Experimental test of Mermin-Peres Magic Square on IBM's 5-qubit Quantum Computer

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Experimentell prövning av Mermin-Peres Magiska Kvadrat på IBMs 5-kvantbitsdator

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Abstract

The validity of the theory of quantum mechanics has been a major topic of discussion for the past century. Einstein, Podolsky, and Rosen famously attempted to show the incompleteness of quantum mechanics, their main criticism hinging on the inherent nonlocality of quantum mechanical phenomena. So-called Hidden-variable theories were proposed as a more fundamental description of reality. The Bell and Kochen-Specker theorems, developed later on, placed limitations on the nature of such Hidden-variable theories. The Kochen-Specker theorem states that no non-contextual Hidden-variable theory is able to reproduce the predictions of quantum mechanics. In this paper an experimental proof of the Kochen-Specker theorem in Hilbert space of dimension four—known as the Mermin-Peres Magic Square—is implemented on IBM’s publicly available five qubit quantum computer. It is shown that the resulting data cannot be explained by a model assuming measurement results can be predetermined, i.e. a model assuming realism.
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1. Introduction

1.1 Quantum Nonlocality

The Einstein-Podolsky-Rosen (EPR) paradox was presented in 1935 as an argument against the completeness of quantum mechanics. It attempted to show that quantum mechanics must be supplemented by additional variables in order to satisfy the defined criteria for completeness:

"Every element of the physical reality must have a counterpart in the physical theory" [12].

The main assumptions are that the physical world is local and realistic, meaning that the propagation of information – and thus causality – is limited to the speed of light (locality) and that measurements only reveal elements of reality already present in the system being measured (reality).

These assumptions, known collectively as local realism, present a clear opposition to the Copenhagen interpretation of quantum mechanics. According to local realism a system’s attributes are predetermined, and a measurement merely reveals this information to the observer. Niels Bohr, however, argues in his response to the EPR paper that, in accordance with Heisenberg’s uncertainty principle, it is impossible for a particle to have e.g. its position and momentum simultaneously defined. Thus it is the act of measuring that makes the attribute come into existence [7].

The EPR Paradox aimed to show that the violation of local realism leads to "spooky action at a distance". One way many physicists hoped to resolve this was by so-called Hidden-variable theories. These theories assume that quantum mechanics is an incomplete theory, and that there is some undiscovered theory of nature for which quantum mechanics acts as a highly accurate statistical approximation. One way of stating this is that the wave function $\Psi$ does not fully describe the state of a system, and that some other quantity $\lambda$, a hidden variable, is required in order to characterize the system [13].

John Stewart Bell later proved that there is no physical theory of local hidden variables that can reproduce the predictions of quantum mechanics [4]. Bell’s conclusion is known as the Bell inequality:

$$|P(a, b) - P(a, c)| \leq 1 + P(b, c)$$

The Bell inequality created an opportunity to experimentally test the validity of a local hidden-variable theory. The violation of the Bell inequality has been verified in atomic physics experiments [3, 29]. A variety of different experiments have also been conducted to validate the Bell inequality violations, by closing loopholes in the experimental setups [16, 17, 28].

1.2 Quantum Contextuality

A complementary theorem to Bell’s theorem is the Kochen-Specker (KS) theorem, derived from the works of Gleason and Bell [14, 5]. The KS theorem enforced stricter restrictions than locality; that hidden variables are exclusively associated with the quantum system being measured and not with the measurement device, meaning that when a quantum system is measured the obtained value will be independent of how the measurement was made [20]. This is called the assumption of non-contextuality. The KS theorem states that no non-contextual hidden-variable (NCHV) model can reproduce the predictions of quantum theory when the dimension of the Hilbert space is three or more. More recent proofs involving less observables – which are considerably simpler – have
been presented, however these proofs only establish a contradiction in four dimensions or higher. This is a weaker result than the KS theorem, as every contradiction in a three dimensional space is also a contradiction in higher dimensions [19, 18, 8].

A simplified proof of the KS theorem in a three dimensional Hilbert space is presented in [25], along with a simple thought experiment that presents arguably the simplest proof to the KS theorem, in the special case of a Hilbert space of dimension four [24, 21, 22]. The thought experiment is known as the Mermin-Peres Magic Square. Unlike other proofs of the the KS theorem, the two-level system measured may be in any initial state, not necessarily a singlet state. It has been shown that this proof can be converted into an empirical test in a four dimensional Hilbert space using a measurement-based quantum computer [26].

1.3 Quantum Computing

Quantum computing is a relatively new field concerned with computation using quantum mechanical phenomena. Generally this is divided into two categories: analog quantum computers, which use concepts like quantum annealing to solve optimization problems, or digital quantum computers, which use quantum logic gates to perform computation (though of course are still very much analog in operation) [23]. It is the latter form that will be used in this paper, as they are well suited to implement the measurements we are interested in, as well as being publicly available in the cloud [27].

In classical computing data is stored in bits, binary states represented by discrete voltage levels in a circuit, and operations on bits are performed using logic gates implemented in transistors. In quantum computing, data is stored in classical bits (quantum memory storage is a field of active research), but operations are done on qubits, which are linear combinations (superpositions) of a distinct set of basis states in a two-level system. Thus a qubit can take on any of an infinity of possible combinations of its basis states. Operations on qubits are represented by quantum logic gates, which are reversible, unitary (i.e. norm preserving) quantum mechanical operators. This makes it possible to perform some quantum mechanical experiments on quantum computers. In this paper a physical implementation of the Mermin-Peres Magic Square – intended to prove that two-level quantum systems are contextual – will be performed on a 5-qubit measurement-based quantum computer using IBM’s Quantum Experience.
2. Theory & Methodology

2.1 Eigenvalues and Eigenvectors

An eigenvector of an operator $\hat{Q}$ is defined to be a non-zero vector that changes by a scalar factor when the operator $\hat{Q}$ is applied to it. The scalar factor is defined to be the eigenvalue of the corresponding eigenvector. In Eq. (2.1) $\hat{Q}$ is the operator, $\lambda$ is the eigenvalue and $\xi$ is the eigenvector

$$\hat{Q}\ket{\xi} = \lambda \ket{\xi}.$$  

(2.1)

2.2 Observables & Pauli Matrices

In quantum mechanics physically observable quantities (observables) are represented by operators, and the possible values of the quantity correspond to the eigenvalues of the operator. The expectation value of any observable $Q$ can be expressed using the inner product notation of an $L_2$ (Hilbert) space. The inner product of two functions in the Hilbert space is given by Eq. (2.2)

$$\langle f | g \rangle = \int_a^b f(x)^* g(x) dx.$$  

(2.2)

Thus the expectation value of an observable is given by Eq. (2.3), where $\psi$ is the wave function and $\hat{Q}$ is the operator representing the observable

$$\langle Q \rangle = \int \psi^* \hat{Q} \psi dx = \langle \psi | \hat{Q} | \psi \rangle.$$  

(2.3)

Furthermore the outcome of any measurement must be real and therefore it can be shown that the operator $\hat{Q}$ is Hermitian, due to the following property:

$$\langle Q \rangle = \langle Q \rangle^\dagger$$

In a spin-$\frac{1}{2}$ system the Hermitian operators which represent the observable corresponding to spin along an axis in $\mathbb{R}^3$ is given by the Pauli matrices shown in Eq. (2.4) where the operators are expressed in terms of $\hbar/2$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.4)

The eigenvalues of these matrices are $\pm 1$. Furthermore the Pauli matrices are also Unitary which means that the eigenvectors belonging to distinct eigenvalues of the operator are orthogonal, which in turn shows that the eigenvectors of the Pauli matrices span the 2-dimensional complex Hilbert space. These results hold for any two-level system, not just a spin-$\frac{1}{2}$-system; any qubit (See Section 2.4.1) implementation can be described using this formalism.

2.2.1 Determinate States

A determinate state for an observable $Q$ is a state such that every measurement of $Q$ returns the same value. Such states are given by the eigenvectors of the operator $\hat{Q}$, where the corresponding eigenvalues are the measurement results of the determinate state.
2.3 Mermin-Peres Magic Square

Peres states that the result of a measurement is dependent on the context in which the measurement is being performed [24]. Even if two operators \(A\) and \(B\) corresponding to measurements of observables commute (\([A, B] = AB - BA = 0\)), the result of these measurements depend on previous measurements performed on the system. In other words, the result of a measurement is contextual. To shed some more light on this unintuitive – even in the context of quantum mechanics – phenomenon, we will present a thought-experiment known as The Mermin-Peres Magic Square (MP square).

The MP square is a \(3 \times 3\) square where each cell represents a measurement on a two-level system (See Fig.2.1). The measurements are mathematically described by the tensor product of the Pauli operators. For example a measurement of the \(x\)-component of spin on particle A and \(y\)-component on particle B is mathematically represented by the operator \(X \otimes Y\) applied on the product state \(A \otimes B\). The tensor product, \(\otimes\), of two matrices is given by (2.5a), and the product state of two vectors is given by (2.5b).

\[
\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \otimes \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{12}y_{21} & x_{12}y_{22} \\ x_{21}y_{11} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{22}y_{21} & x_{22}y_{22} \end{pmatrix} \quad (2.5a)
\]

\[
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix} \quad (2.5b)
\]

The possible measurement results of a two-level system is given by the eigenvalues of the tensor product of the operators. Since the eigenvalues of the Pauli operators are \(\pm 1\) the eigenvalues of any tensor product between the Pauli operators must also be \(\pm 1\). Furthermore if several measurements are made in sequence, the operators on each side of the tensor product are matrix multiplied in the same order. For example given a \(X \otimes Y\) measurement followed by a \(Y \otimes Z\) measurement can be written as \(XY \otimes YZ\), and so on.

2.3.1 Commutation Relations

Lemma 1.

\[
XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iZ \quad (2.6)
\]

\[
YX = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -iZ \quad (2.7)
\]

From (2.6) and (2.7) it is given that \(XY = -YX\) By permuting the operators and matrix multiplying it can be shown that the following relations also hold true: \(YZ = -ZY\) and \(XZ = -ZX\).

Theorem 1. The operators in each row and column in the MP square commute.
**Proof.** For the first row one can see that:

\[(X \otimes 1)(1 \otimes X) = X \otimes X \implies [(X \otimes 1)(1 \otimes X), X \otimes X] = 0.\]  \hfill (2.8)

Therefore the first and second rows commute (permute $X$ with $Y$). For the third row Lemma 1 can be used to show that it commutes:

\[[X \otimes Y, Y \otimes X] = 0, \quad [X \otimes Y, Z \otimes Z] = 0, \quad [Y \otimes X, Z \otimes Z] = 0.\]  \hfill (2.9)

The first and second columns commutation property can be shown using the same logic that was used for the first two rows. For the third column Lemma 1 can once again be used to show:

\[[X \otimes Y, Y \otimes Y] = 0, \quad [X \otimes X, Z \otimes Z] = 0, \quad [Y \otimes Y, Z \otimes Z] = 0.\]  \hfill (2.10)

### 2.3.2 Determinate measurements in the MP Square

The resulting tensor product for each row and column can also be determined. These results are shown in Equations (2.11). Equations (2.11a-2.11c) are the resulting tensor products for the rows, and (2.11d-2.11f) give us the resulting tensor products for the columns. Lemma 1 is used in (2.11c) and (2.11f). These results are also shown in Fig. 2.2.

\[(X \otimes 1)(1 \otimes X)(X \otimes X) = (XX) \otimes (XX) = 1 \otimes 1 = 1 \quad \hfill (2.11a)\]

\[(1 \otimes Y)(Y \otimes 1)(Y \otimes Y) = (YY) \otimes (YY) = 1 \otimes 1 = 1 \quad \hfill (2.11b)\]

\[(X \otimes Y)(Y \otimes X)(Z \otimes Z) = (XY Z) \otimes (YX Z) = i^2(ZZ) \otimes (ZZ) = 1 \otimes 1 = 1 \quad \hfill (2.11c)\]

\[(X \otimes 1)(1 \otimes Y)(X \otimes Y) = (X \otimes Y)(X \otimes Y) = (XX) \otimes (YY) = 1 \otimes 1 = 1 \quad \hfill (2.11d)\]

\[(1 \otimes X)(Y \otimes 1)(Y \otimes X) = (Y \otimes X)(Y \otimes X) = (YY) \otimes (XX) = 1 \otimes 1 = 1 \quad \hfill (2.11e)\]

\[(X \otimes X)(Y \otimes Y)(Z \otimes Z) = (XY Z) \otimes (XY Z) = -i^2(ZZ) \otimes (ZZ) = -(1 \otimes 1) = -1 \quad \hfill (2.11f)\]

Equations (2.11) show that the product of three successive measurements in every row and column of the MP square are deterministic, regardless of the initial state of the two-level system. The product of every row measurement and the first two columns on the left must be equal to 1, and therefore a constraint is imposed on the measurements. For each of the rows, as well as the two leftmost columns an even number of measurements must be equal to $-1$, whereas for the rightmost column, an odd number of measurements must be equal to $-1$ (either one or all three measurements must result in $-1$). Therefore there must be an odd number of measurements with result $-1$, and a contradiction occurs. We cannot assign the MP square both an even and odd number of $-1$ results. An example is shown in Fig. 2.2, where the initial state of a two-level system is given by two qubits (Qubits are introduced in Section 2.4.1), $|\psi_1\rangle$ and $|\psi_2\rangle$, both in the state:

\[|\psi_i\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad i = 1, 2.\]

The first row in the MP square is deterministic; all measurements are equal to 1. For the second row there is a 50% probability of measuring 1 or $-1$. If the first and second measurements from the left are both equal to $-1$ then the third must be equal to 1, as $(-1)(-1) = 1$. We can see that the first two measurements to the left on the final row must also be equal to $-1$ to satisfy the column constraints. The final measurement, however cannot be determined, because the row
constraint states that the final measurement must be equal to 1 in order to have an even number of $-1$ results, but the column constraint states that the final measurement must be equal to $-1$ since there must be an odd number of $-1$ results in the column.

The result of the final measurement is dependent on the context in which the measurement is made. If the previous measurements were $X \otimes Y$ and $Y \otimes X$ (Row measurement) then the result of the final measurement will be 1, however if the previous measurements were $X \otimes X$ and $Y \otimes Y$ (Column measurement), then the result will be $-1$. Despite the fact that the observables commute the MP square shows that the result of a given measurement is contextual.

The crucial realization here is that we could not have a priori, before any measurement, assigned values to every measurement in the square. No matter the initial state the last measurement is contextual; we could not have predicted the result of each individual measurement by any non-contextual hidden variable theory.

### 2.4 Operations on Qubits

#### 2.4.1 Qubits

A qubit is a two-level quantum system whose state can be written

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the bra-ket notation shorthand for the eigenvectors of the $Z$ Pauli matrix – referred to as the computational basis – and $\alpha, \beta$ are complex numbers. The Born Rule dictates that the probability of observing $|0\rangle$ or $|1\rangle$ as the result of a measurement of the observable $Z$ is $|\alpha|^2, |\beta|^2$ respectively, and because these are the only two possible eigenvectors to observe, we have the condition

$$|\alpha|^2 + |\beta|^2 = 1.$$

Generally, if the state was described in another basis, the result of a measurement of any observable $\hat{Q}$ would be one of the eigenvectors of that observable, with probability equal to the projection of the state on that eigenvector:

$$P(|\xi\rangle) = |\langle \xi | \psi \rangle|^2$$

#### 2.4.2 The Bloch Sphere

Naively it seems there are four degrees of freedom in describing a qubit state: $\alpha, \beta$ are complex coefficients and therefore have two degrees of freedom each. Because of the constraint in Eq. (2.13), however, we can make a suitable change of coordinates (to Hopf coordinates) to eliminate one degree of freedom:

$$\alpha = e^{i\psi} \cos \frac{\theta}{2} \quad \text{and} \quad \beta = e^{i(\psi+\phi)} \sin \frac{\theta}{2}.$$
As the overall phase factor $e^{i\phi}$ is physically meaningless (it does not change the expectation value of any observable), it can be factored out and discarded. We are then left with the state description

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle.$$  \hspace{1cm} (2.15)

This has a natural visualization in the form of a sphere, with polar angle $\theta$ and azimuthal angle $\phi$ (See Fig. 2.3).

Figure 2.3: Bloch Sphere [15]

### 2.4.3 Quantum Gates

To perform computation using qubits, we must define a set of operations on a qubit, referred to as logic gates, that can implement any given quantum algorithm. The general unitary operator $U$ on a single qubit can be written (ignoring overall phase) as a rotation around some axis in the Bloch Sphere:

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos \theta/2 & -e^{i\lambda} \sin \theta/2 \\ e^{i\phi} \sin \theta/2 & e^{i(\lambda+\phi)} \cos \theta/2 \end{pmatrix}.$$  \hspace{1cm} (2.16)

In practice, however, there are finite sets of special cases of this operator (called universal sets) that can be used to arbitrarily well approximate (with a finite amount of gates) any given operation on qubits[10]. In classical computation the AND & NOT gates together form a universal set, equivalent to the NAND gate. The quantum equivalent to this is the CCX (controlled-controlled-not), or Toffoli gate, which is by itself a universal set. For the purposes of this paper we will be using the set of Clifford gates:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = U(\pi/2, 0, \pi),$$  \hspace{1cm} (2.17)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = U(0, 0, \pi/2),$$  \hspace{1cm} (2.18)

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = U(0, 0, -\pi/2).$$  \hspace{1cm} (2.19)

This is a universal set of gates which is available in the IBM’s qiskit quantum programming library for Python. The Hadamard gate $H$ can be visualized as a $\pi/2$ rotation around the $X+Y$ axis, the phase gate $S$ as a $\pi/2$ rotation around the $Z$ axis, and the $S^\dagger$ gate as a $-\pi/2$ rotation around the $Z$ axis. For the purpose of optimization of our quantum circuits we will also be using the $u_2(\phi, \lambda) = U(\pi/2, \phi, \lambda)$ (See Eq. (2.16), $\theta$ variable can be canceled out if $\theta = \pi/2$) gate in qiskit (Qiskit is IBM’s Python library required to use the quantum computer over the cloud).
In addition to this set of single qubit gates we will also be using the $C_x$ (CNOT, or controlled NOT/controlled X) gate, operating on two qubits:

$$C_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ \hspace{1cm} (2.20)

The action of the $C_x$ gate on a product state $A \otimes B$ is to flip the value of (apply an X-gate to) the B (target) qubit if the A (control) qubit is in the $|1\rangle$ state.

2.4.4 Measurement, Decoherence & Gate Infidelity

**Measurement** To perform a measurement on a physical qubit in the computer, we cannot simply 'apply' some operator corresponding to the observable we are interested in and record the eigenvalue. Instead we have at our disposal only the Measure gate $M_z$. This is in fact a projection of the qubit state on the computational (Z) basis; the next section will cover how to use rotations to make projective measurements on the X and Y basis. In order to implement measurements of the type described in Section 2.3, we must find a way to store the result of the product measurements without directly measuring the individual qubits. To accomplish this we can apply a $C_x$ gate with each of the two qubits as control, and a third ancilla qubit as the target. We can then identify (using the eigenvalues of the Pauli $Z$ matrix)

$$|0\rangle \equiv 1, \quad |1\rangle \equiv -1,$$ \hspace{1cm} (2.21)

and application of the $C_x$ gate corresponds to multiplication of the ancilla qubit’s ‘value’ by 1 or -1. Measurement of this qubit will then give the result of the product measurement. As we are essentially only interested in the product of the results of each row or column, one ancilla qubit measured at the end is sufficient to obtain the relevant results.

**Decoherence** In an ideal noise-free quantum measurement a measurement in the Z basis on the determinate state $|\psi\rangle = |1\rangle$ will always yield the same result; there is a 100% chance of measuring the state to be $|1\rangle$. Because a physical qubit can never be completely isolated from the environment, however, the result of a set of measurements on identically prepared states of the kind described will generally be some distribution of $|1\rangle$ and $|0\rangle$. There are two primary sources of this noise: relaxation, characterized by the relaxation time $T_1$, and dephasing, characterized by the dephasing time $T_2$. Relaxation time is a property of all superconductive qubit implementations, as the states of the qubit are nondegenerate; $|0\rangle$ is the ground state and $|1\rangle$ is the first excited state. Infinitesimal energy exchange with the environment can then lead to the spontaneous relaxation of the $|1\rangle$ state to the $|0\rangle$ state. The expectation value of the time that this takes is the $T_2$ time. Dephasing is the process wherein environmental coupling causes the relative phase between two eigenstates of the total qubit state to become randomized. In other words, the qubit becomes entangled with the environment and the single qubit state is no longer a pure state. The dephasing time $T_2$ can then be described as a combination of the relaxation time and a ‘pure dephasing’ term from the environmental coupling [9]:

$$1/T_2 = 1/T_1 + 1/\tau_\phi.$$ \hspace{1cm} (2.22)

**Gate Infidelity** The error caused by the physical implementation of quantum gates is referred to as gate infidelity. In short, a superconducting phase qubit uses electromagnetic pulses to induce Rabi oscillations in the qubit, thereby causing a state transition with a given probability dependent on the frequency and time of the pulse. In practice, this operation does not perfectly correspond to the theoretical gate, but rather some perturbed (but still unitary) gate. The error caused by this perturbation is called the gate error.

2.5 Quantum Circuits

In order to perform a measurement on a qubit along an axis other than the Z-axis, such as the X- or Y-axis, a rotation must first be performed. This is done in order to align the axis to the
computational Z-axis. An ancilla qubit is introduced. After the axes are rotated a CNOT gate is applied to the control and ancilla qubit. After each measurement the axes must be returned to their original alignment before another measurement is carried out. Each CNOT gate used corresponds to one 'virtual measurement' of the control qubit. The X- and Y-measurements as quantum circuits are given by Equations (2.23-2.24).

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{H} M_z
\end{align*}
\]

(2.23)

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{S^\dagger H H S} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{S^\dagger H H S} M_z
\end{align*}
\]

(2.24)

Quantum circuits for each row and column in the MP square can be constructed by using combinations of Eq. (2.23) and (2.24).

### 2.5.1 Gate Elimination

Each gate corresponds to an operator which is both Hermitian and Unitary. Therefore it is possible to eliminate gates when applied in a certain order, as they cancel each other out. Two gate combinations which can be canceled out in this way are \(SS^\dagger\) and \(HH\). For example the quantum circuit for a \(XX\otimes\ldots\) measurement (two X measurements on the first qubit) is given by:

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_2 : |0\rangle & \xrightarrow{S^\dagger H H S} M_z
\end{align*}
\]

(2.25)

The two Hadamard gates in between the CNOT gates can be eliminated, because the gates are two rotations back and forth to the same position in the Bloch sphere. The resulting circuit is given by:

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_2 : |0\rangle & \xrightarrow{S^\dagger H H S} M_z
\end{align*}
\]

(2.25)

### 2.5.2 Quantum Circuits for the MP Square

Quantum circuits for each row and column in the MP square can now be constructed using (2.23) and Eq. (2.24), and reduced using the elimination concept described in Section Eq. 2.5.1. The reduced quantum gates for each row and column of the MP square is given by Equations (2.25a-2.25f), where the first three equations give the circuits for the rows and the last three equations are the circuits for the columns from left to right.

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{H} |0\rangle \\
q_2 : |0\rangle & \xrightarrow{M_z}
\end{align*}
\]

(2.25a)

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{S^\dagger H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{S^\dagger H} |0\rangle \\
q_2 : |0\rangle & \xrightarrow{M_z}
\end{align*}
\]

(2.25b)

\[
\begin{align*}
q_0 : |0\rangle & \xrightarrow{S^\dagger H} |0\rangle \\
q_1 : |0\rangle & \xrightarrow{S^\dagger H} |0\rangle \\
q_2 : |0\rangle & \xrightarrow{M_z}
\end{align*}
\]

(2.25c)
2.5.3 Optimizing Quantum Circuits

An assumption that is made is that the noise in a measurement is dependent on the number of gates in a given circuit. Therefore further reduction of the number of gates in a quantum circuit may improve results. A way of doing this is merging two or more gates into a single gate. The possible combinations that can be merged in the circuits for the MP square are the \( S^\dagger H \), \( H S H \) and \( H S^\dagger H \) gate combinations.

All quantum gates in qiskit can be constructed from the general unitary operator, Eq. (2.16) described in Section 2.4.3.

For the first gate combination the combined gate is given by the matrix multiplication between \( H \) and \( S^\dagger \). If we multiply from left to right then the order must be \( HS \).

By identifying terms in Eq. (2.16), we see that a corresponding gate is given by \( U(\pi/2, 0, \pi/2) \).

For the remaining two gate combinations the matrix multiplications are shown in Equations (2.26).

The quantum circuits can now be optimized using these gates. The optimized circuits are given by equations (2.29a-2.29f).
2.5.4 Quantum Computer Architectures

IBM’s two five qubit quantum computers have differing architectures and qubit connectivity [11]. Therefore we must pay attention to which qubits we use as control and target qubits when applying CNOT gates. As described in [11], a qubit shown at the tail of an arrow can only work as a control qubit and a qubit at the head of an arrow can only work as a target qubit. Thus when using the `ibmqx4` architecture (Fig. 2.4a) the ideal candidates to have as a target qubit is qubit 0 or qubit 4, whereas with the `ibmqx2` architecture (Fig. 2.4b) the only ideal candidate is qubit 2. Thus we must make some minor changes to the quantum circuits depending on which quantum computer the code is running on. When running the code on the `ibmqx2` architecture, no changes are needed on the current quantum circuit. When running the code on `ibmqx4` we can optimize the quantum circuit by exchanging qubits 0 and 2.

Figure 2.4: Qubit connectivity for IBM Q’s two 5-qubit quantum computers
3. Results

3.1 Optimized and Non-optimized Measurements

Each row and column measurement was performed 8192 times, the maximum number of runs allowed by IBM for each quantum circuit. The quantum circuits given by Eq. (2.25) in Section 2.5.2 are referred to as the non-optimized quantum circuits, and the quantum circuits described in Section 2.5.3 by Eq. (2.29) are referred to as the optimized circuits. The results for measurements with an initial state of \( |0\rangle \otimes |0\rangle \) (both qubits in the \( |0\rangle \) state) is shown in Fig. (3.1-3.2). The mean value denotes the sum of 1 and \(-1\) results weighted by their relative frequency in the 8192 shots:

\[
\langle M \rangle = \frac{\sum x_i r_i}{8192}.
\]  (3.1)

![Images of bar charts showing results for MP Square using Optimized and Non-optimized Quantum Circuits with an initial state of \( |0\rangle \otimes |0\rangle \). The number in each bar represents counts. The measurements were performed on the ibmqx4 'Tenerife' quantum computer.]

Table 3.1: Properties of qubits in ibmqx4 'Tenerife' on 6 May 2019.

<table>
<thead>
<tr>
<th></th>
<th>Frequency [GHz]</th>
<th>( T_1 ) [( \mu s )]</th>
<th>( T_2 ) [( \mu s )]</th>
<th>( U_1 ) Error</th>
<th>( U_2 ) Error</th>
<th>( U_3 ) Error</th>
<th>Readout Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q0</td>
<td>5.24658</td>
<td>43.81494</td>
<td>22.58358</td>
<td>0</td>
<td>0.00086</td>
<td>0.00172</td>
<td>0.0935</td>
</tr>
<tr>
<td>Q1</td>
<td>5.29831</td>
<td>49.50879</td>
<td>10.96442</td>
<td>0</td>
<td>0.0012</td>
<td>0.0024</td>
<td>0.09075</td>
</tr>
<tr>
<td>Q2</td>
<td>5.33833</td>
<td>50.67352</td>
<td>62.86297</td>
<td>0</td>
<td>0.00086</td>
<td>0.00172</td>
<td>0.0395</td>
</tr>
<tr>
<td>Q3</td>
<td>5.42612</td>
<td>16.54371</td>
<td>40.23874</td>
<td>0</td>
<td>0.00232</td>
<td>0.00464</td>
<td>0.35175</td>
</tr>
<tr>
<td>Q4</td>
<td>5.17451</td>
<td>53.94072</td>
<td>6.4971</td>
<td>0</td>
<td>0.00137</td>
<td>0.00275</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table 3.1 presents the properties of qubits in ibmqx4, including characteristic decoherence times \( T_1 \) \( T_2 \), the gate error of the general unitaries \( U_1 \), \( U_2 \), & \( U_3 \) (The \( U_3 \) gate is the general unitary operator \( U \) described in Section 2.4.3), as well as the readout error (the gate error of the measurement gate combined with relaxation during measurement).
3.2 Varied Initial States

Measurement results for a variety of initial states are shown in Fig. 3.3. Data labels follow the convention $a \otimes a$ where $a$ is the eigenstate pointing along the positive $a$ direction in the Bloch Sphere:

$$x = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad y = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle), \quad z = |0\rangle.$$  (3.2)
3.3 Repeated Identical Measurements

Measurements were performed on the same quantum computer with the same initial state, \((z \otimes z)\) on different dates. The results are shown in Fig. 3.4.

![Image of bar chart](image_url)

Figure 3.4: Mean values over 8192 shots per measurement for the same initial states performed on different dates. Measured on *ibmqx4 'Tenerife'*. 

<table>
<thead>
<tr>
<th>Date</th>
<th>Row 1</th>
<th>Row 2</th>
<th>Row 3</th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>17-Apr-2019</td>
<td>0.524</td>
<td>0.532</td>
<td>0.403</td>
<td>0.498</td>
<td>0.521</td>
<td>-0.558</td>
</tr>
<tr>
<td>21-Apr-2019</td>
<td>0.629</td>
<td>0.622</td>
<td>0.532</td>
<td>0.613</td>
<td>0.619</td>
<td>-0.589</td>
</tr>
<tr>
<td>6-May-19</td>
<td>0.894</td>
<td>0.899</td>
<td>0.834</td>
<td>0.883</td>
<td>0.887</td>
<td>-0.528</td>
</tr>
</tbody>
</table>
4. Discussion

4.1 Analysis of Results

Figs. 3.1 & 3.2 show that there is a small improvement of results when using an optimized circuit to measure Equations (2.11a-2.11f) in the MP square, however it is difficult to determine if these results are a statistically significant improvement or within the range of the fluctuation of different measurements. An interesting discrepancy is seen in the results of the third column for both the optimized and non-optimized quantum circuits. The mean value of the third column has a larger discrepancy from the theoretically predicted mean of $-1$ than the mean values of all rows and the first two columns. The third column has 1886 and 1933 counts respectively for optimized and non-optimized circuits, which is approximately three times as many incorrect counts compared to the other measurements. The questions is why the third column produces seemingly less accurate results compared to the other measurements. A proposed explanation is that these results can be explained by the choice of states which are measured. In Section 2.4.4 the measurement results are defined to be the states given by Eq. (2.21), thus a $-1$ result is defined to be the $|1\rangle$ state. This state is subject to decoherence and can relax to the ground state, $|0\rangle$ (See $T_1$ time described in Section 2.4.4). The hypothesis is that in the time taken for the quantum computer to complete the measurements, some of the runs will have relaxed into the $|0\rangle$ state before the final measurement of the ancilla qubit and thus registering as a $1$ result. In order to verify that this is a correct hypothesis and source of error, further experimental validation is required.

Fig. 3.4 shows the results of measurements on the MP square, on different dates, using the same initial state. Here we see a large variation, despite seemingly no difference in the experimental setup. The only differing parameter is the time elapsed since the previous calibration of the quantum computer, which ideally should not affect the results strongly. Furthermore when considering the results with differing initial states the results appear to be independent of the initial state (see Fig. 3.3).

4.2 Conclusions

The results obtained evidently deviate from the ideal theoretical case. The obvious question is then how this deviation affects the conclusion of the MP square experiment. This is in a sense a fundamentally different question than that posed by the thought-experiment form of the MP square. Where the MP square (and more generally the KS theorem) seeks to determine which class of hidden-variable theory could or could not reproduce the predictions of quantum mechanics, this practical implementation instead concerns the question of what theories could explain the actual results of an experiment on a physical quantum mechanical system.

4.2.1 Constructing a Hidden-Variable Model For MP Square Results

It turns out that the question of whether a given series of noisy measurement results and their averages can be explained by any hidden-variable model is nontrivial. To even begin to answer this question we must define what kind of restrictions define a hidden-variable model. If we recall, the primary intent behind formulating a HVT is to allow for realism, i.e. that the attributes of a system are predetermined. If we define our notion of realism to strictly mean that all measurements of systems can be predetermined before any measurement is done, then we implicitly assume non-contextuality (realism $\implies$ non-contextuality). The null hypothesis is then that the MP square results are realistic and non-contextual.
As any given row or column result is independent of the other results (this is true both classically and quantum mechanically), we can freely associate it with any MP square. The question is, then, if we can construct any given result vector \( \vec{r} \in \mathbb{R}^6 \) of averaged MP square row and column results, from a statistical mixture of predetermined squares, each fulfilling the requirement imposed by predetermined values—that there is either an odd amount of \(-1\) elements or an even number. This is the most general predetermined-value model we can construct, assuming only the minimum requirement of internal consistency.

The question of whether a given result vector \( \vec{r} \) can be explained by this model then boils down to determining if this vector is in the convex hull of predetermined MP-squares, i.e. if it can be written as a linear combination of these squares such that the weights sum to 1. It can be shown [6] that this convex hull consist of 32 squares in \( \mathbb{R}^6 \), thus the problem reduces to determining if a solution \( \vec{p} \) exists to the matrix equation

\[
\vec{r} = \vec{p} \cdot \vec{M}, \quad \vec{r} \in \mathbb{R}^6, \quad \vec{p} \in \mathbb{R}^{32}, \quad |\vec{p}| = 1.
\] (4.1)

The graphical intuition behind this equation is illustrated in a simplified schematic example in Fig. 4.1. Here the convex hull is spanned by 4 possible square result vectors. If a measured result vector lies inside the convex hull then the results can be explained by a mixture of the predetermined squares which span the hull. If the result vector lies outside the hull, however, the results cannot be explained by a predetermined model. In other words, there exists a solution \( \vec{p} \) to Eq. (4.1) if the result vector \( \vec{r} \) lies within the convex hull, but not if it lies outside.

![Figure 4.1: Convex Hull spanned by a mixture of four two dimensional predetermined square results. The blue and black vectors represent result vectors \( \vec{r} \) which can and cannot, respectively, be explained by a predetermined model.](image)

Furthermore it can be shown that this solution exists if and only if

\[
R_1 + R_2 + R_3 + C_1 + C_2 - C_3 \leq 4, \quad \text{where} \quad R_i := \text{Row } i, \quad C_i := \text{Column } i.
\] (4.2)

For our best result (Fig. 3.2), we have that

\[
0.939 + 0.887 + 0.833 + 0.907 + 0.917 - (−0.504) = 4.987.
\] (4.3)

This is larger than 4, which implies that the result vector could not be explained by the model we constructed. In other words, we reject our null hypothesis.

The null hypothesis is that the MP-square is both realistic and non-contextual. Knowing that realism (as defined above) implies non-contextuality, we cannot say with certainty whether the rejection of the null hypothesis is due to rejection of realism or rejection of non-contextuality (i.e. acceptance of contextuality). What we can say, is that accepting contextuality implies rejecting realism. The implication of our result is then that nature is either non-realistic and non-contextual, or non-realistic and contextual. In other words, we cannot reject non-contextuality, but we can reject realism. Despite the specificity of this experiment, it is a stunning result that there exists even a single phenomenon that is unexplainable by assuming that attributes of a system exist in advance.
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Bibliography


