use for this project. Another method that is researched but not used is a dimensionality reduction method, principal component analysis. It is described and demonstrated on a Libor market model in appendix A.

2.1 Skewness and kurtosis

Many models of stock or forward rate dynamics, such as the Black-Scholes model and Libor market model, assume a normal (Gaussian) distribution. However, historical stock prices typically exhibit higher probabilities for large changes, i.e. skewed distributions [2]. To check if a random variable follows a standard normal distribution both skewness and kurtosis is studied. Skewness, \( S \), is typically defined as the third moment of the random variable \( \xi \),

\[
S = \mathbb{E}[(\xi - \mu)^3] / \sigma^3
\]

and kurtosis, \( K \), is typically defined as the fourth moment,

\[
K = \mathbb{E}[(\xi - \mu)^4] / \sigma^4
\]

where \( \mu \) is the expected value of \( \xi \) and \( \sigma^2 \) is the variance of \( \xi \). Kurtosis can also be given in terms of excess kurtosis, \( \tilde{K} \), which is defined by

\[
\tilde{K} = K - 3.
\]

The definition of excess kurtosis is the kurtosis relative to that of a standard normal distribution which is three. When the skewness is non-zero, i.e. \( S \neq 0 \), the distribution is not symmetric. If the skewness is positive, the distribution is skewed to the right and if the skewness is negative, the distribution is skewed to the left.

When investigating the kurtosis of a distribution, excess kurtosis is commonly used. If the excess kurtosis is positive, this indicates that the distribution has fatter tails than a standard normal distribution and is pointier around the mean [4]. This type of distribution is called leptokurtic. An excess kurtosis that is negative indicates that the probability distribution is flatter around the mean and these distributions are called platykurtic. Distributions that have no excess kurtosis are called mesokurtic [3]. These distributions are illustrated in Figure 2.
2.2 Normal log-normal mixture random variable

Yang [5] introduces a normal log-normal mixture random variable, $u$, defined as the product of a normal and a log-normal random variable. It is given by

$$u = e^{\frac{1}{2}\eta \varphi}$$

where $\eta$ and $\varphi$ are random variables sampled from a multivariate normal distribution. There is, in general, a correlation between the normal and the log-normal random variable. This correlation is denoted by $\rho$, the variance of the log-normal random variable is assumed to be $\sigma^2$ and the variance of the normal random variable is one. The multivariate normal distribution that $\varphi$ and $\eta$ follow is given by

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \sigma \\ \rho \sigma & \sigma^2 \end{bmatrix} \right).$$

(2)

The expected value of the normal log-normal mixture random variable is given by

$$\mathbb{E}[u] = \frac{1}{2} \rho \sigma e^{\frac{3}{2} \sigma^2}$$

and the variance by

$$\mathbb{E} \left[ (u - \mathbb{E}[u])^2 \right] = e^{\frac{3}{2} \sigma^2} \left[ 1 + \rho^2 \sigma^2 \left( 1 - \frac{1}{4} e^{-\frac{1}{2} \sigma^2} \right) \right].$$

If $\rho$ is small in the way that terms associated with $\rho^2$ can be ignored, the skewness of $u$ is given by

$$S \approx \frac{1}{2} \rho \sigma e^{\frac{3}{2} \sigma^2} (9 - 3e^{-\frac{1}{2} \sigma^2})$$

and the kurtosis of $u$ is given by

$$K \approx 3e^{\sigma^2}.$$ 

The excess kurtosis, $\tilde{K}$, is simply given by

$$\tilde{K} = K - 3.$$
One property of the normal log-normal mixture random variable is that the expected value,
\[ \mathbb{E}(e^{au}) = \infty, \]
for any \( a \neq 0 \). However, by normalizing \( u \), the expected value in equation (3) is finite and will not tend to infinity, see section 3.2 where this is shown. The normalized random variable \( u_{\text{norm}} \) is given by
\[ u_{\text{norm}} = \frac{u - \mathbb{E}[u]}{\sqrt{\text{Var}(u)}}. \]
Its expected value is
\[ \mathbb{E}[u_{\text{norm}}] = \mathbb{E} \left[ \frac{u - \mathbb{E}[u]}{\sqrt{\text{Var}(u)}} \right] = \frac{\mathbb{E}[u] - \mathbb{E}[u]}{\sqrt{\text{Var}(u)}} = 0 \]
and its variance is
\[ \text{Var}(u_{\text{norm}}) = \text{Var} \left( \frac{u - \mathbb{E}[u]}{\sqrt{\text{Var}(u)}} \right) = \text{Var} \left( \frac{u}{\sqrt{\text{Var}(u)}} \right) = \frac{\text{Var}(u)}{\text{Var}(u)} = 1. \]

The analytical expression for the marginal density function of the normal log-normal mixture random variable is unknown. However, it is possible to determine the function by numerical integration. The marginal density function is given by
\[ \text{pdf}_{\text{u}}(u|\sigma, \rho) = \int_{-\infty}^{\infty} \text{pdf}_{\text{u}}(u|\eta)\text{pdf}_{\eta}(\eta)d\eta. \]
By substituting \( \eta = \sigma y \), the above integral can be written as
\[ \text{pdf}_{\text{u}}(u|\sigma, \rho) = \int_{-\infty}^{\infty} f(u, y|\rho, \sigma)\Phi(y)dy \]
where \( \Phi(\cdot) \) is the standard normal density and the function \( f \) is given by the following equation [5]
\[ f(u, y|\rho, \sigma) = \left[ 2\pi(1 - \rho^2)e^{\sigma y} \right]^{-\frac{1}{2}} \exp \left( -\frac{(u - \rho^2 ye^{\frac{1}{2}\sigma^2})^2}{2(1 - \rho^2)e^{\sigma y}} \right). \]

2.3 The regulatory forward rate curve

There are two types of forward rate curves, one that is governed by the market and one which is a regulatory curve calculated in accordance with the SFSA’s guidelines [6]. This section will describe how the regulatory forward rates and discount factors are obtained from par rates and market forward rates. The par rate is the coupon rate at which the value of the bond is equal to its par value (face value) [7]. The market forward rate curve from 10 to 20 years is adjusted according to SFSA where an ultimate forward rate of 4.2% affects the regulatory forward rates. This means that, from ten years and forward, the forward rates get less sensitive to changes in the market. Forward rates from 20 years forward, are not at all sensitive to the market. Assume that a set of par rates are given. Denote the par rate at time \( t \), with maturity at time \( T \) by \( \text{par}_t(T) \). These par rates will be transformed to forward
rates where the first step is to perform a credit spread adjustment, $a_{cs}$. This is done using the following expression,

$$\text{par}_{\text{adj.},t}(T) = \text{par}_t(T) - a_{cs}, \quad T \geq t$$

where the adjustment is set to, $a_{cs} = 0.0035$. The discount factor from time $t$ to time $T$, $DF^T_t$, is calculated recursively by

$$DF^T_t = \frac{1 - \text{par}_{\text{adj.},t}(T) \sum_{j=t+1}^{T-1} DF^j_t}{\text{par}_{\text{adj.},t}(T)}, \quad T \geq t$$

for $T = t + 1, t + 2, \ldots, t + y$, if par rates for $y$ consecutive years are available. It is assumed that the forward rate is constant between the years for which data is missing. An example of this could be forward rates between year 10 and 15 if par rates for only year 10 and 15 are available. The zero coupon rate for time $T$, evaluated at time $t$, $\tilde{z}_t(T)$, is defined by

$$\tilde{z}_t(T) = \left( \frac{1}{DF^T_t} \right)^{1/T} - 1, \quad T \geq t \quad (10)$$

and the market forward rate between time $T-1$ and $T$ valued at time $t$, $f^T_{\text{market},t}$ is given by

$$f^T_{\text{market},t} = \frac{DF^{T-1}_t}{DF^T_t} - 1, \quad T \geq t. \quad (11)$$

The discount factor between $t$ and $t$, $DF^t_t$, is one. Assume that par rates for year 10 and 15 are given. The forward rate valid between 10 and 15 years will be constant. Denote this forward rate by $f^T_{\text{market},t}$, since it is evaluated at time $t$. The discount factors for the years in between, i.e. 11, ..., 14, are determined from the following equations

$$DF^{t+11}_t = \frac{DF^{t+10}_t}{1 + f^T_{\text{market},t}}$$

$$DF^{t+12}_t = \frac{DF^{t+10}_t}{(1 + f^T_{\text{market},t})^2}$$

$$DF^{t+13}_t = \frac{DF^{t+10}_t}{(1 + f^T_{\text{market},t})^3} \quad (12)$$

$$DF^{t+14}_t = \frac{DF^{t+10}_t}{(1 + f^T_{\text{market},t})^4}$$

$$DF^{t+15}_t = \frac{DF^{t+10}_t}{(1 + f^T_{\text{market},t})^5}.$$  

Equation (12) is used to determine the discount factors $DF^{t+11}_t$, $DF^{t+12}_t$, $DF^{t+13}_t$ and $DF^{t+14}_t$ in terms of only $DF^{t+10}_t$ and $DF^{t+15}_t$ as

$$DF^{t+10+i}_t = (DF^{t+10}_t)^{(5-i)/5} (DF^{t+15}_t)^i, \quad i = 1, \ldots, 4. \quad (13)$$

By rewriting equation (10), the discount factor $DF^{t+15}_t$ is given by

$$DF^{t+15}_t = \frac{1}{1 + \tilde{z}_t(t + 15)} \quad (14)$$

where $\tilde{z}_t(T)$ is the zero coupon rate for time $T$ evaluated at time $t$. Equations (12) and (14)
are used to minimize the following expression with regards to \( \tilde{z}_t(t+15) \) in \( DF_t^{t+15} \),

\[
\left( \text{par}_t(t+15) \cdot \sum_{i=1}^{15} DF_t^{t+i} - (1 - DF_t^{t+15}) \right)^2.
\]

When \( \tilde{z}_t^{\text{min}}(t+15) \) corresponding to the minimum is determined, the discount factor at 15 years ahead, \( DF_t^{t+15} \), is obtained by equation (14). The discount factor at year 15, \( DF_t^{15} \) can then be used to determine the intermediate discount factors according to equation (13). Equivalently, the discount factor for 20 years and the ones between year 15 and year 20 are obtained, given that the par rate for 20 years is known as well.

To shift the curve according to the ultimate forward rate (UFR), which is 4.2% after 20 years, the weights, \( w(T) \) for \( T_1 = 10 \) to \( T_2 = 20 \) years are needed and they are determined by

\[
w(T) =
\begin{cases} 
0, & T \leq T_1 \\
\frac{T - T_1}{T_2 - T_1 + 1}, & T_1 < T \leq T_2 \\
1, & T > T_2.
\end{cases}
\]

The weight for one to ten years ahead are equal to zero. After 20 years, the weights are equal to one. The weighted average of the regulatory forward rate evaluated at time \( t \), \( f_{reg,t}^T \) between time \( T - 1 \) and \( T \) is obtained from the following

\[
f_{reg,t}^T = (1 - w(T)) \cdot f_{market,t}^T + w(T) \cdot UFR
\]

where \( f_{market,t}^T \) is given by equation (11). The corresponding discount factors, \( DF_{reg,t}^T \), are obtained as

\[
DF_{reg,t}^T = \frac{DF_{reg,t}^{T-1} \cdot f_{reg,t}^T}{1 + f_{reg,t}^T} = \prod_{n=T}^{T-1} \frac{1}{1 + f_{reg,t}^T}
\]

since the discount factor, \( DF_{reg,t}^T \) is equal to one by definition \[6\].

### 2.4 Monte Carlo

The Monte Carlo method can be used to value options where the method depends on sampling a large number of possible scenarios to estimate the payoff of the option. It is especially useful when the payoff depends on the path that the underlying asset follows but also when the payoff depends solely on the final value of the underlying asset. It is a qualitative method with regards to accommodating any type of stochastic process and when the payoff from the derivative depends on several underlying market variables. One disadvantage of the method is that computationally, it can be quite time consuming since a large number of simulations are needed to get sufficient convergence and accuracy.

The first step of the Monte Carlo method is to sample a random path for the underlying asset, \( S_T \)[7]. Consider an option with payoff on the form,

\[
\text{payoff} = f(S_T),
\]

where \( S_T \) is the value of the stock at time \( T \) and \( f \) is a function. Assume that the function for determining the value of the stock \( S_T \) is known and that the variable \( S_T \) is stochastic.
The next step is to sample $S_T$ and calculate the payoff this corresponds to according to equation (18). This is repeated several times to generate several possible payoffs from the option. Denote the payoff from the $i$:th sample, $S_T^{(i)}$, by $f(S_T^{(i)})$. The expected payoff from the option defined by equation (18) can be approximated by the following

$$
E[f(S_T)] \approx \frac{1}{n} \sum_{i=1}^{n} f(S_T^{(i)})
$$

(19)

where the accuracy increases as the number of simulations, $n$, increases [7].

2.4.1 Convergence of the Monte Carlo method

Consider the following integral of a function $f$ over the unit interval,

$$
\alpha = \int_{0}^{1} f(x)dx.
$$

Let the integral be represented by an expectation $E[f(U)]$ where $U$ is uniformly distributed between zero and one. Evaluating the function $f$ at $n$ random points drawn from the uniform distribution and calculating the average of these results in the Monte Carlo estimate,

$$
\tilde{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} f(U_i).
$$

If $f$ is integrable over the interval from zero to one, then by the law of large numbers, $\tilde{\alpha} \longrightarrow \alpha$ with probability one as $n \longrightarrow \infty$. The Law of large numbers assures that the estimate converges to a correct value when the number of iterations are increased [8].

2.4.2 Quasi-Monte Carlo

In the standard Monte Carlo method, pseudo random numbers are sampled. To generate random numbers in practise, deterministic functions are used. This means that the random numbers are not truly random but instead determined by a deterministic function. However, they resemble random numbers. These types of random numbers are called pseudo random numbers. The standard Monte Carlo method gives qualitative results but quite a large number of simulations are needed to get convergence. An alternative version of the Monte Carlo method is the so called quasi-Monte Carlo method. It is the same algorithm as the standard Monte Carlo but instead of sampling pseudo random numbers, Sobol’ random numbers are generated which are demonstrated in Figure 3.

A Sobol’ sequence is a low discrepancy sequence. Discrepancy is a measure of how inhomogeneously distributed, in the unit space, a set of multidimensional vectors are. A Sobol’ sequence is a deterministic sequence where the values are generated in an ordered way. A detailed description for generating this sequence can, e.g. be found in Jäckel [10]. A disadvantage with using pseudo random numbers is that some parts of the unit space will not be evaluated, whereas a Sobol’ sequence covers more spread out scenarios, see Figure 3. This contributes to faster convergence.
In the standard Monte Carlo method, the sampling averages of random quantities are used to get an estimate of the corresponding expectations. This is justified by the law of large numbers. In the standard Monte Carlo method, the convergence rate is $O\left(\frac{1}{n}\right)$, where $n$ is the number of points or paths generated, but when using the quasi-Monte Carlo method, the convergence can be decreased to $O\left(\frac{1}{n^q}\right)$. It is therefore common that the quasi-Monte Carlo method produces more accurate results for the same number of simulations than the standard method [8].

When generating a set of Sobol' random numbers and the dimension of the set is small, the correlation between the random numbers will be close to zero. However, if the dimension is sufficiently large such that there exists a correlation between the random numbers that are generated, external methods need to be applied to make the generated numbers uncorrelated.

2.5 Bates model

In 1996, Bates [11] proposes a model that incorporates both a jump process and stochastic volatility in the dynamics of the stock movements. The dynamics of the stock at time $t$, $S_t$, is given by

$$dS_t = (\mu - \lambda \bar{k})S_t dt + \sqrt{V_t}S_t dW_t + kS_t dq_t$$

where $\mu$ is the expected average return, $\lambda$ is the annual frequency of jumps, $V_t$ is the variance of the stock’s movements, $k$ is a random variable, $\bar{k}$ is the average jump size, $W_t$ is a Wiener process and $q_t$ a Poisson-counter with intensity $\lambda$. The dynamics for the volatility of the stocks is given by

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^{V_t}$$

where $\kappa$ is the mean reversion rate, $\theta$ is the long run variance, $\sigma_V$ is the volatility of the variance and $W_t^{V_t}$ is another Wiener process. The covariance between the Wiener processes used in the dynamics of the stock and the Wiener process used in the dynamics of the variance of the stock is given by
\[ \text{Cov}(dW_t, dW_t^V) = \rho dt \]  

(22)

where \( \rho \) is the covariance coefficient. In Bates model, it is also stated that

\[ \text{Pr}(dq_t = 1) = \lambda dt \]  

(23)

and

\[ \ln(1 + k) \sim \mathcal{N}(\ln(1 + \bar{k}) - \frac{1}{2} \delta^2, \delta^2) \]  

(24)

where \( \delta^2 \) is the variance of \( (1 + k) \). No closed form solution describing the value of \( S_t \) over discrete time intervals is derived, instead a method for pricing American options on Deutschemark exchange rate is developed by Bates [11]. Bates model takes into account that some stock return distributions have fat tails, i.e. higher probability of large changes, by including jumps. The stochastic volatility dynamics of the model incorporate varying volatility over time amongst stock returns.

There are several parameters in the model that need to be estimated to fit the model to the data. Cape et al. [12] provide a method to estimate all parameters by performing Markov chain Monte Carlo simulations. In their paper, posterior distributions for each parameter are determined in terms of the other parameters and then sampled from. When a sample has been drawn, the other parameters are sampled using the new drawn sample. After a large number of iterations, the values of the parameters can be determined [12].

### 2.6 Libor market model

A Libor market model can be used to model \( n \) correlated forward rate processes and can be described using the following definition. The idea is to define the Libor forward rates such that, for the \( i \)th forward rate, \( L_i(T) \) is log-normal under it’s measure \( Q^i \). The Libor process \( L_i \), for every \( i = 1, \ldots, n \), is a martingale under the corresponding forward measure \( Q^i \) on interval \( [0, T_{i-1}] \). To do this, the following objects are considered as given a priori

- A set of resettlement dates \( T_0, \ldots, T_n \).
- An arbitrage free market bond with maturities \( T_0, \ldots, T_n \).
- A \( k \)-dimensional \( Q^n \)-Wiener process \( W^n \).
- For each \( i = 1, \ldots, n \), a deterministic function of time \( \sigma_i(t) \).
- An initial non-negative forward rate term structure \( L_1(0), \ldots, L_n(0) \).
- For each \( i = 1, \ldots, n \), \( W^i \) is defined as the \( k \)-dimensional \( Q^i \)-Wiener process generated by \( W^n \) under the Girsanov transformation \( Q^n \rightarrow Q^i \).

**Definition 2.1** If the Libor forward rates have the dynamics

\[ dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), i = 1, \ldots, n, \]  

(25)

where \( W^i \) is a \( Q^i \)-Wiener process as described above, then we say that we have a discrete tenor Libor Market Model with volatilities \( \sigma_1, \ldots, \sigma_n \) [13].
3 Model construction

In this chapter the balance sheet option, which consists of stocks, bonds and liabilities, is presented in section 3.1. The model for the stocks is described in section 3.2 and the model for the bonds and liabilities is described in section 3.3. As both bonds and liabilities are based on forward rates, a model for these are described in section 3.3.2. The models stated in this section are under the real-world measure \( \mathbb{P} \).

3.1 Balance sheet option

Consider a balance sheet consisting of stocks, bonds and liabilities. At time \( t \), the value of the stocks is denoted by \( S_t \), the bonds by \( B_t \) and the liabilities by \( L_t \). Denote the value of the balance sheet consisting of these stocks, bonds and liabilities at time \( t \) by \( BS_t \). This value is the same as the value of the own funds and is defined by the following,

\[
BS_t = S_t + B_t - L_t. \tag{26}
\]

Assume that there is a solvency requirement stating that the solvency level of the balance sheet may not be lower than a certain level, \( s_{\text{min}} \). The solvency level of a balance sheet at time \( t \), \( s_t \), is given by

\[
s_t = \frac{S_t + B_t}{L_t}. \tag{27}
\]

Therefore, the goal is to fulfill

\[
s_t \geq s_{\text{min}}, \quad \forall t \quad \text{a.s.}
\]

which might not always be achievable. The balance sheet option should be modelled in such a way that if the pension company’s solvency level goes below \( s_{\text{min}} \), the option should yield as much capital as it would require to get \( s_t = s_{\text{min}} \) again. When \( s_t \geq s_{\text{min}} \), the option yields nothing. The optimal option would be exercisable at any time and also multiple times. This is however, a more complex option and therefore the option will only be exercisable at maturity time \( T \) which implies that only the solvency level at time \( T \) will be considered. This can be seen as a sort of put option with strike price \( K_T = K_T(s_{\text{min}}, S_T, B_T, L_T) \) where the payoff, \( \Phi \), is given by

\[
\Phi = \max(K_T - BS_T, 0).
\]

The strike price varies with time and is stochastic since it depends on the value of the different parts of the balance sheet which vary over time and are stochastic. The strike price is derived in section 3.1.1 and the lower solvency level is discussed in section 3.1.2.

3.1.1 Strike price of the option

The option is to be modelled as to be worth zero when the pension company remains above a certain level of solvency. The goal is to find a strike price that represent the value of the balance sheet at the lower solvency level limit. An alternative way to view this is to consider what the option should yield given the value of the balance sheet.

Suppose that a lower solvency level limit, \( s_{\text{min}} \), is chosen. When the pension company’s
balance sheet is solvent, the option is worth zero. The pension company’s balance sheet is solvent if
\[
\frac{S_T + B_T}{L_T} \geq s_{\min} \iff S_T + B_T - s_{\min}L_T \geq 0. \tag{28}
\]
However, in the case when the solvency level goes below the minimum solvency level at time \( T \), i.e.
\[
\frac{S_T + B_T}{L_T} < s_{\min} \iff S_T + B_T - s_{\min}L_T < 0
\]
the option should yield \( x \) such that
\[
\frac{S_T + B_T + x}{L_T} = s_{\min}
\]
since this represents the amount that would be needed to add to the assets to get the pension company’s solvency level above the minimum solvency level again. This amount is given by
\[
x = s_{\min}L_T - S_T - B_T
\]
which is always larger than zero when the balance sheet has a lower solvency level than the lower limit level. The corresponding value of \( x \) is smaller than zero when the balance sheet is above the lower level, which can be seen from equation (28). Hence the payoff from the balance sheet option, \( \Phi \), can be expressed as follows,
\[
\Phi(S_T, B_T, L_T) = \max(s_{\min}L_T - S_T - B_T, 0) = \max(s_{\min}L_T - (BS_T + L_T), 0) = \max((s_{\min} - 1)L_T - BS_T, 0) \tag{29}
\]
where equation (26) is used. This is the equivalent of a sort of put option with strike price \( K_T = (s_{\min} - 1)L_T \).

### 3.1.2 Solvency level

The absolute minimum level of solvency required for a pension company according to SFSA is 104% \([1]\). However, at this point, the company is in a place where the SFSA has already given warnings and shut down their business. Therefore, companies need to make sure to have a buffer for their solvency ratio. The pension company will act when they reach their buffered solvency ratio so that they will not risk ending up at such a low level of solvency as 104%. The buffered solvency level is individually determined by each pension company.

### 3.2 Modelling stocks

The underlying stocks in the option can be modelled using different approaches. What is important, for both forward rates and stocks, is that the model includes the possibility of jumps or higher likelihood of large changes. When studying historical data it is clearly seen that jumps need to be included when modelling stocks as well as forward rates. It is possible to use for example Bates model or a discrete model based on normal log-normal mixture random variables to ensure that jumps are incorporated in the process.

The advantages of the Bates model is that it incorporates both a stochastic volatility as well as jumps. This reflects the behaviour of the stocks very well. However, one issue with
Bates model is that it is complex to determine the parameters in the model and several assumptions need to be made in order for the model to work. Making these assumptions leads to the model being less like the real world. It is also possible to use a simpler version of Bates model called Merton jump-diffusion model. This model is very similar with the exception of not having stochastic volatility. Although this model is simpler it would still require non-realistic assumptions in order to estimate parameters.

An alternative to Bates model is a discrete-time model based on normal log-normal mixture random variables. The model described below is under the real-world measure $\mathbb{P}$, i.e. all the random variables are modelled under this measure. The value of the stock at time $T$ is given by

$$S_T = E[S_T]e^{-\gamma_S + \varepsilon_S} \quad (30)$$

where $\varepsilon_S$ is a random variable and $\gamma_S$ is an adjustment factor. The random variable $\varepsilon_S$ could follow a normal distribution which would yield the standard Black-Scholes model. However, the objective is to use a model that has non-zero skewness and kurtosis and therefore $\varepsilon_S$ is chosen to be

$$\varepsilon_S = \sigma \sqrt{T} u_{S,\text{norm}}. \quad (31)$$

Here, $\sigma$ is the volatility of the stock movements, $T$ is the time to maturity and $u_{S,\text{norm}}$ is a normalized normal log-normal random variable given by the following, c.f. equation (4) in section 2.2,

$$u_{S,\text{norm}} = \frac{u_S - \mathbb{E}[u_S]}{\sqrt{\text{Var}(u_S)}}$$

where $u_S$ is a normal log-normal mixture random variable with parameters $\rho_S$ and $\sigma_S$, c.f equation (2). The expected value of $S_T$ in equation (30) is given by

$$\mathbb{E}[S_T] = S_0 e^{\mu_S T} \quad (32)$$

where $S_0$ is the value of the stock at time zero and $\mu_S$ is the drift. Since the expectation of the left hand side of equation (30) should equal the expectation of the right hand side of equation (30), $\gamma_S$ can be determined from

$$\mathbb{E}[S_T] = \mathbb{E}\left[\mathbb{E}[S_T]e^{(-\gamma_S + \varepsilon_S)}\right] = \mathbb{E}[S_T]e^{-\gamma_S} \mathbb{E}[e^{\varepsilon_S}]$$

where the second equality applies because $\mathbb{E}[S_T]$ and $\gamma_S$ are constants. This yields the following,

$$\gamma_S = \log \left(\mathbb{E}[e^{\varepsilon_S}]\right). \quad (33)$$

The final model can be written as

$$S_T = S_0 e^{\mu_S T} e^{-\gamma_S + \sigma \sqrt{T} u_{S,\text{norm}}} \quad (35)$$

where equations (30), (31) and (32) are used. The model will be calibrated to monthly data and therefore describe one month's movement. To simulate the stock’s value in $m$ months, discrete time steps will be taken, where the value of the stocks will be calculated one month
at a time. The price from month zero to month one is obtained by, according to equation (35),
\[ S_1 = S_0 e^{\mu S/12} e^{-\gamma_S + \sigma \sqrt{1/12} u_{S,norm}^1} \]
where the random variable \( u_{S,norm}^1 \) fulfills \( u_{S,norm}^1 \sim u_{S,norm} \) where \( u_{S,norm} \) is defined in equation (47). The value at month two is given by,
\[ S_2 = S_1 e^{\mu S/12} e^{-\gamma_S + \sigma \sqrt{1/12} u_{S,norm}^2} = S_0 e^{2\mu S/12} e^{-2\gamma_S + \sigma \sqrt{1/12}(u_{S,norm}^1 + u_{S,norm}^2)} \]
and by repeating this \( m \) times yields the value at month \( m \),
\[ S_m = S_0 e^{m\mu S/12} e^{-m\gamma_S + \sigma \sqrt{1/12} \sum_{j=1}^{m} u_{S,norm}^j}, \quad m \in \mathbb{Z}_+ \]
(36)
where \( u_{S,norm}^j \) are random variables and fulfill \( u_{S,norm}^j \sim u_{S,norm} \), \( j = 2, \ldots, m \). The random variables \( u_{S,norm}^j \) are pairwise independent of each other for \( j = 1, \ldots, m \).

### 3.3 Modelling bonds and liabilities

 Bonds and liabilities evolve over time as well as the stocks. The driving stochastic process of the value of the bonds and liabilities originates from the processes of the forward rates.

The choice of modelling the forward rates that govern the values of the bonds and liabilities stands between two options. The first is the Libor market model, see section 2.6, which describe continuous processes for all highly correlated forward rate processes. One advantage of using the Libor Market Model is that the forward rate processes are strongly correlated. However, it does not capture jumps which, when studying historical data of forward rates, is an important factor contributing to their behaviour. The second model is a discrete model based on normal log-normal random variables which includes jumps and correlation between the set of forward rates. Based on historical data on forward rates, some sort of jump-process is appropriate to include in the forward rate model and therefore the discrete model with normal log-normal random variables is chosen.

Historical par rates, \( \text{par}_{T_i} \), determined by the market for \( T_i = i, \quad i = 1, \ldots, 10, 15, 20 \) are given a priori. These are used to calculate the market forward rates, \( f_{\text{market},T_i} \), according to section 2.3. The market forward rate, \( f_{\text{market},T_i} \), is valued at time \( t \) and is valid between time \( T_i - 1 \) and \( T_i \).

#### 3.3.1 Value of bonds and liabilities

Both bonds and liabilities have coupon payments and when the forward rates change so will the present value of the future coupon payments. This is what governs the value of the bonds and liabilities. The difference in valuing the bonds from valuing the liabilities lies in different coupon values due to different maturity times. Liabilities are modelled with a longer time to maturity whilst the bonds have a shorter time to maturity.

It is assumed that the coupon payments for both bonds and liabilities are paid annually at times \( T_1, \ldots, T_{n-1} \) and at maturity time \( T_n \), together with the face value. Here, \( n \) corresponds to the number of years to maturity and takes different values for bonds, denoted by \( n_B \), and liabilities, denoted by \( n_L \). Denote the coupon for liabilities and bonds by \( c_L \) and \( c_B \).
respectively. The coupons can be determined from the regulatory discount factors at time zero to time $T_i$, denoted by $DF_{reg,0}^{T_i}$, c.f. equation (17) in section 2.3,

$$c_B \sum_{i=1}^{n_B-1} DF_{reg,0}^{T_i} + (1 + c_B)DF_{reg,0}^{T_B} = 1$$

$$c_L \sum_{i=1}^{n_L-1} DF_{reg,0}^{T_i} + (1 + c_L)DF_{reg,0}^{T_L} = 1.$$

The coupons are given by

$$c_B = \frac{1 - DF_{reg,0}^{T_B}}{\sum_{i=1}^{n_B} DF_{reg,0}^{T_i}}$$

$$c_L = \frac{1 - DF_{reg,0}^{T_L}}{\sum_{i=1}^{n_L} DF_{reg,0}^{T_i}}$$

(37)

The value of the liabilities at time $T$, $L_T$, can be calculated as follows,

$$L_T = L_0 \left( c_L \sum_{i=1}^{n_L-1} DF_{reg,T}^{T_i} + (1 + c_L)DF_{reg,T}^{T_L} \right)$$

(38)

where the regulatory discount factors $DF_{reg,T}^{T_i}$ applies between times $T$ and $T_i$. These regulatory discount factors are determined from the regulatory forward rates, as described in section 2.3. The value of the bonds can be calculated in a similar fashion,

$$B_T = B_0 \left( c_B \sum_{i=1}^{n_B-1} DF_{reg,T}^{T_i} + (1 + c_B)DF_{reg,T}^{T_B} \right).$$

(39)

As will be described in section 3.3.2 all historical forward rates are shifted up one percentage point, i.e. one percentage point is added to all forward rates, before calibrating the model. This is an engineering fix which ensures that the model can be used since there are negative historical forward rates and the model can only handle positive forward rates. However, before the discount factors can be determined from the simulated forward rates, the forward rates need to be shifted back one percentage point by simply subtracting one percentage point. In this thesis liabilities with maturity in $n_L = 20$ years and bonds with maturity in $n_B = x$ years where $x$ depends on which portfolio is studied.

In order to simulate values for $B_T$ and $L_T$, the discount factors $DF_{reg,T}^{T_i}$ are necessary. To calculate these, the market forward rates, $f_{market,T}^{T_i}$, need to be known. These are determined from the shifted forward rates $f_{reg,T}^{T_i}$, c.f. equation (46), which are simulated using the model described below, in section 3.3.2. The relationship between the simulated market forward rates, $f_{market,T}^{T_i}$ and the regulatory forward rates, $f_{reg,T}^{T_i}$, is as follows, c.f. equation (16),

$$f_{reg,T}^{T_i} = (1 - w(T_i)) \cdot f_{market,T}^{T_i} + w(T_i) \cdot UFR$$

(40)

where $w(T_i)$ is given by equation (15) and $UFR = 4.2\%$. Once the regulatory forward rates are known, the regulatory discount factors are calculated according to section 2.3. The regulatory discount factors, $DF_{reg,T}^{T_i}$, in terms of the regulatory forward rates, $f_{reg,T}^{T_i}$, are given by, c.f. equation (17).
The discount factors are determined annually. If coupon payments are due at intermediate times, the corresponding discount factors are determined by cubic spline interpolation as described in section 3.3.3.

3.3.2 Model of the forward rates

Denote the forward rate at time $t$, that applies to the time interval $[T_i - 1, T_i]$ by $f_{T_i}^T$. The following model models the forward rates under the real-world measure $\mathbb{P}$, i.e. the random variables will be $\mathbb{P}$-random variables. Assume that there are $n$ terms to maturity denoted by $T_1, T_2, \ldots, T_n$ and let $T_0$ be the current time $t$. This yields $n$ discrete-time stochastic processes for the forward rate, which are given by

$$f_{T_i}^T = \mathbb{E}[f_{T_i}^T e^{-\gamma_i \tilde{u}_i}], \quad i = 1, \ldots, n$$

(42)

where $\gamma_i$ is an adjustment factor, similar to $\gamma_S$ for the stocks, that ensures that the expected value of the forward rates is unbiased. Similarly to stocks, this model also includes normal log-normal random variables which are denoted by $\tilde{u}_i$. The random variables $\tilde{u}_i$ are strongly correlated for all forward rates $f_{T_i}^T, \quad i = 1, \ldots, n$. It is assumed that the forward rate $f_{T_i}^T$ has drift $\mu_i$ which yields the expected value of the forward rate as

$$\mathbb{E}[f_{T_i}^T] = e^{\mu_i T_i} f_{T_i}^T, \quad i = 1, \ldots, n$$

(43)

for $T > 0$. Since the random variables are strongly correlated, the goal is to reduce the model with correlated normal log-normal random variables to a model depending on a set of fewer uncorrelated normal log-normal random variables. The goal is to find constants, $\beta_{i,j}$, such that the forward rate model can be approximated by

$$f_{T_i}^T = f_{T_i}^T \exp \left( \mu_i T_i - \gamma_i + \sum_{j=1}^{m} \beta_{i,j} u_j \right)$$

where $u_j, \quad j = 1, \ldots, m$ is a set of uncorrelated normal log-normal random variables and $m < n$ is the number of uncorrelated random variables the initial model is approximated by.

As shown in appendix A and by Fusai and Roncoroni [14], the Libor market model can be reduced by PCA. They are able to transform a set of correlated normal random variables to a linear combination of uncorrelated normal random variables. This is done by eigenvalue decomposition of the covariance matrix between all transformed forward rate processes. This transformation is possible since a sum of normal random variables has a normal distribution as well. This implies that each Wiener process in the Libor market model can be described by a sum of uncorrelated Wiener processes corresponding to the orthogonal eigenvectors [14].

The task of reducing the model with normal log-normal random variables is more complex. First of all, for the PCA method to be applicable to this problem, the main requirement is that the sum of normal log-normal random variables has a normal log-normal distribution, possibly with other parameters. This, however, is difficult to prove since the probability distribution function, described in equation (7) to (9), for a normal log-normal random variable does not have an analytical expression. If this was the case, PCA could be used to reduce the initial model, given by equations (42) and (43), to a model consisting of fewer normal
log-normal random variables.

Often, it is sufficient to approximately explain the forward rate processes with three uncorrelated normal random variables, i.e. with the three largest principal components. These explain most of the variance in the movements. The largest principal components correspond to different types of movements [15].

In this thesis, it is not shown what distribution a sum of normal log-normal random variables has. Instead, a model inspired by the PCA-reduced Libor market model determined by Fusai and Roncoroni [14] is used. The model for the forward rate processes \( f_{Ti} \), \( i = 1, \ldots, n \), at time \( T \), is as follows

\[
f_{Ti} = f_{Ti}^0 e^{(\mu_i T - \gamma_i + \varepsilon_i)}, \quad i = 1, \ldots, n
\]  

(44)

where \( \varepsilon_i \) is a linear combination of three uncorrelated normal log-normal random variables given by

\[
\varepsilon_i = \sum_{j=1}^{3} u_{j,\text{norm}} \sqrt{\lambda_j} \sqrt{T} v_{j,i}, \quad i = 1, \ldots, n
\]  

(45)

where \( \lambda_j \) is the \( j \)th largest eigenvalue of the covariance matrix \( C \), described in greater detail below, and \( v_{j,i} \) is the \( i \)th component of the corresponding \( j \)th eigenvector. The covariance matrix, \( C \), is based on the on the historical log-returns of the forward rate, i.e. \( \log \left( \frac{f_{T_{i+1}}}{f_{T_i}} \right) \), denoted by \( z_{i,T} \). Hence, the \( i,j \)th entry in \( C \) is \( \text{Cov}(z_i, z_j) \). However, before performing these steps, it is necessary to ensure that the forward rates are non negative. Since, in recent times, some forward rates are negative, all forward rates are shifted upwards with one percentage point,

\[
f_{Ti} = f_{i,\text{market},T} + 0.01, \quad i = 1, \ldots, n
\]  

(46)

where \( f_{i,\text{market},T} \) are the forward rates determined by the market. Now, the matrix \( C \) can be determined from the shifted forward rates, \( f_{Ti} \). The shifting of the forward rates is an engineering fix, where the size of the shift needs to assure that all historical rates are positive. Also, the shift has some margin such that the shifted points are non-zero since the logarithm of zero is not finite.

The variable \( u_{j,\text{norm}} \), in equation \(45\), is a normalized normal log-normal random variable defined as the following, c.f. equation \(4\) in section \(2.2\)

\[
u_{j,\text{norm}} = \frac{u_j - \mathbb{E}[u_j]}{\sqrt{\text{Var}(u_j)}}, \quad j = 1, 2, 3
\]  

(47)

where \( u_j \) is a normal log-normal mixture random variable with parameters \( \rho_f \) and \( \sigma_f \), c.f. equation \(2\) in section \(2.2\). For the normal log-normal mixture random variable, one sample of \( \varphi \) and \( \eta \) are drawn per uncorrelated random variable which, by section \(2.2\), results in one \( u_j \) with a normal log-normal distribution. In equation \(14\), \( \gamma_i \) is an adjustment factor which is defined such that the expectation of the exponential term in equation \(14\) is equal to one, i.e.

\[
\gamma_i = \log \left( \mathbb{E}[e^{\varepsilon_i}] \right) = \log \left( \mathbb{E}[e^{\sum_{j=1}^{3} u_{j,\text{norm}} \sqrt{\lambda_j} \sqrt{T} v_{j,i}}] \right), \quad i = 1, \ldots, n.
\]  

(48)

The forward rate model, as well as the stock model, will be calibrated to monthly data. Therefore, the calibrated model describes one month’s evolution. To simulate the forward
rates in \( m \) months, discrete time steps are taken such that the forward rates can be calculated one month at a time. The method of obtaining the model describing the forward rates in \( m \) months is the same as described for stocks in section 3.2. The model is given by the following, where \( f_{T_i}^m \) is the forward rate between time \( T_i - 1 \) and \( T_i \), valued at month \( m \),

\[
f_{T_i}^m = f_0^{T_i} \exp \left( \mu_i m/12 - m \gamma_i + \sum_{k=1}^{m} \sum_{j=1}^{3} u_{j,norm}^k \sqrt{\lambda_j} \sqrt{1/12 v_{j,i}} \right), \quad m \in \mathbb{Z}_+. \tag{49}
\]

The random variables \( u_{j,norm}^k \) are equivalent to \( u_{j,norm} \), i.e. \( u_{j,norm}^k \sim u_{j,norm} \), for \( k = 1, \ldots, m \), which means that \( u_{j,norm}^k \) are normal log-normal random variables with parameters \( \rho_f \) and \( \sigma_f \). The random variables, \( u_{j,norm}^k \) are independent of each other for all \( k \) and \( j \).

### 3.3.3 Intermediate discount factors

As mentioned in section 3.3.1, regulatory discount factors are given as one for each year, for years \( T_i = i, \ i = 1, \ldots, 10, 15, 20 \). To determine the value of the bonds and liabilities, regulatory discount factors corresponding to intermediate times are necessary. To find these discount factors some sort of interpolation of the existing discount factors is useful. For this purpose, the cubic spline interpolation method is applied to find the regulatory discount factors at time \( T \) and valid until time \( T_i, DF_{reg}^{T_i} \), for all \( T_i \) that are needed in equations (38) and (39). The cubic spline interpolation method finds the best third degree polynomial for each interval between a priori given discount factors. These piece-wise functions are fit together to create a smooth curve, called a spline. Two constraints assuring smoothness are that both the first and second derivatives of the piece-wise functions need to agree at each a priori given point [16].

Other interpolation methods, such as polynomial interpolation, i.e. one polynomial function over the entire interval, would also work on this problem. However, cubic spline interpolation results in better fits than polynomial interpolation, since the curve intersects with the a priori given points.
4 Calibration and simulation

A method for calibrating the parameters needed when modelling stocks and forward rates is given in section 4.1. Since there has historically been a correlation between stocks and forward rates, a method for correlating stock and forward rate movements is introduced in section 4.2. When the calibration of the parameters and correlation is determined, the price of the option is calculated according to section 4.3 where a transformation from the real-world measure \( P \) to a risk-neutral measure \( Q \) is performed. The test portfolios used for simulations are presented in section 4.4. Finally, any assumptions made are presented in section 4.5.

4.1 Calibration of model parameters

In both the stock and forward rate model, there are parameters that need to be calibrated to the given data. The one period model describing the value of the stock is the following, c.f. equation (35),

\[
S_T = S_0 e^{\mu S T + \sigma S \sqrt{T} u_{S,norm}}. \tag{50}
\]

There are several parameters that need to be determined when calibrating to the historical data. Two of these are the parameters for the random variable \( u_{S,norm} \), i.e. \( \rho_S \) and \( \sigma_S \), which is done in section 4.1.1. Also, the volatility \( \sigma \) needs to be determined which is done in section 4.1.2. The computation of parameter \( \gamma_S \) is described in section 4.1.3. The one period model describing the forward rates is given by equations (44) and (45) combined, and is the following

\[
f_{T_i}^T = f_{T_i}^T (\mu_i T - \gamma_i + \sum_{j=1}^3 u_{j,norm} \sqrt{T} u_{j,i}), \quad i = 1, \ldots, n. \tag{51}
\]

The parameters that need to be estimated in this model are \( \gamma_i \) and the parameters for the random variables \( u_{j,norm} \). The estimation of the parameters of \( u_{j,norm} \), i.e. \( \rho_f \) and \( \sigma_f \) for \( j = 1, 2, 3 \), are done similarly to the estimation of \( \rho_S \) and \( \sigma_S \), which is described in section 4.1.1. The parameters \( \gamma_i \) are calculated in the same way as \( \gamma_S \) which is demonstrated in section 4.1.3.

4.1.1 Estimation of normal log-normal parameters

The parameters \( \rho \) and \( \sigma \), see equation (2), for the normal log-normal random variables for both stocks, \( \rho_S \) and \( \sigma_S \), and forward rates, \( \rho_f \) and \( \sigma_f \), are calibrated by simulating sets of samples from the normal log-normal distribution for different values of the parameters. Each set is compared with the historical data on log-returns for both the stocks and forward rates using a two-sample Kolmogorov-Smirnov test. This test is used to explore if samples from the two sets are from the same continuous distribution. The samples are assumed to be independently generated and are denoted by \( x_1, \ldots, x_n \). The null hypothesis, \( H_0 \), stating that the two distributions are identical, and the hypothesis that the two distributions are different, \( H_1 \), are the following

\[
H_0 : \quad F_1(x_i) = F_2(x_i), \quad \forall x_i
\]

\[
H_1 : \quad F_1(x_i) \neq F_2(x_i), \quad \text{for at least one } x_i
\]

where \( F_1(x) \) and \( F_2(x) \) are the distribution functions of the first respectively second distribution to test similarity between. The test statistics are
\[ D = \sup_{x_i} |F_2(x_i) - F_1(x_i)| \]

where for the observed value \( d \) of \( D \), the null hypothesis, \( H_0 \), is rejected if
\[ d > d_{1-\alpha} \]

where the \( d_{1-\alpha} \) is a critical value for the significance level \( \alpha \) \[17\]. This test is performed for the log-returns of the stocks and forward rates, respectively, with different normal log-normal distributions to find which distribution parameters are best suited for the historical data.

The distributions, with parameters chosen in accordance with the Kolmogorov-Smirnov test, are also verified using a quantile-quantile (Q-Q) plot. A Q-Q plot is a plot that compares the quantiles of two distributions. If the set of samples lie on a straight line, this would indicate that the samples come from the same distribution. The function \texttt{qqplot} in Matlab is used to generate Q-Q plot of two set of samples. These plots are dependent on the parameters \( \rho_S, \sigma_S, \rho_f \) and \( \sigma_f \) and therefore also the set of samples generated which yields slightly different plots for different parameters and different sets of samples \[18\]. Performing these two tests will yield estimates for the parameters corresponding to the normal log-normal random variable for stocks, \( \hat{\rho}_S \) and \( \hat{\sigma}_S \), and for forward rates, \( \hat{\rho}_f \) and \( \hat{\sigma}_f \).

### 4.1.2 Volatility estimation for the stock model

Since monthly historical data is available and the stock model in equation \[35\] applies to a time period from 0 to \( T \), the model is used between times \( T_i \) and \( T_{i+1} \). The time step between \( T_i \) and \( T_i+1 \) here is one month, i.e. \( T_i+1 - T_i = 1/12 \). According to equation \[35\], the monthly model for the stock is given by the following,
\[ S_{T_{i+1}} = S_{T_i} e^{\mu_S(T_{i+1} - T_i)} e^{-\gamma_S + \sigma \sqrt{T_{i+1} - T_i} u_{S,norm}}. \] \[52\]

As described in section \[3.2\], \( u_{S,norm} \) is a normal log-normal random variable with parameters \( \hat{\rho}_S \) and \( \hat{\sigma}_S \) estimated in section \[4.1.1\]. The monthly stock log-returns are given by
\[ \ln \left( \frac{S_{T_{i+1}}}{S_{T_i}} \right) = \mu_S(T_{i+1} - T_i) - \gamma_S + \sigma \sqrt{T_{i+1} - T_i} u_{S,norm}. \] \[53\]

To estimate the volatility \( \sigma \), the variance of both sides of equation \[53\] are computed, yielding
\[ Var \left( \ln \left( \frac{S_{T_{i+1}}}{S_{T_i}} \right) \right) = \sigma^2(T_{i+1} - T_i) Var(u_{S,norm}). \]

Since \( Var(u_{S,norm}) = 1 \), c.f. equation \[5\], the volatility \( \sigma \) is given by the following
\[ \sigma = \sqrt{\frac{Var \left( \ln \left( \frac{S_{T_{i+1}}}{S_{T_i}} \right) \right)}{T_{i+1} - T_i}}. \]

Define the log return over period \( T_i \) to \( T_{i+1} \), \( z_i \), by
\[ z_i = \ln \left( \frac{S_{T_{i+1}}}{S_{T_i}} \right) \]
and denote the general log return variable by $z$. The variance of the log returns is given by the variance of $z$ which is given by the following,

$$Var(z) = \frac{1}{n-1} \sum_{i=1}^{n} (z_i - \bar{z})^2$$

where $\bar{z}$ is the mean over all $z$ and $n$ is the number of data points available. From this, an estimate of the volatility, $\hat{\sigma}$, can be determined by

$$\hat{\sigma} = \sqrt{\frac{Var(z)}{\Delta T}}$$

where $\Delta T$ is the time step between each measured $S_T$.

### 4.1.3 Calculation of adjustment factor $\gamma$

In the models for stocks and forward rates, there is an adjustment factor $\gamma_S$ for the stocks, see equation (34), and one adjustment factor $\gamma_i$ for forward rate $f^i_T$, see equation (48). These adjustment factors depend on the expected value of the exponential of normalized normal log-normal random variables, c.f. equation (34) and (48). The expected value of the exponential of a normal log-normal random variable is infinite, see equation (3). However, it can be shown through simulations that the expected value of the exponential of a normalized normal log-normal random variable is finite, see section 5.2. This implies that $\gamma_S$ and $\gamma_i$, $i = 1, \ldots, 10, 15, 20$, are finite. For more details regarding normal log-normal random variables and the definition of the normalized normal log-normal random variable, see section 2.2.

The following procedure for calculating $\gamma$ is applicable to both stocks, $\gamma_S$, and each forward rate, $\gamma_i$, but will only be demonstrated for the stocks. The adjustment factor $\gamma_S$ is given by equation (34), i.e.

$$\gamma_S = \log (\mathbb{E}[e^{\xi_S}]) = \log \left( \mathbb{E} \left[ e^{\sigma \sqrt{T} u_{S,\text{norm}}} \right] \right) \quad (54)$$

where equation (31) is used in the last equality. The adjustment factor $\gamma_i$ can be determined from equation (48). The random variable $u_{S,\text{norm}}$ is a normalized normal log-normal random variable with properties $\mathbb{E}[u_{S,\text{norm}}] = 0$ and $Var(u_{S,\text{norm}}) = 1$ according to equations (5) and (6) in section 2.2 respectively. The probability distribution function for a normal log-normal random variable is complicated and does not have an analytical expression, see equations (7) to (9). Therefore, $\gamma_S$ is estimated numerically. After the parameters for $u_{S,\text{norm}}$, i.e. $\mu_S$ and $\sigma_S$, and $\sigma$ have been estimated according to section 4.1.1 and 4.1.2 respectively, $\gamma_S$ is determined by sampling a large number, $i = 1, \ldots, n$, of values for $u_{S,\text{norm}}$, denoted by $u_{S,\text{norm}}^i$. Estimating the expected value of the exponential by an average over all sampled values according to the following,

$$\mathbb{E} \left[ e^{\sigma \sqrt{T} u_{S,\text{norm}}} \right] \approx \frac{1}{n} \sum_{i=1}^{n} e^{\sigma \sqrt{T} u_{S,\text{norm}}^i}.$$ 

This applies due to the law of large numbers, see section 2.4.1 where increasing the number of samples will result in a more accurate value for the expected value. Note that the expected value of the exponent of a normalized normal log-normal random variable is finite while for a normal log-normal random variable, the corresponding expected value is infinite.
4.2 Correlation between first principal component and stocks

The historical correlation between stocks and forward rates is given by

$$\rho_{hist} = \text{corr}(S_T, f_T^T) = \frac{\text{cov}(S_T, f_T^T)}{\sigma_f \sigma}$$

(55)

where \(\text{cov}(S_T, f_T^{T-1})\) is the covariance between the log-returns of the forward rate, between times \(T_i - 1\) and \(T_i\), and the log-returns of the stocks. Variables \(\sigma\) and \(\sigma_f\) are the standard deviations of the log-returns of the stocks and forward rates. When studying the historical correlation between stocks and forward rates, depending on which stock is used, there is often a correlation between these processes. After the sets of variables \(\varphi\) and \(\eta\) that govern the normal log-normal random variables, have been generated for both stocks and forward rates, it is desired to correlate these variables to make the stocks and forward rates processes correlated. The first component, corresponding to the largest eigenvalue of \(C\) in section 3.3.2, \(u_{1,\text{norm}}\), is the most significant since it explains the largest variance. Due to this, the corresponding \(\varphi_{f_1}\) and \(\eta_{f_1}\) are correlated with \(\varphi_S\) and \(\eta_S\) that are generated for the stocks. The goal is to correlate \(\varphi_S\) with \(\varphi_{f_1}\) and \(\eta_S\) with \(\eta_{f_1}\) with a correlation factor \(\rho_{f-S}\),

$$\begin{bmatrix} \varphi_S \\ \eta_S \end{bmatrix} \leftarrow \rho_{f-S} \begin{bmatrix} \varphi_{f_1} \\ \eta_{f_1} \end{bmatrix}. \quad \text{(56)}$$

The correlation between the variables in equation (56) is made by keeping \(\varphi_S\) and \(\eta_S\) fixed and recalculating \(\varphi_{f_1}\) and \(\eta_{f_1}\) with the correlation \(\rho_{f-S}\) using the following formulas

$$\varphi_{f_1,\text{corr}} = \sigma_f \left( \rho_{f-S} \varphi_S + \left(\sqrt{1 - \rho_{f-S}^2}\right) \varphi_{f_1} \right)$$

$$\eta_{f_1,\text{corr}} = \sigma_f \left( \rho_{f-S} \eta_S + \left(\sqrt{1 - \rho_{f-S}^2}\right) \eta_{f_1} \right).$$

This should yield a correlation between the simulated processes for the stocks, \(S_T\), and the forward rates where the aim is to be as close as possible to the historical correlation determined according to equation (55). Since the correlation factor \(\rho_{f-S}\) has an upper boundary of one and a lower boundary of minus one, \(n\) values within this interval is used to calculate the RMSE between the resulting correlation between the simulated stock process and the forward rates processes and the historical correlation. The RMSE for the \(n\):th correlation factor \(\rho_{f-S,n}\) is determined as the following,

$$\text{RMSE}_n = \sqrt{\frac{1}{12} \sum_{i=1}^{12} (\rho_{f-S,n} - \rho_{hist})^2}.$$

The \(\rho_{f-S,n}\) which results in the smallest RMSE is chosen. Since, according to equation (2), \(\varphi_{f_1,\text{corr}}\) should have variance equal to one and \(\eta_{f_1,\text{corr}}\) should have variance equal to \(\sigma_f^2\). The variance of \(\varphi_{f_1,\text{corr}}\) is given by

$$\text{Var}(\varphi_{f_1,\text{corr}}) = \rho_{f-S}^2 \text{Var}(\varphi_S) + (1 - \rho_{f-S}^2) \text{Var}(\varphi_f).$$

Since \(\text{Var}(\varphi_S) = 1\) and \(\text{Var}(\varphi_f) = 1\), this yields \(\text{Var}(\varphi_{f_1,\text{corr}}) = 1\) and no adjustment is needed. The variance of \(\eta_{f_1,\text{corr}}\) is given by

$$\text{Var}(\eta_{f_1,\text{corr}}) = \rho_{f-S}^2 \text{Var}(\eta_S) + (1 - \rho_{f-S}^2) \text{Var}(\eta_f), \quad \text{(57)}$$

where \(\text{Var}(\eta_S) = \sigma_S^2\) and \(\text{Var}(\eta_f) = \sigma_f^2\). The variance of \(\eta_{f_1,\text{corr}}\) is desired to be given by,
$\text{Var}(\mathbf{\xi}_f) = \sigma_f^2$ and hence $\eta_f_{\text{corr}}$ should be multiplied by a factor $\mathbf{\xi}$ such that

$$\text{Var}(\mathbf{\xi}_f \eta_{f_{\text{corr}}}) = \mathbf{\xi}^2 \left( \rho_f - \mathbf{\sigma} f^2 + (1 - \rho_f - \mathbf{\sigma}) \mathbf{\sigma}^2 \right).$$

(58)

The adjustment factor, $\mathbf{\xi}$, that is multiplied with $\eta_f_{\text{corr}}$ is therefore determined by

$$\mathbf{\xi} := \frac{\sigma_f}{\sqrt{\rho_f - \mathbf{\sigma} f^2 + (1 - \rho_f - \mathbf{\sigma}) \mathbf{\sigma}^2}}.$$

4.3 Calculation of option price

The models for stocks and forward rates in section 3 are under the real-world measure $\mathbb{P}$. In order to retrieve a risk-neutral price of the option, a risk-neutral measure $\mathbb{Q}$ is chosen and the models are defined under this measure.

4.3.1 Choosing a risk-neutral measure

Suppose that there is a risk-free bank account with value $BA_m$ at month $m$ and interest rate $r$. The interest rate is chosen as the three month Stibor rate and the value of the bank account at month $m$, $BA_m$, is given by

$$BA_m = BA_0 e^{r m}, \quad m \in \mathbb{Z}_+,$$

(59)

where $BA_0$ is the initial value of the bank account. At each month, there are an infinite number of possible values that the stocks, bonds and liabilities can assume. This is due to the fact that the values are functions of continuous random variables. Since the values are calculated at discrete times, the market is incomplete. This means that it is not possible to find a unique risk-neutral measure $\mathbb{Q}$ and therefore, one $\mathbb{Q}$ is chosen amongst several possible. There are different possible approaches to pricing the option. The approach used in this thesis is based on the assumption that stocks and forward rates are the only tradable instruments on the market. Ideally the only tradable instruments on the market would be stocks, bonds and liabilities. However, since the forward rates are simulated and the bonds and liabilities depend of the forward rates, this task becomes more complex. The measure $\mathbb{Q}$ should therefore be chosen such that $\frac{S_m}{BA_m}$ and $\frac{\mathbb{E}_m^Q}{BA_m}$ for $i = 1, \ldots, 10, 15, 20$ are martingales under this measure. Since there are several martingale conditions, there will be several conditions on the risk-neutral measure $\mathbb{Q}$. The first one originates from the martingale condition on $\frac{S_m}{BA_m}$ given by

$$\mathbb{E}_m^Q \left[ \frac{S_m}{BA_m} \right] = \mathbb{E}_0^Q \left[ \frac{S_0}{BA_0} \exp \left( (\mu_m - r) \frac{m}{12} - m \gamma_m + \sigma \sqrt{1/12} \sum_{j=1}^{m} \mathbb{I}_{n_{m,norm}} \right) \right] =$$

$$= \frac{S_0}{BA_0} \mathbb{E}_0^Q \left[ \exp \left( (\mu_m - r) \frac{m}{12} - m \gamma_m + \sigma \sqrt{1/12} \sum_{j=1}^{m} \mathbb{I}_{n_{m,norm}} \right) \right] =$$

$$= \{\text{want} \} = \frac{S_0}{BA_0}.$$  

(60)

For this to be fulfilled, one condition on the risk-neutral measure $\mathbb{Q}$ is given by

$$\mu_m \frac{m}{12} + \sigma \sqrt{1/12} \sum_{j=1}^{m} \mathbb{I}_{n_{m,norm}} \sim r \frac{m}{12} + \sigma \sqrt{1/12} \sum_{j=1}^{m} \mathbb{I}_{n_{m,norm}},$$

(61)
where $\tilde{u}_{j,norm}^Q \overset{\text{NLN}}{=} \text{NLN}(\rho_S, \sigma_S)$, i.e. a normal log-normal random variable under $Q$ with the same parameters as $u_{j,norm}^S$, $\rho_S$ and $\sigma_S$. A confirmation that this condition on $Q$ ensures that $\frac{S_m}{BA_m}$ is a martingale is as follows,

$$E^Q \left[ \exp \left( (\mu_S - r) \frac{m}{12} - m \gamma_S + \sigma \sqrt{1/12} \sum_{j=1}^{m} \tilde{u}_{j,norm}^S \right) \right] =$$

$$= E^Q \left[ \exp \left( -m \gamma_S + \sigma \sqrt{1/12} \sum_{j=1}^{m} \tilde{u}_{j,norm}^S \right) \right] =$$

$$= e^{-m \gamma_S} E^Q \left[ \exp \left( \sigma \sqrt{1/12} \sum_{j=1}^{m} \tilde{u}_{j,norm}^S \right) \right] = \{ \tilde{u}_{j,norm}^S \text{ indep of each other} \} = \text{equation (54)}$$

$$= e^{-m \gamma_S} \prod_{j=1}^{m} e^{\gamma_S} = e^{-m \gamma_S} e^{m \gamma_S} = 1.$$  

According to equation (62), $\frac{S_m}{BA_m}$ is a martingale under $Q$ if a condition on $Q$ is the one stated in equation (61). The second condition on $Q$ originates from the martingale condition on $\frac{f_{T_i}}{BA_m}$ given by

$$E^Q \left[ \frac{f_{T_i}}{BA_m} \right] = E^Q \left[ \frac{f^{T_i}_{0}}{BA_0 e^{r \frac{m}{12}}} \exp \left( \mu_i m/12 - m \gamma_i + \sum_{k=1}^{3} \sum_{j=1}^{m} u_{j,norm}^k \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] =$$

$$= \frac{f^{T_i}_{0}}{BA_0} E^Q \left[ \exp \left( \mu_i m/12 - m \gamma_i + \sum_{k=1}^{3} \sum_{j=1}^{m} u_{j,norm}^k \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] =$$

$$= \{ \text{want} \} = \frac{f^{T_i}_{0}}{BA_0}. \text{ equation (63)}$$

For this martingale condition to be satisfied, the second condition on the risk-neutral measure $Q$ is given by

$$\mu_i \frac{m}{12} + \sum_{k=1}^{3} \sum_{j=1}^{m} \left( u_{j,norm}^k \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \sim \frac{m}{12} + \sum_{k=1}^{3} \sum_{j=1}^{m} \left( \tilde{u}_{j,norm}^k \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right), \forall i \text{ equation (64)}$$

where $\tilde{u}_{j,norm}^k \overset{\text{NLN}}{=} \text{NLN}(\rho_f, \sigma_f)$, i.e. a normal log-normal random variable under $Q$ with the same parameters as $u_{j,norm}^k$. This condition on $Q$ ensures that $\frac{f_{T_i}}{BA_m}$ is a martingale which is demonstrated by the following,
\[
E^Q \left[ \exp \left( \mu_i m/12 - m \gamma_i + \sum_{k=1}^{m} \sum_{j=1}^{3} \tilde{u}_{j,norm} \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] = \\
= E^Q \left[ \exp \left( -m \gamma_i + \sum_{k=1}^{m} \sum_{j=1}^{3} \tilde{u}_{j,norm} \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] = \\
= e^{-m \gamma_i} E^Q \left[ \exp \left( \sum_{k=1}^{m} \sum_{j=1}^{3} \tilde{u}_{j,norm} \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] = \{\tilde{u}_{j,norm} \text{ indep of each other}\} = \\
= e^{-m \gamma_i} \prod_{j=1}^{m} E^Q \left[ \exp \left( \sum_{k=1}^{3} \tilde{u}_{j,norm} \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right) \right] = \{\text{equation (48)}\} = \\
= e^{-m \gamma_i} \prod_{j=1}^{m} e^{\gamma_i} = e^{-m \gamma_i} e^{m \gamma_i} = 1. \quad (65)
\]

By equation (65), \( f_i^{T_m} \) is a martingale under \( Q \) if a condition on \( Q \) is the one stated in equation (64).

In summary, if \( Q \) is chosen such that the conditions in equations (61) and (64) are fulfilled, \( Q \) will be the risk-neutral measure that can be used for computing a risk-neutral price. Since the risk-neutral measure \( Q \) is not unique, the price will not be unique either. The stock model under \( Q \) is given by,

\[
S_m = S_0 \exp \left( \frac{m}{12} - m \gamma_S + \sigma \sqrt{1/12} \sum_{j=1}^{m} \tilde{u}_{j,norm} \right), \quad m \in \mathbb{Z}_+. \quad (66)
\]

and the forward rates under \( Q \) are modelled as

\[
f_i^{T_m} = f_i^{T_0} \exp \left( r \frac{m}{12} - m \gamma_i + \sum_{k=1}^{m} \sum_{j=1}^{3} \tilde{u}_{j,norm} \sqrt{\lambda_j} \sqrt{1/12} v_{j,i} \right), \quad m \in \mathbb{Z}_+. \quad (67)
\]

4.3.2 Option price simulation

To determine what it will cost to remain above the lower boundary solvency level \( s_{min} \) for a year, four options with maturity in three months will be bought. Since the first option is valid for the first quarter, the second one for the second quarter and so forth, the price of the four options corresponding to entering into the contract today, in three, six respectively nine months need to be discounted. The price of the one year option, \( p_{1Y} \), is approximated using the following discount formula,

\[
p_{1Y} = p_{3M} \left( 1 + e^{-r \frac{3}{12}} + e^{-r \frac{6}{12}} + e^{-r \frac{9}{12}} \right),
\]

where \( p_{3M} \) is the value of the three month option and \( r \) is the interest rate of the bank account. This is a simplification of the problem since the value of the building blocks of the balance sheet in three months will not be the same as today and therefore a new option should be priced based on the new initial portfolio in three months. The same reasoning applies for the option bought in six months and in nine months. To be completely correct, an option lasting 12 months and that would be exercisable at any time as soon as the solvency level of the balance sheet goes below the chosen minimum solvency level, would be ideal.
This would also require that the option is valid all 12 months, meaning that the option should be exercisable multiple and an unlimited number of times during the year. This is a complex option to model, hence the version determining the cost by four options, solely exercisable on maturity date, are used to prove that there is a cost and to indicate its size.

The initial discount factors, $DF_{reg,0}^{T_i} \ i = 1, 2, \ldots, 10, 15, 20$, are determined according to section 2.3 using the historical regulatory forward rates, $f_{reg,0}^{T_i}$ from the historical par rates as input. Coupons are then determined by equation (67) for the bonds and liabilities based on their different maturity times. The initial solvency ratio, $s_0$, of the balance sheet is determined from the following, c.f. equation (27),

$$s_0 = \frac{S_0 + B_0}{L_0}$$

where $S_0, B_0$ and $L_0$ are the initial values of the stocks, bonds and liabilities respectively. The lower solvency boundary is then determined as

$$s_{\min} = s_0 - b$$

where $b$ denotes the amount that the pension company will allow their solvency level to decrease by before actions need to be taken.

The risk-neutral price of the balance sheet option with maturity in three months is given by the following [13],

$$p_{3M} = \mathbb{E}^Q \left[ \Phi(S_{3M}, B_{3M}, L_{3M}) \right] = \mathbb{E}^Q \left[ \max \left( s_{\min}L_{3M} - S_{3M} - B_{3M}, 0 \right) \right]$$

(68)

where the payoff function $\Phi$ is given in equation (29), $S_{3M}$ denotes the value of the stock in three months, $B_{3M}$ the value of the bonds in three months, $L_{3M}$ the value of the liabilities in three months and $BA_{3M}$ the value of the bank account in three months. The value of the bank account in three months is deterministic and given by equation (59). The price of the three month option can be approximated using quasi-Monte Carlo simulations, as described in section 2.4,

$$p_{3M} \approx \frac{1}{BA_{3M}} \sum_{l=1}^{n} \max \left( s_{\min}L_{3M}^l - S_{3M}^l - B_{3M}^l, 0 \right)$$

(69)

where $S_{3M}^l, B_{3M}^l$ and $L_{3M}^l$ are simulated values of the stock, bonds and liabilities in three months under the risk-neutral measure $\mathbb{Q}$, respectively. The value of the bank account is $BA_{3M} = \exp(r \frac{n}{12})$ and $n$ is the total number of simulations. All simulated values are under the risk-neutral measure $\mathbb{Q}$. The value of the stock is simulated using the model under $\mathbb{Q}$ which is described in equation (66). The stock and forward rates are generated such that they are correlated with a correlation that is as similar to the historical one as possible. This is described in section 4.2.

The values of the bonds and liabilities are implicitly determined by the simulated forward rates in three months, $f_{3M}^{T_i,l}$. These forward rates are simulated under $\mathbb{Q}$ according to the model described in equation (67). The simulated forward rates $f_{3M}^{T_i,l}$ need to be shifted back by one percentage point to get the market forward rates $f_{market,3M}^{T_i,l}$, which is done according to equation (46). The market forward rates, $f_{3M}^{T_i,l}$, are converted to regulatory forward rates, $f_{reg,3M}^{T_i,l}$, according to equation (40) and from the regulatory forward rates the discount factors, $DF_{reg,3M}^{T_i,l}$ are calculated using equation (41). The discount factors corresponding to
intermediate times, i.e. times between which discount factors are known, are interpolated using a cubic spline interpolation according to section 3.3.3. Using the discount factors together with the interpolated discount factors, the value of the bonds and of the liabilities can be retrieved from equation (39) and (38), respectively.

Sobol’ random numbers, see section 2.4.2, are used to generate normal log-normal samples from $u_{S,norm}$ and $u_{j,norm}$, $j = 1, 2, 3$ in accordance with section 2.2. These normal log-normal samples determine values of $S_{iM}^t$ and $f_{3M}^{T_i}$. These samples are generated as a set. When the dimension of this set is small, the correlation between the random numbers will be close to zero which is the case for these simulations and therefore no extensive methods are needed.

The relative price of the option with regards to the initial value of the balance sheet, i.e. own funds, is given by

$$p_{rel}^{1Y} = 100 \cdot \frac{p_{1Y}}{BS_0} [%] = 100 \cdot \frac{p_{1Y}}{S_0 + B_0 - L_0} [%],$$

where equation (26) is used in the second equality.

### 4.4 Test portfolios

Two portfolios, provided by Captor, are used for the simulations in this thesis. The data for these portfolios consists of monthly par rates for SEK Swap 3M for 1Y-10Y, 15Y, 20Y and monthly stock prices for different stocks from 2009-01-30 to 2019-02-28. The first portfolio, portfolio A, consists of stocks, bonds and liabilities as shown in Table 1 and the allocation for the second portfolio, portfolio B, is shown in Table 2.

<table>
<thead>
<tr>
<th>Share of total asset value</th>
<th>Name or maturity time</th>
</tr>
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<tbody>
<tr>
<td>Global stocks</td>
<td>20%</td>
</tr>
<tr>
<td></td>
<td>MSCI World Developed</td>
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<tr>
<td></td>
<td>(global equity index)</td>
</tr>
<tr>
<td>Swedish stocks</td>
<td>20%</td>
</tr>
<tr>
<td></td>
<td>SBX</td>
</tr>
<tr>
<td></td>
<td>(swedish equity index)</td>
</tr>
<tr>
<td>Bonds</td>
<td>60%</td>
</tr>
<tr>
<td></td>
<td>Maturity time = 5 years</td>
</tr>
<tr>
<td>Liabilities</td>
<td>$100_{1.25}$ %, $100_{1.5}$ % or $100_{1.75}$ %</td>
</tr>
<tr>
<td></td>
<td>Maturity time = 20 years</td>
</tr>
</tbody>
</table>

Table 1: Assets and liabilities allocation for portfolio A.
<table>
<thead>
<tr>
<th></th>
<th>Share of total asset value</th>
<th>Name or maturity time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global stocks</td>
<td>20%</td>
<td>Captor Scilla Global Equity (global equity fund)</td>
</tr>
<tr>
<td>Swedish stocks</td>
<td>20%</td>
<td>Captor Scilla Nordic Equity (nordic equity fund)</td>
</tr>
<tr>
<td>Bonds</td>
<td>60%</td>
<td>Maturity time = 4 or 11 years</td>
</tr>
<tr>
<td>Liabilities</td>
<td>$100%$, $100%$ or $100%$</td>
<td>Maturity time = 20 years</td>
</tr>
</tbody>
</table>

Table 2: Assets and liabilities allocation for portfolio B.

Stocks and bonds correspond to the left side of the balance sheet, i.e. the assets, and the liabilities correspond to the right side of the balance sheet where, except for liabilities, there are own funds. See illustration in Figure 4.

![Figure 4: Illustration of the balance sheet. Here $x$ denotes the share of the total asset value which corresponds to the value of the liabilities.](image)

The liabilities are assumed to have a maturity time equal to 20 years for both portfolios and different initial values are used to give the portfolios different initial solvency levels. Both portfolios consists of 40% stocks and 60% bonds.

### 4.5 Assumptions

In this thesis, it is, for simplicity, assumed that the volatility for the stocks and the forward rates, based on historical data, is constant over time. It is also assumed that the correlation between the stocks and the forward rates is constant over time for the same reason. The assets are assumed to solely consist of bonds and stocks and the liabilities of FTA (actuarial provisions). A small part of the liabilities for pension companies apart from the FTA is dependent on life expectancy and not market risk which the FTA is. However, there is no correlation between that part and the FTA or the assets consisting of stocks and bonds. The associated risk is very small as well which is why it is neglected in this thesis.
5 Results

In this section, the results from previous sections are demonstrated with plots and associated explanations for the test portfolios. How the parameters in the model are calibrated is found in section 5.1. The results regarding convergence of the $\gamma$ parameter and correlation between stocks and forward processes are found in section 5.2 and 5.3, respectively. Simulation of paths for the stocks and forward rates compared to historical paths are found in section 5.4. The final price of the option for different initial solvency levels is demonstrated in section 5.5.

5.1 Calibration of parameters

Calibration of the parameters for the normal log-normal random variables is performed according to section 4.1.1 and the parameters are chosen in accordance with the Kolmogorov-Smirnov test. Q-Q plots are used to demonstrate the results.

5.1.1 Portfolio A

The Q-Q plot for the stocks and for the mean of the forward rates in portfolio A can be found in Figure 5. By studying these Q-Q plots, it can be seen that they are distributions that fits normal log-normal distributions well, even though there are a few outliers.

![Figure 5: Q-Q plot of samples from the normal log-normal distribution against the log returns from the stocks (left) and forward rates (right), based on historical data.](image)

5.1.2 Portfolio B

The Q-Q plot for the stocks and for the mean of the forward rates in portfolio B can be found in Figure 6. These Q-Q plots shows a distribution that fits a normal log-normal distribution well but has a few outliers.
5.2 Convergence of $\gamma$-parameter

The convergence of the $\gamma$-parameter is demonstrated by studying how the value of the parameter changes when the number of iterations increase. As mentioned in section 4.1.3, $\gamma$ does not converge when the normal log-normal random variable, $u$, is not normalized. However, since $u$ is normalized, it is clearly seen in below figures that $\gamma$ converges.

5.2.1 Portfolio A

The convergence of $\gamma_S$ and $\gamma_i$ for forward rate $f_{T_i}^T$ can be found in Figure 7. Note that the scale of the y-axis is small, making the fluctuations look large.

The changes in the expected value appears to be small for the larger numbers of iterations and therefore, it can be seen that the $\gamma$-parameter for both the stocks and for all 12 forward rates converges. The value of $\gamma$ appears to be close to one for both the stocks and for all forward rates.
5.2.2 Portfolio B

The convergence of $\gamma_S$ and $\gamma_i$ for forward rate $f_{T_i}$ for portfolio B can be found in Figure 8.

![Convergence of expected value, stocks](chart1.png)  ![Convergence of expected value, forward rates](chart2.png)

**Figure 8:** Illustration of how the adjustment factor $\gamma_S$ and $\gamma_i$ for forward rate with maturity time $T_i$ converges.

Similarly to portfolio A, the changes in the expected value are small for the larger numbers of iterations indicating that the $\gamma$-parameters for both the stocks and for all 12 forward rates converges. The value of $\gamma$ appears to be close to one for all forward rates and for the stocks.

5.3 Correlation

Correlation between the stocks and forward rates is determined according to section 4.2. The forward rates are the same for portfolio A and B, however, since different stocks are used in the two portfolios, there will be different correlations between the different assets for the portfolios.

5.3.1 Portfolio A

In Figure 9, the root mean square error, RMSE, between the historical correlation and the simulated correlation between the stocks and forward rates is illustrated. The RMSE is shown for different values of the correlation factor between the stocks and the first principal component for the forward rates. The correlation factor corresponding to the smallest RMSE is chosen since this results in a correlation between the forward rates and the stocks which is closest to the historically observed correlation.

In Figure 10, the historical correlation between the forward rates and the stocks is plotted against the simulated correlation between the forward rates and the stocks. The correlation factor corresponding to the smallest RMSE is used. The figure shows that the simulated correlation follows the historical correlation quite well.
Figure 9: The root mean square error between the historical and simulated correlation as a function of the correlation factor between the stocks and the first principal component for the forward rates.

Figure 10: Illustration of how well the simulated correlation between the stocks and the 12 forward rates imitates the historical correlation.

In this case, the correlation appears to be close to zero. This could be due to the specific stock indices used for this portfolio.
5.3.2 Portfolio B

In Figure [11], the RMSE between the correlation determined from historical data and the correlation between the simulated stocks and forward rates values is illustrated. It is plotted as a function of the correlation factor between the stocks and the first principal component for the forward rates. The correlation factor which results in the smallest RMSE is chosen.

![Diagram showing RMSE between historical and simulated correlation as a function of the correlation factor between the stocks and the first principal component for the forward rates.]

Figure 11: The root mean square error between the historical and simulated correlation as a function of the correlation factor between the stocks and the first principal component for the forward rates.

In Figure [12] the historical correlation between the forward rates and the stocks is compared to the correlation between the forward rates and the stocks. The correlation factor which results in the smallest RMSE is used. As can be seen in the figure, the simulated correlation follows the historical correlation quite well.
Figure 12: Illustration of how well the simulated correlation between the stocks and the 12 forward rates imitates the historical correlation.

The correlation is slightly negative for portfolio B which indicates that there is a connection between the behaviour of the stocks and forward rates and that their changes tend to occur in opposite directions.

5.4 Comparison between simulated and historical paths

The values of the stocks and forward rates are simulated according to section 3.2 and 3.3 respectively. Below are examples of how the simulated paths can look. These paths are demonstrated together with the historical paths in order to put the simulated paths in perspective.

5.4.1 Portfolio A

An illustration of the simulation of the stock price and forward prices in comparison with the historical prices can be found in Figure 13.
The figure shows that the simulated behaviour fits quite well with the historical behaviour of the process. As desired, movements similar to jumps for both processes can be observed.

### 5.4.2 Portfolio B

The simulation of the stock price and forward prices in contrast to the historical prices for portfolio B can be found in Figure 14. By studying the figure, it seems that the simulated behaviour agrees with the historical behaviour relatively well.

### 5.5 Price of balance sheet option

The price of the balance sheet option is simulated according to section 4.3. The price in relation to the initial value of the own funds indicates how many percent of the own funds that the pension company would need for the solvency level to stay above \( s_{\min} \) for a year. The relative price, i.e. price of the option over own funds, is calculated for several initial solvency levels where the lower solvency bound varies from 15 percentage points less than the
initial solvency level up to the initial solvency level. This is to demonstrate how the choice or determination of $s_{\text{min}}$ affects the price of the option. The following results are based on 500,000 simulations per price, i.e. per lower solvency boundary. To simulate the price for very low solvency levels would require a lot of simulations to obtain a non-zero value since the likelihood of large enough changes of value over the studied time-period would be close to zero.

5.5.1 Portfolio A

The relative prices are calculated with initial solvency levels 125%, 150% and 175%. The corresponding relative prices can be found in the top figure, middle figure and lower figure of Figure 15 respectively. The prices for respective initial solvency level are also given in Table 3.

<table>
<thead>
<tr>
<th>Lower solvency bound</th>
<th>Price</th>
<th>Lower solvency bound</th>
<th>Price</th>
<th>Lower solvency bound</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.10</td>
<td>0.01%</td>
<td>1.35</td>
<td>0.03%</td>
<td>1.60</td>
<td>0.10%</td>
</tr>
<tr>
<td>1.12</td>
<td>0.04%</td>
<td>1.37</td>
<td>0.08%</td>
<td>1.62</td>
<td>0.22%</td>
</tr>
<tr>
<td>1.14</td>
<td>0.12%</td>
<td>1.39</td>
<td>0.21%</td>
<td>1.64</td>
<td>0.46%</td>
</tr>
<tr>
<td>1.16</td>
<td>0.37%</td>
<td>1.41</td>
<td>0.55%</td>
<td>1.66</td>
<td>0.93%</td>
</tr>
<tr>
<td>1.19</td>
<td>1.10%</td>
<td>1.44</td>
<td>1.36%</td>
<td>1.69</td>
<td>1.83%</td>
</tr>
<tr>
<td>1.21</td>
<td>3.16%</td>
<td>1.46</td>
<td>3.13%</td>
<td>1.71</td>
<td>3.45%</td>
</tr>
<tr>
<td>1.23</td>
<td>8.49%</td>
<td>1.48</td>
<td>6.52%</td>
<td>1.73</td>
<td>6.15%</td>
</tr>
<tr>
<td>1.25</td>
<td>18.80%</td>
<td>1.50</td>
<td>12.20%</td>
<td>1.75</td>
<td>10.28%</td>
</tr>
</tbody>
</table>

Table 3: The relative prices of the option for portfolio A shown for different lower solvency bounds and initial solvency levels.
Figure 15: Portfolio A. All figures show price of option in relation to initial value of balance sheet as a function of different lower solvency level limits. The initial solvency level is 125% in top, 150% in middle and 175% in bottom figure. The maturity time of the bonds is five years.
5.5.2 Portfolio B and maturity time 4 year

The relative prices are calculated for the same three initial solvency levels as for portfolio A. The prices can be found in the top figure, middle figure and lower figure of Figure [16] respectively. The prices for respective initial solvency level can also be found in Table [4].

<table>
<thead>
<tr>
<th>Initial solvency: 1.25</th>
<th>Initial solvency: 1.50</th>
<th>Initial solvency: 1.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower solvency bound</td>
<td>Price</td>
<td>Lower solvency bound</td>
</tr>
<tr>
<td>1.10</td>
<td>0.02%</td>
<td>1.35</td>
</tr>
<tr>
<td>1.12</td>
<td>0.05%</td>
<td>1.37</td>
</tr>
<tr>
<td>1.14</td>
<td>0.18%</td>
<td>1.39</td>
</tr>
<tr>
<td>1.16</td>
<td>0.56%</td>
<td>1.41</td>
</tr>
<tr>
<td>1.19</td>
<td>1.56%</td>
<td>1.44</td>
</tr>
<tr>
<td>1.21</td>
<td>4.07%</td>
<td>1.46</td>
</tr>
<tr>
<td>1.23</td>
<td>9.74%</td>
<td>1.48</td>
</tr>
<tr>
<td>1.25</td>
<td>20.54%</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Table 4: The relative prices of the option for portfolio B, where the bonds have maturity time four years, shown for different lower solvency bounds and initial solvency levels.
Figure 16: Portfolio B. All figures show price of option in relation to initial value of balance sheet as a function of different lower solvency level limits. The initial solvency level is 125% in top, 150% in middle and 175% in bottom figure. The maturity time of the bonds is four years.
5.5.3 Portfolio B and maturity time 11 years

The same initial solvency levels as for portfolio A is used to determine the relative prices which can be found in the top, middle and bottom of Figure 17 respectively. These prices are also given in Table 5.

<table>
<thead>
<tr>
<th>Initial solvency: 1.25</th>
<th>Initial solvency: 1.50</th>
<th>Initial solvency: 1.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower solvency bound</td>
<td>Price</td>
<td>Lower solvency bound</td>
</tr>
<tr>
<td>1.10</td>
<td>0.00%</td>
<td>1.35</td>
</tr>
<tr>
<td>1.12</td>
<td>0.00%</td>
<td>1.37</td>
</tr>
<tr>
<td>1.14</td>
<td>0.01%</td>
<td>1.39</td>
</tr>
<tr>
<td>1.16</td>
<td>0.03%</td>
<td>1.41</td>
</tr>
<tr>
<td>1.19</td>
<td>0.20%</td>
<td>1.44</td>
</tr>
<tr>
<td>1.21</td>
<td>1.00%</td>
<td>1.46</td>
</tr>
<tr>
<td>1.23</td>
<td>4.56%</td>
<td>1.48</td>
</tr>
<tr>
<td>1.25</td>
<td>14.47%</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Table 5: The relative prices of the option for portfolio B, where the bonds has maturity time eleven years, shown for different lower solvency bounds and initial solvency levels.
Figure 17: Portfolio B. All figures show price of option in relation to initial value of balance sheet as a function of different lower solvency level limits. The initial solvency level is 125% in top, 150% in middle and 175% in bottom figure. The maturity time of the bonds is 11 years.
5.5.4 Convergence of price of balance sheet

The number of simulations is chosen such that there is an approximate convergence. In Figures 18 and 19, two examples of convergence is shown for the two different portfolios with different initial solvency levels and different lower solvency bounds. In these two plots, the price is demonstrated for all simulations. The price is also given as a factor and is not to be confused with the price in percent in the previous sections. Notice that the scales of the y-axis are different between Figures 18 and 19 and therefore the convergence in the first figure may not seem as significant as in the second figure.

The deviation in price for incremental number of simulations are small, in the magnitude of $10^{-6}$. This means that the price fluctuates with a couple of $10^{-4}$ percentage points when the number of simulations approaches 500,000 and the corresponding amounts are small in comparison to the value of the portfolio. This accuracy is sufficient to prove that the price exists and to give an accurate enough price.

Figure 18: The price of the balance sheet as a function of the number of simulations for portfolio A with initial solvency level 150% and lower solvency bound 137%. The price is given as a factor in this figure and is not to be confused with the percentages in the previous figures.
Figure 19: The price of the balance sheet as a function of the number of simulations for portfolio B where the bond has maturity time four years, initial solvency level is 125% and lower solvency bound is 119%. The price is given as a factor in this figure and is not to be confused with the percentages in the previous figures.
6 Discussion

A drawback with modelling the stocks and forward rates with normal log-normal random variables is that these are quite difficult to calibrate to. At times, parameter values for the distributions could lead to distributions not matching the historical log-values of stocks or forward rates in accordance with the Kolmogorov-Smirnov test and the visualization through Q-Q plots. However, suitable parameters were found which also generated reasonable simulated paths.

The model describing the forward rate processes may not be suitable for long time horizons as there is no mean-reverting term in the process. As a long time passes, a mean-reverting model would ensure that the value does not drift away from a reasonable value. However, the time to maturity of the option studied is short, three months, and hence the chosen model was used.

The price of the option decreases with a decreasing lower solvency level bound, i.e. further from the initial solvency level. This is intuitive since the probability of insolvency is lower when the lower limit is further away from the initial solvency level, i.e. the company has a large buffer, since the probability of large changes is lower than the probability of small changes in the market. This pattern is clearly exhibited for all initial solvency levels and portfolios. Worth noticing is that the price increases in an exponential manner as the lower bound solvency level gets closer to the initial solvency level. This means that it is significantly more expensive when the initial solvency level is close to the lower solvency bound. This is intuitive since when the buffer is larger, the market can move significantly before the option is in the money. However, when the buffer is small, it does not take a large stochastic change in market for the option to be in the money. This means that a larger share of the simulations will result in a price not equal to zero. Furthermore, for relatively higher lower solvency bounds, the relative price decreases as the initial solvency level increases. For the relatively lower solvency bounds, the relative price increases as the initial solvency level increases. When studying portfolio B where the bonds have different maturity times it is noticeable that there is a significant decrease in the price when the maturity time is longer.

The choice of number of simulations was limited by the amount of time that was feasible to run simulations to obtain one price. The amount of time it took to complete one price simulation was around 30 minutes when the number of simulated paths was 500,000 and the total time consumed for all simulations was several days. The choice of number of simulations was a trade-off between the amount of time the simulations would consume versus that accuracy that would be retrieved. For 500,000 simulations the error in price is of the magnitude of $10^{-4}$ percentage points. This is considered sufficient precision for the price of remaining above a certain solvency level bound since there might be other factors affecting the price to a larger extent than the precision from the Monte Carlo simulations. Such a factor might be the parameters governing the normal log-normal random variables or the choice of option model. The option is modelled as to solely be exercisable at a certain time $T$. If the solvency of the portfolio falls below the lower bound before time $T$ and then recovers at time $T$ it means that the option’s price will be zero when in fact, it would have been in the money during the length of the option. To incorporate this, options that are exercisable at any time before $T$ and are exercisable at multiple occasions, ensuring that the option covers the entire time interval, would be preferred. These are however more complicated to implement.
The assets that are possible to trade on this market are solely stocks and forward rates but it would be more realistic if bonds and liabilities are also tradable. However, finding a risk-neutral measure $Q$ for this market would be more complex. The tradable assets, i.e. the stocks and forward rates, in combination with the discrete model makes the market incomplete. This implies that one risk-neutral measure $Q$ has to be chosen amongst several possible. Since the measure is not unique, the price of the balance sheet option will not be unique either.
7 Conclusion and further developments

7.1 Conclusion

In this thesis, different possible models to model the balance sheet option have been investigated. The model that appeared to be the most appropriate was a discrete model based on normal log-normal random variables. By using the chosen model, it is shown that there exists a cost for remaining above a certain solvency level and that this cost is significant for lower solvency bounds that are near the initial solvency level, i.e. if the balance sheet’s solvency starts near the bound. The price of the option depends on the relationship between the assets and liabilities but also on the total initial value of the balance sheet. The lower solvency bound that is chosen is also of large significance for the price of the balance sheet option.

The purpose of this thesis is to prove that the price is of significance and where the price, i.e. the cost, originates from by modelling a balance sheet option. The price is not exact by any means, which was not the purpose of the thesis either, but the model used indicates the size of the cost in relationship to the initial value of the balance sheet and how it changes for different portfolios and solvency bounds. The knowledge of the price for remaining above a certain solvency level creates awareness amongst the pension companies regarding the cost that arises when selling assets cheap and buying them expensive in order to remain above the solvency level.

7.2 Further developments

The most important future development would be to validate the model and method against other models to see if similar results are obtained. An example of another model would be Bates or the simpler Black-Scholes model for the stocks and Libor market model for the forward rates. These are similar to the models used in this thesis, with the main difference lying in the example models having normal random variables and the model in this thesis having normal log-normal random variables.

There is a wide range of other possible options that could be used to model the cost of remaining above a certain solvency level. Examples of these are options that are exercisable at any given time, which would be as soon as the option is in the money when the solvency level drops below the lower solvency bound. Another example is an option that is exercisable at multiple times, which would be useful if the value of the balance sheet falls below the lower solvency limit multiple times during the life of the option.

Other improvement areas would be to incorporate other types of assets in the balance sheet, such as derivatives or real estates to make the model more extensive. One could also run a larger number of simulations to improve accuracy of the price even more.
References


A Principal Component Analysis

One way to reduce the dimensionality of the Libor market model, described in section 2.6, is to perform a principal component analysis (PCA). This is a technique that reduces a set of strongly correlated variables to a linear combination of uncorrelated variables. By decreasing the dimensionality and thereby number of variables, fewer simulations are needed when performing Monte Carlo simulations. The method is based on an eigenvalue decomposition of a covariance matrix. The covariance matrix originates from observations of the correlated processes. PCA works for any system of stationary processes where rewriting a random variable as a linear combination of other random variables is possible. However, the method works best when the processes in the system are highly correlated such as for a term structure [19].

The algorithm for performing the PCA on a Libor market model is provided by Fusai and Roncoroni [14] and is briefly outlined here. Denote the historical data on the Libor forward rates by \( L_i^p(t) \) for process \( i \) and \( t = \delta, \ldots, \delta N \) where \( \delta \) is the time step between each measurement and \( N \) is the total number of measurements. Calculate the mean of the forward rates by

\[
\mu_i = (\delta N)^{-1} \sum_{k=1}^{N} \left( \frac{L_i(k\delta) - L_i((k-1)\delta)}{L_i((k-1)\delta)} \right), \quad i = 1, \ldots, n
\]

and the centered annual yields by

\[
\Delta_i(t) = \delta^{-1} \left[ \frac{L_i(t + \delta) - L_i(t)}{L_i(t)} \right] - \mu_i, \quad i = 1, \ldots, n \text{ and } t = \delta, \ldots, N\delta.
\]

Define \( \Delta(t) = [\Delta_1(t), \ldots, \Delta_n(t)] \) and \( \Delta = [\Delta(\delta), \ldots, \Delta(N\delta)]^\top \). Also, define the covariance matrix of the central yields by \( C \) where the \( i,j \)-th entry is given by

\[
C_{i,j} = \text{Cov}(\Delta_i, \Delta_j) = N^{-1} \sum_{t=1}^{N} \Delta_i(\delta t)\Delta_j(\delta t).
\]

The next step is to compute the eigenvalues and eigenvectors of the covariance matrix \( C \). This matrix will have \( n \) positive eigenvalues, \( \lambda_1, \lambda_2, \ldots, \lambda_n \). They are ordered in a descending fashion, where \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). The eigenvalues correspond to the respective variance that each eigenvector explains. The corresponding eigenvectors are denoted by \( u_1, u_2, \ldots, u_n \) where the eigenvector matrix \( U \) is defined by \( U = [u_1 | \ldots | u_n] \) [13].

**Definition A.1** A principal component is a linear combination of the columns of \( \Delta \), where the weights are chosen in such a way that:

- the principal components are uncorrelated with each other; and
- the first principal component explains the most variation (i.e. the greatest amount of the total variation in \( \Delta \)) and the second component explains the greatest amount of the remaining variation, etc [19].

Using this definition of principal components, denote the principal component vector by \( f \), which is determined by

\[
f = \text{Diag} \left( \sqrt{\frac{1}{\lambda_1}}, \ldots, \sqrt{\frac{1}{\lambda_n}} \right) U^\top \Delta.
\]

Fusai and Roncoroni show that the variance of \( f \) is the identity matrix, indicating that all
principal components are orthogonal and hence describe uncorrelated processes. This equation can be solved for $\Delta$ which yields

$$\Delta = U \text{Diag} \left( \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n} \right) f$$

where the equality $U^\top = U^{-1}$ holds. This yields a transformed Libor market model which is given by

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=1}^{n} \sqrt{\lambda_j} u_j^i d\tilde{W}_j(t)$$

where $u_j^i$ is the $i$:th element of the $j$:th eigenvector and $\tilde{W}_j(t) = f_j$ are uncorrelated Wiener-processes. The sum can be approximated by the $k$ first principal components. The total variation that is explained by them is the sum of the corresponding eigenvalues and the share of total variation explained by the chosen PCA model is given by

$$\frac{\lambda_1 + \lambda_2 + \ldots + \lambda_k}{\lambda_1 + \lambda_2 + \ldots + \lambda_k + \ldots + \lambda_n}.$$

It is desired to choose $k$ which is as small as possible while keeping the total explained variation as high as possible and above a chosen limit. The $k$ first principal components together with the $k$ first eigenvalues now describe the system of correlated variables and may be used to simulate values for the variables. The approximate dynamics are given by the following

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=1}^{k} \sqrt{\lambda_j} u_j^i d\tilde{W}_j(t).$$

This can be solved for $L_i(t)$ to yield an analytical expression for the movements of the forward rates \footnote{Note that in Fusai and Roncoroni \cite{14} this equation is wrongfully stated.}.