\[
x^{7p} = r + \xi^3 a_3^\perp + \xi^3 w^{5p} + \xi^3 w^{6p} + (\xi^3)^2 w^{7p}
\]

 Implementation and Validation of an Isogeometric Hierarchic Shell Formulation

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Master Thesis in Civil Engineering

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Abstract

Within this thesis, thin walled shell structures are discussed with modern element formulations in the context of the Isogeometric Analysis (IGA). IGA was designed to achieve a direct interface from CAD to analysis. According to the concept of IGA, Non-Uniform Rational B-Splines (NURBS) are used as shape functions in the design and the analysis. Depending on the polynomial order, NURBS can come along with a high order continuity. Therefore, the curvature of a shell surface can be described directly by the shape function derivatives which is not possible within the classical Finite Element Analysis (FEA) using linear meshes. This description of the curvature gives rise to the application of the Kirchhoff-Love shell formulation, which describes the curvature stiffness with the differentiation of the spatial degrees of freedom. Based upon this, the formulation can be enhanced with further kinematical expressions as the shear difference vector, which leads to a 5-parameter Reissner-Mindlin formulation. This kinematic formulation is intrinsically free from transverse shear locking due to the split into Kirchhoff-Love and additional shear contributions. The formulation can be further extended to a 7-parameter three-dimensional shell element, which considers volumetric effects in the thickness direction. Two additional parameters are engaged to describe the related thickness changes under load and to enable the use of three-dimensional material laws. In general, three-dimensional shell elements suffer from curvature thickness and Poisson’s thickness locking. However, these locking phenomena are intrinsically avoided by the hierarchic application of the shear difference vector and the 7th parameter respectively. The 3-parameter Kirchhoff-Love, the 5-parameter Reissner-Mindlin and the 7-parameter 3D shell element build a hierarchic family of model-adaptive shells.

This hierarchic family of shell elements is presented and discussed in the scope of this thesis. The concept and the properties of the single elements are elaborated and the differences are discussed. Geometrically linear and non-linear benchmark examples are simulated. Convergence studies are performed and the results are validated against analytical solutions and solutions from literature, taking into account deflections and internal forces. Furthermore, the different locking phenomena which occur in analyses with shell formulations are examined. Several test cases are designed to ensure a validated implementation of the hierarchic shell elements. The element formulations and further pre- and postprocessing features are implemented and validated within the open-source software environment *Kratos Multiphysics*.

Keywords

Isogeometric Analysis; Shell; Locking; Kirchhoff-Love; Reissner-Mindlin; Three-Dimensional Shell
Kurzfassung


Schlüsselwörter

Isogeometrische Analyse; Locking; Schale; Kirchhoff-Love; Reissner-Mindlin; Dreidimensionale Schale
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Nomenclature

The following nomenclature does not list all used abbreviations and symbols, but it only clarifies particular terms which are seldom used in the corresponding literature.

Abbreviations

car Cartesian
con contravariant
cov covariant
hier hierarchic
KL Kirchhoff-Love
PK2 Piola-Kirchhoff stress of second kind
stand standard
RM Reissner-Mindlin
3p 3-parameter
5p 5-parameter
6p 6-parameter
7p 7-parameter
Symbols

$a_i$ covariant base vector at the mid-surface of a shell in the current configuration

$A_i$ covariant base vector at the mid-surface of a shell in the reference configuration

$E$ geometrically non-linear Green-Lagrange strain

$g_i$ covariant base vector at an arbitrary point of the shell body in the current configuration

$G_i$ covariant base vector at an arbitrary point of the shell body in the reference configuration

$r$ position vector of a point on the mid-surface of a shell in the current configuration

$R$ position vector of a point on the mid-surface of a shell in the reference configuration

$x$ position vector of a point of the shell body in the current configuration

$X$ position vector of a point of the shell body in the reference configuration

$\epsilon$ geometrically linear Green-Lagrange strain

(¯) value refers to a Cartesian coordinate system

(˘) discrete value

(˘) discretized value
1 Introduction

1.1 Motivation

Shell structures find a broad usage in different fields of engineering, such as tanks or vaults in civil engineering and aeroplanes or vehicles in mechanical engineering. Shells are reduced three-dimensional continua for which one dimension is much smaller than the other two. Their curved shape prevents the occurrence of high bending forces and facilitates to carry transversal loads in membrane action. This typology makes them efficient with respect to the ratio between stiffness and material use. Thus, shells are often optimized slender structures.

The high slenderness makes shells sensitive for instabilities due to small imperfections. Therefore, it is advantageous to simulate shell structures with methods which are able to yield results with a high accuracy. Nowadays, the Finite Element Method (FEM) is widely used for the structural analysis of complex structures. Shear rigid, shear deformable or three-dimensional shell elements are available in order to compute shell structures. Each element type has certain theoretical assumptions and in consequence specific application limits.

In the classical Finite Element Analysis (FEA), a CAD model is generated in the design process and has to be transformed by a meshing procedure to an analysis model in order to run simulations. The meshing procedure plays a crucial role with respect to the computational time and the quality of the results. T. Hughes et al. proposed the Isogeometric Analysis (IGA) which makes the transformation between design and analysis model obsolete [11]. The key idea of IGA is to use the same geometric description for the design and the analysis model. CAD tools commonly use Non-Uniform Rational B-Splines (NURBS) as shape functions. These NURBS functions normally have a higher polynomial order and a higher continuity compared to linear shape functions used in the classical FEA. The shear rigid Kirchhoff-Love (KL) shell is not applicable straight-forward for the classical linear FEA because it requires at least a $C^1$ continuity across element boundaries. In contrast, the standard shear rigid Reissner-Mindlin (RM) shell, parameterized with independent translations and rotations, only requires $C^0$ continuity between elements and is therefore mostly used in the classical FEA.

J. Kiendl initially developed a NURBS-based isogeometric Kirchhoff-Love shell element at the Chair of Structural Analysis of the Technical University of Munich [14]. The usage of NURBS as shape functions gives rise to a higher polynomial order, a higher continuity and a point-wise exact shell director in the whole patch domain. Making use of these beneficial properties, a hierarchic family of shells was presented by R. Echter at the Institute of Structural Mechanics of the University of Stuttgart [10]. The hierarchic shells consist of a 3-parameter KL, a 5-parameter RM and a 7-parameter 3D shell, where the number of parameters indicates how
many degrees of freedom the particular shell has per node respectively control point in the context of IGA. The notion of hierarchy means that the three formulations build up on each other in a model-adaptive way. In common RM element formulations, either the rotations or a shear difference vector which is directly added to the undeformed normal director are chosen as primal variables. In contrast, a hierarchic shear difference vector which is added to the deformed normal director is introduced in order to capture the shear deformation of the hierarchic shear deformable shell elements. An important advantage of this concept is that the hierarchic RM and the hierarchic 3D shell are intrinsically free from transverse shear locking due to this specific parameterization. The 3D shell is also intrinsically free from curvature thickness locking and Poisson thickness locking which normally affect standard 3D shells.

The hierarchic shells are derived for geometrically linear problems in the study of Echter et al. [10]. An extension to geometrically non-linear kinematics is developed by B. Oesterle for the RM shell and in the scope of a master thesis for the 3D shell at the same institute ([16] and [20]). This extension facilitates the simulation of complex shell structures with large rotations.

Within this thesis, all implementations and simulations are performed in the IgaApplication of the open-source software Kratos Multiphysics [12]. Shell elements have the advantage in the context of development of simulation tools that they are versatile with respect to their fields of application. They can be used for a range of different structures including typical beam and plate problems as well as complex curved shells.

1.2 Objective

The objective of this thesis is the implementation and validation of the above described hierarchic family of isogeometric shell elements in the module IgaApplication of the open-source software Kratos Multiphysics. The shell formulations shall be able to simulate complex structures. Therefore, they are implemented with a geometrically non-linear kinematic which facilitates the simulation of large rotations. Furthermore, the shells are also discretized in thickness direction and solved by Gauss integration. This allows the application of complex plastic material laws. The 3-parameter KL shell uses two-dimensional material laws considering only in-plane strains. The 5-parameter RM shell is statically condensed neglecting transverse normal strains. However, the 7-parameter 3D shell enables the use of full three-dimensional material laws. Since the simulations are performed for structural applications, the computation of stresses and internal forces plays a crucial role. Therefore, stress recovery procedures shall be added to the postprocessing.

The KL shell has already been implemented. Within this thesis, a stress recovery procedure shall be added. The other two shells shall be implemented from scratch in the scope of the present work. Since the focus lies on the element formulations themselves, multiple patches and complex support as well as coupling conditions are not considered because they would require special care with respect to the additional parameters of the RM and 3D shell. Only fixed or moving supports are used where solely the translations of the mid-surface of the shell are fixed.
The shell formulations are validated for different numerical examples against analytical solutions and reference solutions provided by [9], [13] and [16]. The referenced researches themselves also compared their results to simulations with further shell formulations which facilitates the discussion of advantages and disadvantages of the hierarchic shell concept. Respective references are given to the literature where needed. The numerical examples are chosen in such a way that different structural effects can be elaborated separately.

1.3 Overview

In the following, a short overview about the three main chapters of this thesis is given. In chapter 2, the fundamentals concerning the Finite Element Method for structural shell analysis are described. At first, the typology of a shell and the different shell theories are introduced in general. Afterwards, the main principles, equations and procedures are explained in order to build up a system of equations for a structural shell analysis. Special focus is put on the description of the differential geometry which is of high importance since a shell is a free-form object. The hierarchic shell family is developed on the basis of the Isogeometric Analysis. Therefore, the specialties and key terms of IGA are summarized. At the end of the chapter, the locking phenomena which occur in analysis with shell elements are elaborated. They are of special interest because the hierarchic shells promise to intrinsically avoid some locking types.

Chapter 3 describes the hierarchic shell formulation. The shells are presented in the same order as they build up on each other: first the 3-parameter KL shell, then the 5-parameter RM shell and as last the 7-parameter 3D shell. Each shell element is introduced in a separate section. The main components which are needed for the computation of the stiffness matrix and the load vector are presented for each element. The solution of an integral over the whole shell body is required in order to set up the system of equations. The integration over the mid-surface is performed by a Gauss integration. In thickness direction, it is possible to apply a pre-integration or a Gauss integration. Both alternatives are described and the differences are discussed. Furthermore, the stress recovery procedure is elaborated because the computation of stresses and internal forces plays a crucial role in structural analysis. In addition, the static condensation of strains from the system of equations is explained because the transverse normal strain has to be removed by means of this method in case of the RM shell element in order to apply three-dimensional material laws.

Chapter 4 discusses the numerical examples which are engaged in order to validate the shell implementation. The examples are separated in geometrically linear and non-linear problems. The obtained results are compared to analytical solutions and solutions from literature. The quality of the results is discussed in detail, taking into account deformations and internal forces. Furthermore, transverse shear locking and membrane locking are investigated by means of two examples.
2 Finite Element Method for Structural Shell Analysis

2.1 Shell Simulation

Shells can be defined as arbitrarily shaped structures, where one dimension is much smaller than the two others. The curvature is optimally chosen such that the bending forces of the structure are alleviated and that the shell mainly bears in membrane action. The high slenderness of a shell in combination with compression forces may yield to systems which are very sensitive with respect to instability. Therefore, an exact prediction of the shell’s behaviour is necessary. Since there are no analytical solutions available for many cases, approximate solutions are needed (e.g. by means of the Finite Element Method). Different computational formulations are known in practice. The shell can be understood as reduction from a solid where the thickness is very small. The main reason for this reduction is the decreased computational effort.

Depending on further assumptions, shell theories can be distinguished in shear rigid (Kirchhoff-Love), shear deformable (Reissner-Mindlin) and three-dimensional (3D) shells. The three different formulations are discussed in detail in Chapter 3. Starting from the shear rigid over the shear deformable to the 3D shell, the respective shell formulations yield more accurate results. However, correct results are obtained with the Kirchhoff-Love shell for many problems and the application of a simpler shell is concurrently more efficient with respect to the computation time. For thin shells, the Kirchhoff-Love shell is appropriate [13, section 3.2], whereas a Reissner-Mindlin shell should be used for thick shells in order to achieve reliable results. The thickness describes a critical parameter, on which the application limits of the two theories are based. In the literature, the slenderness \((R/t)\), which is the ratio between the radius \(R\) and the thickness \(t\) of a shell, is also often chosen as critical parameter. The slenderness of shells can be quantified as shown in table 2.1. In contrast to the Kirchhoff-Love and the Reissner-Mindlin shell, three-dimensional shells additionally consider load induced thickness changes by means of a semi-discretization of the thickness direction and are therefore closer related to volumetric elements.

<table>
<thead>
<tr>
<th>(R/t)</th>
<th>thick</th>
<th>thin</th>
<th>very thin</th>
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<tr>
<td>&lt; 20</td>
<td>20 to 1000</td>
<td>&gt; 1000</td>
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Table 2.1: Distinction of shell slenderness ratios (based on [5, pg. 1.23])
2.2 Finite Element Analysis based on Continuum Mechanics

In this section, the governing equations of a finite element procedure for shells are presented. At first, the differential geometry of free-form surfaces is discussed and special features of shells are investigated (subsection 2.2.1). Coming from these fundamentals, the kinematic equations are derived, whereby the Green-Lagrange strain tensor is used as strain measure (subsection 2.2.2). The constitutive equations used in this thesis are shortly presented. The Piola-Kirchhoff stress (PK2) is energetically conjugate to the Green-Lagrange strain. The transformation rule from the PK2 to the Cauchy stress, which represents physical meaningful values, is also determined (subsection 2.2.3). In subsection 2.2.4 the equilibrium equations in the strong and the weak form and the discretization are discussed. In the last subsection 2.2.5 the solving strategy, which is the Newton-Raphson-Method in the scope of this thesis, is shortly described.

2.2.1 Differential Geometry

Shells as free-form surfaces are described in the parametric space. In this subsection, all important geometrical formulas are summarized in order to facilitate a finite element analysis. At first, geometry descriptions of surfaces in curvilinear coordinates are discussed in general. Then, the transformation between different coordinate spaces, such as the co- and contravariant, and the Cartesian coordinate systems, is explained. Finally, the geometry description is specified with respect to particular features of shells. The subsequent content follows [13, Chapter 2.4] as well as [16, Chapter 6.1] and is modified to cover specific questions within this thesis. The following convention is used: Latin letters take the values 1, 2, 3 whereas Greek letters only take the values 1, 2.

2.2.1.1 Geometry Description of Surfaces in Curvilinear Coordinates

For the description of arbitrarily curved geometries, it is advantageous to use curvilinear coordinates instead of Cartesian ones. In the curvilinear space there are two important bases - the covariant basis $g_i$ and the contravariant basis $g^i$. The corresponding contravariant and covariant coordinates are called $\theta^i$ and $\theta_i$ respectively. Thus, the position vector $x$ can be expressed as:

$$ x = \theta_i g^i = \theta^i g_i $$

(2.1)

The description of the position vector $x$ by means of co- and contravariant coordinates is illustrated in figure 2.1. The covariant base vectors are defined as:

$$ g_i = \frac{\partial x}{\partial \theta^i} = x_{,i} $$

(2.2)
Figure 2.1: Geometrical description of the position vector $x$ in co- and contravariant coordinates (based on [15, example 1.4])

Figure 2.1 shows graphically the relationship between the covariant and contravariant base vectors. They are pairwise perpendicular to each other and mathematically related by the Kronecker delta $\delta^i_j$:

$$g_i \cdot g_j = \delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$  \hspace{1cm} (2.3)$$

Investigating surfaces, the first two base vectors $g_1$ and $g_2$ can be computed as in equation 2.2. These vectors span the surface. The third base vector $g_3$, which is perpendicular to the surface, is calculated from $g_1$ and $g_2$ as:

$$g_3 = \frac{\mathbf{g}_3}{dA} = \frac{g_1 \times g_2}{|g_1 \times g_2|}$$\hspace{1cm} (2.4)$$

where $\mathbf{g}_3$ is the not-normalized director and $dA$ is the differential area. As $g_3$ is perpendicular to $g_1$ and $g_2$, and normalized, it holds that:

$$g_3 = g^3$$\hspace{1cm} (2.5)$$

An important quantity of surfaces is the metric tensor $g$. The corresponding metric coefficients $g_{\alpha\beta}$ contain important properties of the surface such as the length of the base vectors and the angles between them. The metric tensor can be expressed in the covariant as well as in the contravariant basis:

$$g = g^{\alpha\beta} g_\alpha \otimes g_\beta = g_{\alpha\beta} g^\alpha \otimes g^\beta$$\hspace{1cm} (2.6)$$
Equation [2.6] is called the first fundamental form of surfaces [13, section 2.4]. The covariant metric coefficients can be computed by the scalar product of the covariant base vectors:

$$g_{\alpha\beta} = g_{\alpha} \cdot g_{\beta} \quad (2.7)$$

The covariant and contravariant metric coefficients are the inverse of each other:

$$[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1} \quad (2.8)$$

With the metric coefficients, it is possible to perform a transformation between the covariant and contravariant base vectors:

$$g^{\alpha} = g^{\alpha\beta} g_{\beta} \quad (2.9)$$
$$g_{\alpha} = g_{\alpha\beta} g^{\beta} \quad (2.10)$$

The second fundamental form of surfaces (equation [2.11]) describes the curvature properties of a surface. The curvature tensor coefficients $b_{\alpha\beta}$ are defined as [13, subsection 2.4]:

$$b_{\alpha\beta} = -g_{\alpha} \cdot g_{3,\beta} = -g_{\beta} \cdot g_{3,\alpha} = g_{\alpha,\beta} \cdot g_{3} \quad (2.11)$$

Mostly, constitutive laws are defined with respect to Cartesian coordinate systems as in the open source software used in the present work. Therefore, a transformation from the parametric to a local Cartesian space is needed. The local Cartesian coordinates $e_{1}, e_{2}$ and $e_{3}$ can be defined as follows:

$$e_{1} = g_{1} \frac{g_{1}}{|g_{1}|} \quad (2.12)$$
$$e_{2} = g_{2} \frac{g_{2}}{|g_{2}|}$$
$$e_{3} = g_{3} = g_{3}$$

The covariant and contravariant Cartesian coordinates are identical by definition:

$$e_{i} = e^{i} \quad (2.13)$$

### 2.2.1.2 Transformation between Different Coordinate Systems

The transformation between the different coordinate systems is of high importance with respect to the application of the material laws and the stress conversion as stated in subsection 2.2.3. The material stiffness matrix with its physical values for example refers to a local Cartesian coordinate system, whereas the strain tensor is defined with respect to a curvilinear contravariant coordinate system. Therefore, the strain has to be transformed from one space to the other in order to facilitate the use of the constitutive equation as stated in equation 2.40.
In the following, the transformation from one base to another is described in general. Considering two different bases \(c_i\) and \(d_i\), a tensor \(M\) can be described as:

\[
M = c^i c_j \otimes c^j = d^i d_j \otimes d^j
\]  

(2.14)

The transformation between the contravariant bases \(c^i\) and \(d^i\) is shown as an example:

\[
c^i c^j \otimes c^j = d^i d^j \otimes d^j
\]  

(2.15)

For this purpose, both terms are multiplied by the base vectors \(d_k\) and \(d_l\) from two sides and the condition from equation (2.3) is used:

\[
c^i j d_k (c^i \otimes c^j) d_l = d^i j d_k (d^i \otimes d^j) d_l
\]

(2.16)

\[
c^i j (d_k \cdot c^i) (c^j \cdot d_l) = d^i j (d_k \cdot d^j) (d^l \cdot d_l)
\]

\[
= d^i j \delta_k^i \delta_l^j
\]

The same derivation for all possible combinations of co- and contravariant base vectors yields to four transformation rules:

\[
d_{kl} = c^i j (d_k \cdot c^i) (c^j \cdot d_l)
\]  

(2.17)

\[
d_{kl} = c^i j (d_k \cdot c^i) (c^j \cdot d_l)
\]  

(2.18)

\[
d_{kl} = c^i j (d_k \cdot c^i) (c^j \cdot d_l)
\]  

(2.19)

\[
d_{kl} = c^i j (d_k \cdot c^i) (c^j \cdot d_l)
\]  

(2.20)

An example is discussed in the following in order to make the transformation between different coordinate systems easier understandable. The transformation from the curvilinear contravariant to the local Cartesian coordinate system is examined for a Kirchhoff-Love shell. This transformation is for example needed for the strain vector which is computed with respect to the contravariant coordinate system but needed in the local Cartesian space in order to apply a constitutive equation which is normally given in the local Cartesian coordinate system:

\[
\bar{E}_{\alpha \beta} e_\alpha \otimes e_\beta = E_{\gamma \delta} G^\gamma \otimes G^\delta
\]  

(2.21)

where \(\bar{E}_{\alpha \beta}\) and \(E_{\gamma \delta}\) are the geometrically non-linear Green-Lagrange strain coefficients. The upper bar (\(\bar{\)}\) denotes that the value refers to local Cartesian coordinates. In case of a Kirchhoff-Love shell, the strains can be reduced to the in-plane components. The strain vector in Voigt notation in the contravariant space is defined by:

\[
E_{\gamma \delta} = \begin{bmatrix}
E_{11} \\
E_{22} \\
E_{12}
\end{bmatrix}
\]  

(2.22)
As usual for engineering strains, the shear components are multiplied by a factor 2 in Voigt notation in the local Cartesian coordinate system:

\[
\bar{E}_{\alpha\beta} = \begin{bmatrix}
\bar{E}_{11} \\
\bar{E}_{22} \\
2\bar{E}_{12}
\end{bmatrix}
\]  (2.23)

According to equation 2.17, the transformation from \(E_{\gamma\delta}\) to \(\bar{E}_{\alpha\beta}\) is given by:

\[
\bar{E}_{\alpha\beta} = E_{\gamma\delta}(e_{\alpha} \cdot G^{\gamma})(e_{\beta} \cdot G^{\delta}) = E_{\gamma\delta} T_{\alpha\beta\gamma\delta}^{con-car}
\]  (2.24)

where \(T_{\alpha\beta\gamma\delta}^{con-car}\) are the coefficients of the transformation matrix from the curvilinear contravariant to the local Cartesian space. Using the Voigt notation, the indices can be reduced as follows: \(\alpha\beta \rightarrow i\) and \(\gamma\delta \rightarrow j\). The indices \(i\) and \(j\) refer to the three entries of the strain vectors in Voigt notation of the equations 2.22 and 2.23. Thus, equation 2.24 may be written as:

\[
\bar{E}_{i} = E_{j} T_{ij}^{con-car}
\]  (2.25)

The transformation matrix \(T_{ij}^{con-car}\) has the dimensions 3 \(\times\) 3 in Voigt notation. The coefficients are computed as:

\[
\begin{align*}
T_{11}^{con-car} &= (e_{1} \cdot G^{1})(e_{1} \cdot G^{1}) = (e_{1} \cdot G^{1})^2 \\
T_{12}^{con-car} &= (e_{1} \cdot G^{2})(e_{1} \cdot G^{2}) = (e_{1} \cdot G^{2})^2 = 0 \\
T_{13}^{con-car} &= 2(e_{1} \cdot G^{1})(e_{1} \cdot G^{2}) = 0 \\
T_{21}^{con-car} &= (e_{2} \cdot G^{1})(e_{2} \cdot G^{1}) = (e_{2} \cdot G^{1})^2 \\
T_{22}^{con-car} &= (e_{2} \cdot G^{2})(e_{2} \cdot G^{2}) = (e_{2} \cdot G^{2})^2 \\
T_{23}^{con-car} &= 2(e_{2} \cdot G^{1})(e_{2} \cdot G^{2}) \\
T_{31}^{con-car} &= 2(e_{1} \cdot G^{1})(e_{2} \cdot G^{1}) \\
T_{32}^{con-car} &= 2(e_{1} \cdot G^{2})(e_{2} \cdot G^{2}) = 0 \\
T_{33}^{con-car} &= 2 \left((e_{1} \cdot G^{1})(e_{2} \cdot G^{2}) + (e_{1} \cdot G^{2})(e_{2} \cdot G^{1})\right) = 2(e_{1} \cdot G^{1})(e_{2} \cdot G^{2})
\end{align*}
\]  (2.26)

Some coefficients become zero due to the specific definition of the local Cartesian basis depending on the curvilinear contravariant basis as given in equation 2.12. The underlined deuces arise due to the different consideration of the shear components with and without a factor of 2 as stated in the equations 2.22 and 2.23.

### 2.2.1.3 Geometry Description of Shells

So far, all discussed properties are generally valid for parametric descriptions of surfaces. The shell theory reduces a structure to a two-dimensional geometry in the parametric space defined by its mid-surface. The curvilinear coordinates \(\xi^{\alpha}\) describe this mid-surface. The third coordinate is defined in the direction of the shell’s thickness by \(\xi^{3} \in [-t/2, t/2]\), whereby \(t\) is the thickness. The base vectors at the mid-surface are denoted by \(A_{i}\) in the reference configuration and \(a_{i}\) in the current configuration, whereas the base vectors at an arbitrary
point of the shell body are called $G_i$ and $g_i$ respectively. The reference and the current configuration refer to the undeformed and the deformed structure respectively. The base and position vectors in the reference configuration are denoted by upper-case characters and the actual configuration by lower-case characters. The two configurations of a shell body and all important vectors are shown in figure 2.2.

![Figure 2.2: Reference and current configuration of a shell body](image)

The position of any point in the undeformed state is defined by the position vector $X$ as:

$$X(\xi^1, \xi^2, \xi^3) = R(\xi^1, \xi^2) + \xi^3 A_3(\xi^1, \xi^2)$$ (2.27)

where $R$ is the position vector of a point on the mid-surface of the shell in the reference configuration. The derivative of the position vector $X(\xi^1, \xi^2, \xi^3 = 0)$ with respect to the coordinates of the shell’s mid-surface $\xi^\alpha$ yields the tangential covariant base vectors:

$$A_\alpha = \frac{\partial X}{\partial \xi^\alpha} \bigg|_{\xi^3=0} = \frac{\partial R}{\partial \xi^\alpha} = R_{\alpha}$$ (2.28)

The third base vector $A_3$ can be computed by equation 2.4. As a consequence of equation 2.27 the base vectors $G_i$ of the shell body are computed as follows:

$$G_\alpha = \frac{\partial X}{\partial \xi^\alpha} = R_{\alpha} + \xi^3 A_{3,\alpha} = A_\alpha + \xi^3 A_{3,\alpha}$$

$$G_3 = \frac{\partial X}{\partial \xi^3} = A_3$$ (2.29)
One should notice that \( G_3 = A_3 \) is not generally valid. The underlying assumption is that the metric is linear in direction of the thickness \( \xi^3 \) (see equation \((2.27)\)). The position vector \( x \) of the deformed system can be computed analogically to equation \((2.27)\):

\[
x(\xi^1, \xi^2, \xi^3) = r(\xi^1, \xi^2) + \xi^3 a_3(\xi^1, \xi^2)
\]

where \( r \) is the position vector of a point on the mid-surface of the shell in the current configuration. It is important to note that the director \( a_3 \) is no longer defined by equation \((2.4)\) but that it is dependent on the shell model. Therefore, the actual definition of the director \( a_3 \) for each shell model presented in this thesis is specified in the respective subsection of chapter \(3\) where the different shell formulations are discussed. The tangential covariant base vectors \( a_\alpha \) of the current configuration are computed as:

\[
a_\alpha = \left. \frac{\partial x}{\partial \xi^\alpha} \right|_{\xi^3=0} = \left. \frac{\partial r}{\partial \xi^\alpha} \right|_{\xi^3=0} = r_{,\alpha} \tag{2.31}\]

In order to perform a stress conversion, which is discussed in the following subsection \((2.2.3)\), the deformation gradient \( F \) is needed. The matrix \( F \) describes the mapping of a differential line element in the reference configuration \( dX \) into a line element in the actual configuration \( dx \):

\[
dx = F \cdot dX \tag{2.32}\]

The deformation gradient is defined by the base vectors in the reference and the actual configuration:

\[
F = g_i \otimes G^i \quad F^T = G^i \otimes g_i \quad F^{-1} = G_i \otimes g^i \quad F^{-T} = g^i \otimes G_i \tag{2.33}\]

### 2.2.2 Kinematics

The kinematic, which describes the deformation of a body, is discussed with respect to shells in the following section. The displacement \( u \) of an arbitrary point of the shell body, which is shown in figure \((2.2)\), is computed from the position vectors of the reference and the actual configuration \( X \) and \( x \):

\[
u = x - X = r + \xi^3 a_3 - R - \xi^3 A_3 = \nu + \xi^3 (a_3 - A_3) \tag{2.35}\]

where \( \nu \) denotes the displacement of the mid-surface of the shell. Different strain measures are used in order to describe deformations excluding rigid body movements. The Green-Lagrange strain is applied in this thesis. It describes a non-linear relation between deformations and strains and it is therefore used for large deformations. The Green-Lagrange strain \( E \) can also be linearised for a formulation which considers only small displacements. Such a simplification is for example used in \([16]\) in order to derive shell models for small displacements. The advantage of such models lie in the better computational performance in comparison to non-
linear ones. There are two possible ways of computing $E$ provided by the literature. In [13, equation (3.8)], the formula for the strain $E$ is given as:

$$E = \frac{1}{2} (g_{ij} - G_{ij}) G^i \otimes G^j \quad (2.36)$$

In [16, equation (6.19)], an alternative formula is provided:

$$E = \frac{1}{2} (u_{,i} \cdot G_j + u_{,j} \cdot G_i + u_{,i} \cdot u_{,j}) G^i \otimes G^j \quad (2.37)$$

It can be shown that the equations 2.36 and 2.37 are identical. Within this thesis, the first alternative is used for the Kirchhoff-Love shell, and the second alternative is chosen for the 5-parameter (Reissner-Mindlin) and the 7-parameter (3D) shell. The reason for this approach is simply that the literature, on which the different formulations are based, uses these two variants in the specified cases. If $E$ is calculated by means of equation 2.37, the derivatives of the displacement with respect to the curvilinear coordinates $u_{,i}$ have to be determined:

$$u_{,\alpha} = v_{,\alpha} + \xi^3 (a_{3,\alpha} - A_{3,\alpha}) \quad (2.38)$$
$$u_{,3} = a_3 - A_3 \quad (2.39)$$

### 2.2.3 Constitutive Equations

A St.Venant-Kirchhoff material model is used in all calculations performed in this thesis. It assumes a linear relation between strains and stresses. The usage of the Gauss integration over the thickness for the RM and 3D shells presented in this thesis allows also the application of plastic material laws. Different strain measures are known, whereby the Green-Lagrange strain $E$ is used in this work. The Green-Lagrange strain is associated with the corresponding Piola-Kirchhoff stress of second kind $S$ (PK2) by the material stiffness tensor $C$:

$$S = C : E \quad (2.40)$$
$$S^{ij} = C^{ijkl} E_{kl} \quad (2.41)$$
$$C = C^{ijkl} G_i \otimes G_j \otimes G_k \otimes G_l \quad (2.42)$$

However, the material stiffness tensor is mostly defined with respect to Cartesian coordinates. The constitutive equation is then given as:

$$\bar{S} = \bar{C} : \bar{E} \quad (2.43)$$
$$\bar{S}^{ij} = \bar{C}^{ijkl} \bar{E}_{kl} \quad (2.44)$$
$$\bar{C} = \bar{C}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l \quad (2.45)$$

It is possible to switch between the different coordinate systems by means of the transformation rules described in subsection 2.2.1.2. It is noteworthy that the PK2 stress is energetically
conjugate to the Green-Lagrange strain. However, \( S \) has to be converted to the Cauchy stress tensor \( \sigma \) in order to obtain physical meaningful stresses. The PK2 stress tensor refers to the covariant coordinate system of the undeformed configuration and the Cauchy stress tensor to that one of the deformed configuration:

\[
S = S^{ij} G_i \otimes G_j
\]

\[
\sigma = \sigma^{ij} g_i \otimes g_j
\]

The rule for the conversion from one stress measure to the other is as follows:

\[
\sigma = (\det F)^{-1} \cdot F \cdot S \cdot F^T
\]

The formula for the computation of the Cauchy stress coefficients can be derived by the means of equation 2.33 and the associative calculation rule:

\[
\sigma_{ij} g_i \otimes g_j = (\det F)^{-1} \cdot F \cdot S_{ij} G_i \otimes G_j \cdot F^T = (\det F)^{-1} S_{ij} (FG_i) \otimes (G_j F^T)
\]

\[
= (\det F)^{-1} S_{ij} g_i \otimes g_j
\]

### 2.2.4 Equilibrium

The strong form of the equilibrium is given as:

\[
\text{div}(F \cdot S) + \rho_0 B = 0
\]

\( F \) and \( S \) are explained in chapter 2.2.3. The scalar \( \rho_0 \) is the density and \( B \) the vector of body forces, both with respect to the reference configuration. In most cases it is not possible to solve this equilibrium equation directly. Therefore, discretization methods are employed, like the Finite Element Method, which is used in this thesis. Following the isoparametric concept, the same shape functions are used for the discretization of the geometry and the displacements. The discretization rule is defined by a linear combination of shape functions and nodal variables:

\[
X = \sum_i N^i \hat{X}^i
\]

\[
u = \sum_i N^i \hat{u}^i
\]

The symbol (\( \hat{} \)) marks that the variables represent discrete values. In combination with equation 2.35, it yields:

\[
x = \sum_i N^i (\hat{X}^i + \hat{u}^i)
\]

In the Finite Element Method, the equilibrium is not fulfilled point-wise but in an integral sense. The resulting equation of equilibrium is called the weak form of the problem. Based
on the Principle of Virtual Work, the sum of the internal and external work has to vanish [3, Chapter 3.1.3]:
\[ \delta W = \delta W_{\text{int}} + \delta W_{\text{ext}} = 0 \]  
\[ (2.54) \]

The equilibrium has to be fulfilled for an arbitrary variation \( \delta D_r \) of the global degree of freedoms \( D_r \):
\[ \delta W = \frac{\partial W}{\partial D_r} \delta D_r = 0 \]  
\[ (2.55) \]
\[ \frac{\partial W}{\partial D_r} = 0 \]  
\[ (2.56) \]

The given equation \[ (2.56) \] represents a non-linear equation system if the virtual work \( \delta W \) depends as function on the degree of freedoms \( D_r \), i.e. \( \delta W(D_r) \). This is the case within this thesis because the presented shell formulations are geometrically non-linear. The Newton-Raphson method is applied as solving strategy. This method solves non-linear systems of equations as iterative sequence of linearised subproblems. Therefore, equation \[ (2.56) \] is linearised:
\[ \frac{\partial W}{\partial D_r} + \frac{\partial^2 W}{\partial D_r \partial D_s} \Delta D_s = 0 \]  
\[ (2.57) \]

The first derivative of the virtual work with respect to the displacement vector is called the residual force vector \( \mathbf{R} \) and can be computed by:
\[ R_r = \frac{\partial W}{\partial D_r} = \left( \frac{\partial W_{\text{int}}}{\partial D_r} + \frac{\partial W_{\text{ext}}}{\partial D_r} \right) = F_{\text{int}}^r + F_{\text{ext}}^r \]  
\[ (2.58) \]

where \( r \) refers to the entry of the residual force vector corresponding to the degree of freedom \( D_r \). The external nodal forces \( \mathbf{F}^{\text{ext}} \) are directly derived from the loads and the internal nodal forces \( \mathbf{F}^{\text{int}} \) are given by:
\[ F_{\text{int}}^r = - \int_{V_{R}} \left( \frac{\partial \mathbf{E}}{\partial D_r} \right)^T \mathbf{S} \ dV_{R} \]  
\[ (2.59) \]

where \( dV_{R} \) is the differential volume of the reference configuration. Its calculation is discussed in chapter [3.2.6] The second derivative of the virtual work with respect to the degrees of freedom \( D_r \) and \( D_s \) yields the stiffness matrix \( \mathbf{K} \). It can be subdivided in an internal and external part:
\[ K_{rs} = - \left( \frac{\partial^2 W_{\text{int}}}{\partial D_r \partial D_s} - \frac{\partial^2 W_{\text{ext}}}{\partial D_r \partial D_s} \right) = K_{\text{int}}^{rs} + K_{\text{ext}}^{rs} \]  
\[ (2.60) \]

The part \( K_{\text{ext}}^{rs} \) describes the derivative of external loads with respect to the degrees of freedom. It only exists if the loads are displacement-dependent. This kind of loads are not used in this
thesis. The coefficients of the internal stiffness matrix $K_{rs}^{\text{int}}$ are computed by deriving $F_r^{\text{int}}$ again with respect to the displacement variables:

$$K_{rs}^{\text{int}} = \int_{V_R} \left( \left( \frac{\partial E}{\partial D_r} \right)^T \frac{\partial S}{\partial D_s} + \left( \frac{\partial^2 E}{\partial D_r \partial D_s} \right)^T S \right) dV_R$$

$$= \int_{V_R} \left( \left( \frac{\partial E}{\partial D_r} \right)^T C \frac{\partial E}{\partial D_s} + \left( \frac{\partial^2 E}{\partial D_r \partial D_s} \right)^T CE \right) dV_R$$

$$= \int_{V_R} \left( B^T CB + \left( \frac{\partial^2 E}{\partial D_r \partial D_s} \right)^T CE \right) dV_R$$

(2.61)

The internal stiffness matrix $K^{\text{int}}$ can be split up in a so called initial stiffness matrix $K_e$ and a geometrical stiffness matrix $K_g$, corresponding to the first and second term of equation (2.61) respectively. The consideration of the geometrical stiffness matrix $K_g$ including the second derivatives of the strain $E$ facilitates an efficient geometrically non-linear analysis but is not necessary. The linearised equation system is finally determined by inserting the equations (2.58 - 2.61) into equation (2.57):

$$K \Delta D = R$$

(2.62)

where $D$ is the global vector containing the degrees of freedom. Further informations about the topics of this subsection are provided by [13] and [16].

### 2.2.5 Solving Strategy

The non-linear system of equations, given in equation (2.56), is solved iteratively by the means of the Newton-Raphson method. In each iteration step, the linearised system of equations given in equation (2.57) is solved. One iteration step $i$ can be shortly described by the following three operations:

1. check: $R(D^i) \approx 0$
2. $\Delta D^{i+1} = -K(D^i)^{-1}R(D^i)$
3. $D^{i+1} = u^i + \Delta D^{i+1}$

Step (1) holds the convergence criterion based on the residual force vector $R$ which is constituted by the unbalanced forces.
2.3 Isogeometric Analysis

The development and implementation of the shell elements which are discussed in this thesis is carried out in the context of the Isogeometric Analysis (IGA). Therefore, the basic ideas of this analysis method are shortly introduced. Many principles of the element formulation are the same as in the classical Finite Element Analysis (FEA).

In the classical FEA the meshing and reconstruction of the CAD-based design geometry in an analysis environment plays a crucial role. This transformation process requires additional computational effort and may lead to approximation errors. The basic concept of IGA is to directly apply the geometrical description which is used in CAD to the analysis procedure. As a consequence, no transformation is needed for pre- and postprocessing, and the analysis procedure.

According to the isoparametric concept, which is known from the classical FEA, the same shape functions $R_i$ are used for the discretization of the geometry $x^h$ and the displacement field $u^h$:

$$x^h = \sum_{i=1}^{n} R_i \hat{x}_i$$
$$u^h = \sum_{i=1}^{n} R_i \hat{u}_i$$

(2.63)

where $(^h)$ denotes a discretized variable. In CAD, mainly Non-uniform rational B-Spline (NURBS) functions are used as shape functions $R_i$ for the geometry description. NURBS have the advantage that they are able to correctly capture complex shapes such as conic sections. In this section, only a short introduction to the description and properties of NURBS is given. More detailed information can be found in the literature ([8], [13], [21]).

An example for the description of a curve by a NURBS function is shown in figure 2.3. NURBS are described by a knot vector $\Xi$, a polynomial degree $p$, a certain number of control points $n$ and weights $w_i$ corresponding to the $i^{th}$ control point. In figure 2.3 the control points are depicted by the points $P_i$ which are spanning the control polygon, marked with the dotted line. The small crosses on the continuous curve show the position of the knots. The relation between the number of knots $m$, the number of control points $n$ and the polynomial degree $p$ is given as:

$$m = n + p + 1$$

(2.64)

The curve has an open knot vector. This means that the knots at the beginning and the end are repeated ’$p+1$’-times. In consequence, the control points at beginning and end are interpolated and the curve is tangential to the control polygon at these points. The NURBS functions $R_{i,p}$ are computed by the B-Spline basis functions $N_{i,p}$ and the weights $w_i$:

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{i=1}^{n} N_{i,p}(\xi)w_i}$$

(2.65)
The B-Spline basis functions are computed by the Cox-De-Boor recursion formula. As it can be seen in equation 2.63, the geometry and the displacement field are discretized at the control points. It is important to notice that the control points are not located on the geometric object itself as shown in figure 2.3. Therefore, the discrete nodal results $\hat{u}_i$ have no physical meaning. The NURBS function has to be evaluated at a point $\xi$, which lies on the object itself, in order to obtain physical meaningful values.

The region between two knots is called knot span. As smallest possible entities, they are used as elements. This is shown in figure 2.4. These elements can be included in the FE procedure in the same way as classical elements. If all weights $w_i$ are equal and the knot spans are uniformly distributed, the NURBS are reduced to so called B-Splines. In this thesis only B-Splines are used as they are sufficient to describe the investigated geometries.

For exact models and appropriate element formulations a high continuity is desired. The continuity within one element is $C^\infty$ and between two elements $C^{p-k}$, where $k$ is the multiplicity of the knot. Within a NURBS patch, knots have normally a multiplicity of one except an interpolation of a certain control point is wanted, which is facilitated by a knot multiplicity of $p$. Thus, a higher continuity across element boundaries is normally given for IGA compared to the classical FEA because of the higher polynomial degree of the NURBS shape functions.

The properties of NURBS curves which were discussed so far can be directly transferred to NURBS surfaces which are obtained by a tensor product of two NURBS functions. Such a NURBS surface in the parametric space is shown in figure 2.4. The influence of the control points on single elements can be observed. One control point $i$ has influence within the region $[\xi_i, \xi_{i+p+1}]$. 

**Figure 2.3:** NURBS curve, $p = 3$, $n_{CP} = 7$, $\Theta = 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1$ and $w_i = 1 \forall i$ (based on [21] figure 2.15)
Isogeometric B-Rep Analysis (IBRA)

For more complex geometries, boundary representations (B-Rep) are commonly used in CAD. They consist of multiple trimmed NURBS patches which are coupled with each other. In the recent years, the isogeometric B-Rep analysis (IBRA) was developed by making use of this kind of CAD modelling. Multiple patches are not used in this thesis and the problems of trimming and coupling are therefore not further described. Specific information about this topic can be found in [6].
2.4 Locking Phenomena of Shells

Locking is a common problem in numerical structural analysis. The most known phenomenon is the transverse shear locking of the Timoshenko beam. Due to this locking phenomenon, shear forces occur in a beam under pure bending. In this thesis, two shell models are implemented that intrinsically prevent certain locking effects: a 5-parameter Reissner-Mindlin shell (5p) and a 7-parameter 3D shell (7p). It has been shown that the 5p shell is free from transverse shear locking and the 7p shell from transverse shear, curvature thickness and poisson thickness locking ([9] and [16]). This promising feature is further discussed by means of two examples (subsection 4.1.3 and 4.2). The different locking phenomena, which affect structural shell analysis, are shortly explained in the following subsections.

The content of this section is mainly based on the work done by the Institute for Structural Mechanics at the University of Stuttgart ([3, section 6.4] and [16, chapter 4]). It is known that finite elements derived by the Principle of Virtual Displacements show bad results and reduced convergence rates in the pre-asymptotic region under certain circumstances. This phenomenon is called locking. Three ways to explain causes of locking effects are available: the mechanical, the numerical and the mathematical perspective. Within this thesis the mechanical perspective, which is probably the most illustrative, is chosen to explain this topic. For further information about the latter two, it is referred to [3, chapter 6.4.2 and chapter 6.4.3].

In the mechanical perspective, locking can be roughly described as the occurrence of parasitic strains or stresses, which are responsible for additional energy in the system. The consequence is an additional artificial stiffness. The parasitic contributions come from a disbalance of the shape functions between strains and their energetic conjugate stresses that yields geometrical constraints. Two symptoms of locking can be identified:

- underestimation of displacement solutions (structure behaves too stiff)
- oscillation of stress solutions

The intensity of the named symptoms depends on a critical parameter. Two groups of locking are distinguished in order to identify the kind of critical parameter: material and geometrical locking. The only discussed locking type which belongs to the group of material locking is the Poisson thickness locking (subsection 2.4.5). The corresponding critical parameter is the Poisson’s ratio $\nu$ or the bulk modulus $K$ which is calculated from the former one. The critical parameter for practically all geometrical locking phenomena, such as for example the transverse shear locking (subsection 2.4.1), is the thickness or the slenderness, which is derived from the thickness [16, section 4.1].

Different shell theories have different application limits and not all of them are affected by the same locking phenomena. The Kirchhoff-Love shell is for example not in danger of transverse shear locking but also not able to achieve accurate results for thick shells (concerning the application limits, it is referred to chapter 2.1).
Locking problems occur in classical FEA as well as in IGA. Methods to avoid locking for classical finite elements are available in various forms. Unfortunately, they are not directly applicable to IGA elements. However, some methods are already transferred to this analysis procedure (for example the NURBS-DSG presented in [10, chapter 6.1]).

Table 2.2 shows which locking types are problematic for the different shell models discussed in this thesis. All locking types which are listed in the table below are shortly examined in the following subsections. The 3p shell is the classical Kirchhoff-Love shell which is also the starting point for the hierarchic shell models as they are presented in this thesis. The 5p-hier and 7p-hier shells denote the hierarchic Reissner-Mindlin (RM) and 3D shell respectively. The elements 5p-stand, 6p-stand and 7p-stand refer to standard shell models. The difference between the standard and the hierarchic formulations with respect to the description of the position vector $\mathbf{x}$ is illustrated in figure 2.5. The standard formulations directly add a shear vector to the undeformed director $\mathbf{A}_3$, whereas the hierarchic description always starts from a deformed director $\mathbf{a}_3$. The shear deformation is alternatively described by a discretization of the total rotations $\varphi^1$ and $\varphi^2$ in case of the standard formulations. The hierarchic formulations 5p-hier and 7p-hier are able to intrinsically avoid certain locking phenomena in contrast to the standard formulations.

<table>
<thead>
<tr>
<th>Locking \ Shell</th>
<th>3p</th>
<th>5p-stand</th>
<th>5p-hier</th>
<th>6p-stand</th>
<th>7p-stand</th>
<th>7p-hier</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transverse shear</td>
<td></td>
<td>X</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>In-plane shear</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Membrane</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Curvature thickness</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Poisson thickness</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

*Table 2.2: Locking types and their appearance in different shell models*
2.4.1 Transverse Shear Locking

Transverse shear locking occurs for shear deformable shell elements. The critical parameter of this locking phenomenon is the slenderness or more directly the thickness. According to [3, Chapter 6.4.7], this locking phenomenon has severe influence on the quality of the results, in particular the shear forces.

The most illustrative way to describe transverse shear locking is an example on the Timoshenko beam where shear strains can be observed under the state of pure bending. The kinematic of a plane Timoshenko beam is shown in figure 2.6.
The kinematic equations are given by:

\[
\gamma = v_x + \varphi \\
\kappa = \varphi_x
\]  \hspace{1cm} (2.66)  \hspace{1cm} (2.67)

where \( \gamma \) is the transverse shear strain, \( \kappa \) the curvature, \( v \) the vertical displacement, \( \varphi \) the rotation and \( x \) the coordinate running along the beam. The displacement \( v \) and the rotation \( \varphi \) are the primal variables and are discretized with the same shape functions in the sense of an equal order interpolation.

In the easiest case, the shape functions are linear. In a state of pure bending no shear strains occur by definition, i.e. \( \gamma^h = 0 \). The derivative of the displacement \( v \) is constant due to the linear shape functions, i.e. \( v^h_x = \text{const} \). Thus, equation (2.66) is only fulfilled for a constant \( \varphi^h \), but the equation (2.67) for \( \kappa \) has to be constant for the state of pure bending. In consequence, it is not possible to fulfill the kinematic equations for this load case because of the disbalance of the shape functions. This disbalance remains also for higher shape functions. Quadratic shape functions are for example able to capture constant curvature, but lead to locking with a linear curvature distribution.

As usual for locking phenomena, the solution is still converging to the true value but with a much slower rate. Convergence originates from the fact that the slenderness as critical parameter does not refer to the whole structure but to each single element. The slenderness decreases for proceeding mesh refinement [3, chapter 6.4.7].

### 2.4.2 In-plane Shear Locking

The in-plane shear locking occurs in particular for in-plane loaded plate elements. This is the reason why it is most simply explained on a bi-linear in-plane loaded plate element under pure bending (see figure 2.7).

![Figure 2.7: Deformation and strain distributions of a bi-linear in-plane loaded plate element under pure bending [16, figure 4.2]](image)

The corresponding displacement field of this example is:

\[
u = [u_1, u_2]^T = [\xi^1 \xi^2, 0]^T \]  \hspace{1cm} (2.68)
From this displacement equations the strain distributions, which are also presented in figure 2.7, can be derived:

\[ \begin{align*}
\epsilon_{11} &= u_{1,1} = \xi^2 \\
\epsilon_{22} &= u_{2,2} = 0 \\
\epsilon_{12} &= \frac{1}{2} (u_{1,2} + u_{2,1}) = \frac{1}{2} \xi^1
\end{align*} \] (2.69)

The in-plane shear strain \( \epsilon_{12} \) should not occur for the state of pure bending and is therefore parasitic. Thus, the structure behaves stiffer than it should.

The described locking phenomenon is a problem for in-plane bending. For shell problems, it is therefore typically of minor interest. In this thesis no tool is provided to avoid in-plane shear locking because the focus lies on the shell element implementation itself. In order to eliminate this locking type, a Mixed-Displacement (MD) concept based on a variational method is presented in [16]. The same tool may be used to prevent membrane locking.

### 2.4.3 Membrane Locking

Inextensional deformations, as shown in figure 2.8, occur for geometrically linear shells under the state of pure bending where no membrane forces are active. The membrane stiffness of shells is often higher than the bending stiffness. Therefore, deformations due to bending are larger than for a mixed membrane-bending state. An everyday example is the rolling up of a piece of paper where large deformations occur with few energy consumption. An artificial stiffening due to membrane locking may prevent inextensional deformations.

![Figure 2.8: Inextensional deformations of shells](5 pg. 1.18)

A curved Bernoulli beam is considered as an example. Membrane locking occurs for all curved beam, shell or volumetric elements, but it is not easy to identify this locking phenomenon.
in shell equations. In figure 2.9 the geometrical description of a curved Bernoulli beam is presented.

Figure 2.9: Geometry of a curved Bernoulli beam [16 figure 4.3]

The kinematic equations read:

$$\epsilon = u_s - \frac{v}{R} \quad \kappa = \left(-v_{ss} + \frac{u}{R^2}\right)_s$$  \hspace{1cm} (2.70)

where $R$ is the radius of the beam and $s$ is the parametric coordinate. The displacements $u$ and $v$ are used as primal variables. For the limit case of a straight beam ($R = \infty$) the kinematic equations reduce to the known equations of the Euler-Bernoulli beam:

$$\epsilon = u_s \quad \kappa = -v_{ss}$$  \hspace{1cm} (2.71)

In this limit case the membrane strain $\epsilon$ and curvature $\kappa$ are independent. Considering now a curved Bernoulli beam - with a clamped edge on one side, under pure bending and constant radius $R$ - the membrane strains should vanish, i.e. $\epsilon = 0$. In consequence, the kinematic equation of the membrane strain can be reformulated:

$$u_s = \frac{v}{R}$$  \hspace{1cm} (2.72)

This equation can be inserted in the kinematic equation of the curvature $\kappa$:

$$\kappa = -v_{ss} + \frac{v}{R^2}$$  \hspace{1cm} (2.73)

Pure bending means that the curvature is constant, i.e. $\kappa = \text{const}$. To fulfill this condition, the primal variable $v$ has to be constant as well. However, $v = 0$ is valid because of the clamped edge. Therefore, the curvature $\kappa$ has to be zero as well. This yields to a contradiction. The curved beam is not able to represent a pure state of bending for any kind of shape function.

According to [16 chapter 4.1.2], membrane locking is no problem for bi-linear finite shell elements. However, in case of an exact geometry description as in IGA the effects of membrane locking have to be considered. As already mentioned in the previous chapter 2.4.2, there is no tool to avoid membrane locking provided in this thesis, but in general the
Mixed-Displacement (MD) concept presented in [16] can be applied to prevent its occurrence.

2.4.4 Curvature Thickness Locking

Curvature thickness locking, as it is already indicated in the name, plays only a role for three-dimensional shell elements with load induced thickness changes (6p or 7p shells, see chapter 3.3) applied to curved structures. Parasitic strains $E_{33}$ may occur in thickness direction due to this locking type.

This locking phenomenon can be again explained by the example of the curved beam under pure bending. The geometry and deformation is shown in figure 2.10. Bi-linear standard shell elements are used. The normal director is expected to remain unchanged under a state of pure bending. The shear vector $\mathbf{w}$ is orthogonally added to the undeformed director $\mathbf{A}_3$ as usual for standard formulations. It is observed that the director in the middle of the element shortens which leads to parasitic transverse normal strains $E_{33}$. In contrast to the standard formulation, the hierarchic shear difference vector is added to the deformed normal director $\mathbf{a}_3$. Thus, the hierarchic 7p shell intrinsically avoids curvature thickness locking. A numerical example, which proves this property, is shown in [10, chapter 5.4.4]. This example cannot be repeated in this work because additional tools are needed to separate curvature thickness locking from other locking phenomena.

![Figure 2.10: Deformation of a trapezoidally shaped element under pure bending, $\nu = 0$ [16, figure 4.4]](image)

2.4.5 Poisson Thickness Locking

Within the scope of this thesis the name poisson thickness locking is used instead of the more general term volumetric locking because the former one is more common in the context of three-dimensional shells. As indicated by the name, this locking type exists only for three-dimensional shells and a non-zero Poisson’s ratio $\nu \neq 0$. The critical parameter is the bulk modulus $K = E/(3 - 6\nu)$ respectively the Poisson’s ratio $\nu$. The limit case is given by incompressible material, where the bulk modulus becomes infinite, i.e. $\nu \to 0.5$. Typical materials, which are in particular affected, are rubbers, metals in the nonlinear regime or undrained soils.
Simple examples to describe this phenomenon are not available. In [1, chapter 8.4] an example based on the velocity of an object is given. The constraint condition for incompressible materials is:

\[ \text{div} \mathbf{u} = u_{j,j} = 0 \quad (2.74) \]

If the element is not able to satisfy this constraint of a volume preserving strain field, non-physical strains and stresses are generated. In the context of shells, parasitic linear strains and stresses in transverse normal direction may occur due to non-zero membrane stresses in combination with \( \nu \neq 0 \) ([9, 5.4.5]).

As for curvature thickness locking, 3p and 5p shell elements are not affected because the locking is directly avoided by an a priori neglection of transverse normal strains and stresses. Problems occur especially for 6p shells. If the shell model cannot be refined in thickness direction, this locking phenomenon is not even resolvable. 7p shells are not affected, if the 7th parameter introduces a linear strain distribution in thickness direction and, thus, the shell element is able to consider quadratic displacement fields.
3 Hierarchic Shell Formulation

NURBS functions, which are used in the Isogeometric Analysis, mostly have a higher continuity across element boundaries as known from the classical Finite Element Analysis. This facilitates the straight-forward use of the Kirchhoff-Love (KL) shell theory, which requires at least a $C^1$ inter-element continuity. Furthermore, it gives rise to a rotation free parameterization of a shear deformable shell by means of a hierarchic shear difference vector, which also requires at least a $C^1$ inter-element continuity. This formulation of a shear deformable shell intrinsically avoids transverse shear locking due to its special parameterization. Locking effects are normally alleviated ex post.

The here presented hierarchic shell family, which consists of a 3-parameter Kirchhoff-Love (3p), a 5-parameter (5p) Reissner-Mindlin (RM) and a 3D 7-parameter (7p) shell, was developed at the Institute for Structural Mechanics of the University of Stuttgart. The parameters describe the degrees of freedom which the particular shell has per node respectively control point in the context of IGA. The hierarchy promises a model-adaptive concept which means that the contributions due to additional parameters can be separated from the terms of the hierarchically lower shell.

Figure 3.1 illustrates the initial and actual configuration of the mid-surface of the hierarchic shells. The vectors $w^{5p}$, $w^{6p}$ and $w^{7p}$ contribute to the position vector $x$ depending on the particular shell parameterization. The model-adaptive concept of the hierarchic shell is revealed in the formula for the position vector $x$ presented in figure 3.1, whereby the separation of the contributions of the different parameterizations is observed. The 6-parameter (6p) and 7-parameter (7p) shells are three-dimensional (3D) shells and take load induced thickness changes into account. The hierarchic 7p shell intrinsically avoids Poisson’s thickness locking in contrast to the 6p shell. This is the reason why only the 7p shell is implemented in the scope of this thesis.

All three formulations have geometrically non-linear kinematics. The isogeometric 3-parameter KL shell was developed by J. Kiendl at the Chair of Structural Analysis of the Technical University of Munich ([14] and [13]). This shell model already existed in the used open-source software Kratos Multiphysics before the start of the thesis. However, postprocessing functionalities are added to compute stresses and internal forces. The RM shell is based on the concept presented by B. Oesterle [16]. R. Echter derived the 7p 3D shell with a geometrically linear kinematic [9]. The formulation was extended to a geometrically non-linear kinematic in a master thesis at the University of Stuttgart [20].
Hierarchic Shell Formulation

Figure 3.1: Concept of the hierarchic shell formulation
3.1 Kirchhoff-Love Shell (3p)

The basis of the hierarchic shells is the 3-parameter Kirchhoff-Love shell. The shell is shear rigid and is parameterized by three translations corresponding to the three directions in space. The three translations are collected in the displacement vector $\mathbf{u}$. The KL shell is only briefly summarized in order to make the formulations with more parameters easier understandable. Further information can be found in [13].

The stress recovery is implemented as part of this work and, therefore, discussed in detail. This new feature is tested in a simple beam example in subsection 4.2.1. The KL shell is validated to assure the correct implementation in the software and is used as comparison in the numerical examples.

The deformed configuration of the mid-surface of the KL shell is illustrated in figure 3.2. The director $\mathbf{a}_3$ is computed as the normalized normal of the mid-surface recalling equation 2.4:

$$
\mathbf{a}^{KL}_3 = \mathbf{\hat{a}}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}
$$

(3.1)

Figure 3.2: Actual configuration of the shell mid-surface of the 3-parameter Kirchhoff-Love shell
3.1.1 Green-Lagrange Strain

Due to its shear rigid definition, the KL shell theory only considers in-plane strains neglecting transverse normal and transverse shear strains. The remaining strains can be written as a vector in Voigt notation:

\[
E_{\alpha\beta} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{bmatrix}
\]  

(3.2)

where \( E_{\alpha\beta} \) is the vector of the strain coefficients in Voigt notation referring to the initial contravariant basis as indicated in equation 2.36. In this thesis, the shear strains are used without the factor 2 known from engineering strains as long as they are in the curvilinear space and contain the factor 2 as soon as they are in the local Cartesian space which is denoted by an upper bar:

\[
\tilde{E}_{\alpha\beta} = \begin{bmatrix} \tilde{E}_{11} \\ \tilde{E}_{22} \\ 2\tilde{E}_{12} \end{bmatrix}
\]  

(3.3)

where \( \tilde{E}_{\alpha\beta} \) is the vector of the strain coefficients in Voigt notation referring to the local Cartesian basis. In this thesis, the different definitions of the strain vector depending on the respective vector space is considered in the transformation matrix which is used for the mapping from the curvilinear contravariant to the Cartesian space and vice versa. This matrix was derived in subsection 2.2.1.2.

The geometrically non-linear Green-Lagrange strain is computed according to equation 2.36 following the approach of [13]:

\[
E = \frac{1}{2}(g_{\alpha\beta} - G_{\alpha\beta}) G^\alpha \otimes G^\beta
\]  

(3.4)

Quadratic contributions in \( \xi^3 \) are neglected for thin and moderately thick shells [13, section 3.2]. With this assumption, the strain coefficients are calculated as:

\[
E_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - G_{\alpha\beta}) = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) + \xi^3 (B_{\alpha\beta} - b_{\alpha\beta}) = \epsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}
\]  

(3.5)

where \( \xi^3 \) is the third curvilinear contravariant coordinate which describes the thickness direction of the shell and is defined in \([−t/2, t/2]\), \( a_{\alpha\beta} \) and \( A_{\alpha\beta} \) are the metric coefficients at the mid-surface, and \( b_{\alpha\beta} \) and \( B_{\alpha\beta} \) are the curvature coefficients referring to the initial and actual configuration respectively. The constant part of the strains, which corresponds to membrane action, is denoted as \( \epsilon_{\alpha\beta} \) and \( \kappa_{\alpha\beta} \) is the linear part of the strains, which corresponds to a change in curvature due to bending. The metric coefficients \( a_{\alpha\beta} \) are computed as follows recalling equation 2.7:

\[
a_{\alpha\beta} = a_\alpha \cdot a_\beta
\]  

(3.6)
The metric coefficients $A_{\alpha\beta}$ are computed in the same way but using the initial base vectors. The curvature coefficients $b_{\alpha\beta}$ are computed by:

\[ b_{\alpha\beta} = a_{\alpha,\beta} \cdot a_3 \]  

(3.7)

The curvature coefficients $B_{\alpha\beta}$ are computed in the same way but using the initial base vectors. Further explanations and a derivation of the provided formulas are available in [13, section 3.2].

### 3.1.2 B-Operator

The first derivative of the strains with respect to the primal variables $D_r$ is:

\[
E_{11,r}^{KL} = \frac{1}{2} a_{11,r} - \xi^3 b_{11,r}
\]

\[
E_{22,r}^{KL} = \frac{1}{2} a_{22,r} - \xi^3 b_{22,r}
\]

\[
E_{12,r}^{KL} = \frac{1}{2} a_{12,r} - \xi^3 b_{12,r}
\]

(3.8)

where the primal variables $D_r$ consist of the three translations of all control points. The derivatives of the base vectors of the initial configuration vanish because they are independent of variations of the displacements $(r)$. Further explanations and exact formulas, how the derivatives are computed, are provided by [13, section 5.1].

### 3.1.3 Geometric Stiffness Matrix

The second derivative of the strain with respect to the primal variables $D_r$ and $D_s$ is:

\[
E_{11,rs}^{KL} = \frac{1}{2} a_{11,rs} - \xi^3 b_{11,rs}
\]

\[
E_{22,rs}^{KL} = \frac{1}{2} a_{22,rs} - \xi^3 b_{22,rs}
\]

\[
E_{12,rs}^{KL} = \frac{1}{2} a_{12,rs} - \xi^3 b_{12,rs}
\]

(3.9)

Further explanations and exact formulas, how the derivatives are computed, are provided by [13, section 5.2].

### 3.1.4 Integration Schemes

The solving of an integral is required in order to solve the system of equations discussed in subsection 2.2.4. In computational engineering, this integration is performed numerically. The Gauss integration scheme is used in this thesis to integrate over the mid-surface. The integration over the thickness is carried out either by a pre-integration or by a Gauss integration. For the KL shell presented within this section, a pre-integration is used following the concept of [13].
3.1.4.1 Gauss Integration of the Mid-Surface

The concept of the Gauss integration says that an integral can be solved by a summation over a discrete number of Gauss points. The position of the Gauss points and the corresponding Gauss weights are clearly defined and can be found for example in [21, table E.2]. The number of Gauss points per direction \( n_{\text{req}} \) which is required in order to obtain exact results is given by:

\[
 n_{\text{req}} = \frac{p + 1}{2} \tag{3.10}
\]

where \( p \) is the polynomial degree of the function which is integrated. Since NURBS functions, which are used in IGA, are rational functions, a higher number of Gauss points \( n_{\text{GP}} \) is used per direction:

\[
 n_{\text{GP}} = p + 1 \tag{3.11}
\]

The Gauss points are defined in the Gaussian space and the shape functions are defined in the parameter space. Therefore, a mapping from the physical via the parameter to the Gaussian space is required. Figure 3.3 shows the mapping steps which are performed by the means of Jacobian matrices.

![Mapping between spaces for Gauss integration](based on [21, figure 4.9])

Considering the named points, the initial stiffness matrix coefficients are for example computed by:

\[
 K_{rs,eu} = \int_{V_R} (B^T CB) \, dV_R = \sum_{i=1}^{n_{\text{GP}}} w_{GP,i} B^T CB \text{det}J_{x-\xi} \text{det}J_{\xi-\tilde{\xi}} = \sum_{i=1}^{n_{\text{GP}}} w_{GP,i} B^T CB \frac{l_{ele,\xi_1} l_{ele,\xi_2}}{4} \tag{3.12}
\]

where \( n_{\text{GP}} \) is the total number of Gauss points, \( w_{GP,i} \) is the Gauss weight of the \( i^{th} \) Gauss point, \( J_{x-\xi} \) is the Jacobian matrix mapping from the physical to the parameter space and \( J_{\xi-\tilde{\xi}} \) is the Jacobian matrix mapping from the parameter to the Gaussian space. The differential area is denoted as \( dA \), which can be computed according to equation 2.4 and \( l_{ele,\xi_1} \) and \( l_{ele,\xi_2} \).
are the length of the element in the parameter space in the two directions $\xi_1$ and $\xi_2$. It has to be mentioned that $\det J_{x-\xi} = dA$ is a simplification.

### 3.1.4.2 Pre-integration over the Thickness

The pre-integration of the KL shell is only briefly discussed in this subsection, but a detailed description can be found in [13, section 3.2]. The stress distribution at the cross-section is assumed to be linear, which is accurate for thin shells with a linear elastic material law. The PK2 stresses $\mathbf{S}$ are replaced by corresponding stress resultants. The stress resultants are separated in membrane forces $\mathbf{n}$ and internal moments $\mathbf{m}$, which complies with the separation of the strains in a membrane and a curvature part in subsection 3.1.1. The membrane forces correspond to the constant part of the stresses and the internal moments to the linear one. In consequence, the membrane forces $\mathbf{n}$ are computed by:

$$
\begin{bmatrix}
\bar{n}_{PK2}^{11} \\
\bar{n}_{PK2}^{22} \\
\bar{n}_{PK2}^{12}
\end{bmatrix} = t \cdot \mathbf{D} \cdot
\begin{bmatrix}
\bar{\epsilon}_{11} \\
\bar{\epsilon}_{22} \\
2\bar{\epsilon}_{12}
\end{bmatrix}
$$

(3.13)

where the upper bar (\bar{\cdot}) again marks the local Cartesian basis, ($_{PK2}$) indicates that the stress resultants correspond to the PK2 stresses $\mathbf{S}$, $t$ is the thickness and $\mathbf{D}$ is the material stiffness matrix in Voigt notation. The internal moments $\mathbf{m}$ are computed by:

$$
\begin{bmatrix}
\bar{m}_{PK2}^{11} \\
\bar{m}_{PK2}^{22} \\
\bar{m}_{PK2}^{12}
\end{bmatrix} = \frac{t^3}{12} \cdot \mathbf{D} \cdot
\begin{bmatrix}
\bar{\kappa}_{11} \\
\bar{\kappa}_{22} \\
2\bar{\kappa}_{12}
\end{bmatrix}
$$

(3.14)

### 3.1.5 Stress Recovery

The PK2 stresses are normally obtained with respect to local Cartesian coordinates because the material stiffness tensor $\bar{\mathbf{C}}$ is defined in the Cartesian space. The underlying reason is that the material stiffness tensor contains physical values which generally require a representation in Cartesian coordinates. Recalling equation 2.44, the constitutive equation is given as:

$$
\bar{\mathbf{S}}_{\alpha\beta} = \bar{\mathbf{C}}_{\alpha\beta\gamma\delta} \bar{\mathbf{E}}_{\gamma\delta}
$$

(3.15)

The PK2 stresses $\bar{\mathbf{S}}_{\alpha\beta}$, which refer to the initial covariant basis $\mathbf{G}_\alpha \otimes \mathbf{G}_\beta$, are obtained from $\bar{\mathbf{S}}_{\alpha\beta}$:

$$
\mathbf{S}_{\gamma\delta} = \bar{\mathbf{S}}_{\alpha\beta}(\mathbf{G}_\gamma \cdot \mathbf{e}_\alpha)(\mathbf{e}_\beta \cdot \mathbf{G}_\delta)
$$

(3.16)

The Cauchy stresses $\sigma_{\alpha\beta}$, which are defined with respect to the actual covariant basis $\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$, are computed from $\mathbf{S}_{\alpha\beta}$ by means of the deformation gradient $\mathbf{F}$:

$$
\sigma_{\alpha\beta} = (\det \mathbf{F}^{-1})S_{\alpha\beta}
$$

(3.17)
Finally, the Cauchy stresses are transformed from the actual covariant to the local Cartesian basis:

$$\bar{\sigma}_{\alpha\beta} = \sigma^{\gamma\delta}(e^\alpha \cdot g_\gamma)(g_\delta \cdot e^\beta)$$  \hspace{1cm} (3.18)

The Cauchy stresses $\bar{\sigma}_{\alpha\beta}$ are physical meaningful stress values. The preceding discussion about stresses is generally valid for every stress distribution in thickness direction. For the calculation of stress resultants, a certain stress distribution has to be assumed. The distribution at the cross-section is linear for thin shells with linear elastic material. The stress resultants $\bar{n}_{\alpha\beta}$ and $\bar{m}_{\alpha\beta}$ are analogically to the pre-integration of subsection 3.1.4.2 computed by:

$$\bar{n}_{\alpha\beta} = \sigma_{\alpha\beta}^{\text{membrane}} \cdot t$$  \hspace{1cm} (3.19)

$$\bar{m}_{\alpha\beta} = \sigma_{\alpha\beta}^{\text{bending}} \cdot \frac{t^3}{12}$$  \hspace{1cm} (3.20)

In case of the 3-parameter KL shell, the transverse shear forces cannot be computed directly since only in-plane stresses are considered. For shallow shells however, they can be computed by the derivative of the moment in compliance with equilibrium considerations ([13, section 3.4]):

$$q^\alpha_{PK2} = \frac{\partial m_{PK2}^{\alpha\alpha}}{\partial s^\alpha} + \frac{\partial m_{PK2}^{\alpha\beta}}{\partial s^\beta}$$  \hspace{1cm} (3.21)

where $s$ is the arc length in the two parameter directions. The differential arc length $ds$ is computed by:

$$ds_\alpha = \sqrt{a_{\alpha\alpha}} d\xi^\alpha$$  \hspace{1cm} (3.22)

Inserting equation 3.22 in equation 3.21 yields to:

$$q^\alpha_{PK2} = \frac{\partial m_{PK2}^{\alpha\alpha}}{\partial s^\alpha} \sqrt{a_{\alpha\alpha}}^{-1} + \frac{\partial m_{PK2}^{\alpha\beta}}{\partial s^\beta} \sqrt{a_{\beta\beta}}^{-1}$$  \hspace{1cm} (3.23)

The moment $m_{PK2}^{\alpha\alpha}$, which is defined with respect to the initial covariant basis $G_\alpha \otimes G_\beta$, is derived from $\bar{m}^{\alpha\beta}_{PK2}$ by means of the fourth order transformation tensor $T_{\alpha\beta\gamma\delta}^{\text{car-cov}}$:

$$m^{\gamma\delta}_{PK2} = \bar{m}^{\alpha\beta}_{PK2}(e^\gamma \cdot e_\alpha)(e^\delta \cdot e_\beta) = \bar{m}^{\alpha\beta}_{PK2} T_{\alpha\beta\gamma\delta}^{\text{car-cov}}$$  \hspace{1cm} (3.24)

In consequence, the derivative of the transformation tensor $T_{\alpha\beta\gamma\delta}^{\text{car-cov}}$ is required in order to compute the derivative of the moment as indicated in equation 3.23. As known for stresses, the transverse shear forces obtained in equation 3.23 are transformed from the initial to the actual covariant basis $g_\alpha \otimes g_\beta$ by means of the deformation gradient:

$$q^\alpha = (det F^{-1}) q^\alpha_{PK2}$$  \hspace{1cm} (3.25)
As a last step, $q^\alpha$ has to be transformed to the local Cartesian coordinates by means of the transformation matrix $T_{\alpha\beta}^{\text{cov-car}}$ in order to obtain real physical values:

$$\bar{q}^\alpha = q^{\beta\alpha}T_{\alpha\beta}^{\text{cov-car}}$$

(3.26)

For more details concerning the used transformation rules, it is referred to subsection 2.2.1.2.
3.2 Reissner-Mindlin Shell (5p)

The Reissner-Mindlin (RM) shell is shear deformable and can be described by five parameters (5p). In the sense of a hierarchic formulation two new parameters are introduced in addition to the parameters of the Kirchhoff-Love (KL) shell which is described in chapter 3.1.

Two different concepts are presented by [16] in order to model the shear deformable RM shell: a formulation with hierarchic rotations and a formulation with hierarchic displacements. Within the first alternative, the additional deformation due to shear is described by an additional rotation, whereas the same phenomenon is modeled by an additional displacement within the second one. In this thesis, the discretization by means of hierarchic rotations is chosen, merely because it is simpler to implement. In the case of the shear deformable formulation with hierarchic displacements, additional constraint and boundary conditions are necessary in order to avoid nonphysical zero energy modes [16]. However, the quality and convergence rate of the solutions for the shear forces are better in the case of hierarchic displacements, which was shown for B-Splines of degree two ($p = 2$) by [16]. This effect may be negligible for higher order B-Splines or NURBS.

The here implemented RM shell with hierarchic rotations is the first formulation without a direct discretization of the total rotations that is able to model a RM plate correctly according to the author of [16] who developed this element description. In contrast to the here presented formulation with hierarchic rotations, the discretization of shear deformable shells by means of total rotations instead of shear rotations can be seen as standard formulation (RM-st).

3.2.1 Geometrically Non-Linear Kinematic

The RM shell considers shear rotations in contrast to the KL shell. The additional rotations are parameterized by means of a hierarchic shear difference vector $\mathbf{w}$ consisting of two independent parameters $w^1$ and $w^2$ spanning in the actual tangential space $\mathbf{a}_a$:

$$\mathbf{w} = w^1 \mathbf{a}_1 + w^2 \mathbf{a}_2$$  \hspace{1cm} (3.27)

The parameters $w^1$ and $w^2$ are the two new primal variables. The shear difference vector is added to the actual normal $\mathbf{a}_1 \times \mathbf{a}_2$:

$$\mathbf{a}^{RM}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{w}}{|\mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{w}|}$$ \hspace{1cm} (3.28)

It has to be mentioned that the director $\mathbf{a}_3$ is not perpendicular to the surface anymore. As indicated in equation 3.28, the normal vector has to be normalized in order to fulfill the condition of disappearing transverse normal strains.

The equation 3.28 is highly non-linear with respect to the primal variables. The consequence is a high complexity of the corresponding algorithm. Furthermore, there is no elegant additive decomposition of strains coming from bending, membrane and transverse shear action.
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with this description of the director \( \mathbf{a}_3 \). However, this is especially desired in a hierarchic concept, where the different shell formulation are thought to build up on each other. The hypothesis stated in [16, pg. 78] says that the shear rotations are small compared to the total rotations. Based on this hypothesis, a linearization of the shear rotations is possible, whereby the description of the bending and membrane action remains exactly geometrically non-linear:

\[
\mathbf{a}^{RM}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} + \mathbf{w} = \mathbf{a}^\perp_3 + \mathbf{w} \tag{3.29}
\]

This description of the director \( \mathbf{a}_3 \) facilitates a decomposition of the strains as mentioned above. Of course, the underlying hypothesis has to be proven. Oesterle discussed this simplification at two numerical examples: a non-linear beam problem ([16, Chapter 8.2.1]) and the snap-through of a ring ([16, Chapter 8.2.4]). Both examples compared the results to the shell element SHELL181 of ANSYS16, which is completely non-linear, and underlined the usability of the conjectured hypothesis.

The deformed configuration of the mid-surface of the RM shell with linearised shear rotations is illustrated in figure 3.4. It has to be mentioned that the director \( \mathbf{a}^\perp_3 \) is not identical by value to that one of the KL shell because the deformation is different for the two shell types. The displacement of the shell body \( \mathbf{u} \) is computed by:

\[
\mathbf{u} = \mathbf{x} - \mathbf{X} = \mathbf{v} + \xi^3(\mathbf{a}^{RM}_3 - \mathbf{A}_3) = \mathbf{v} + \xi^3(\mathbf{a}^\perp_3 + \mathbf{w} - \mathbf{A}_3) \tag{3.30}
\]

where only the specification of the director \( \mathbf{a}_3 \) differs from that one used for the KL shell. The derivatives of the displacement field are needed in order to compute the Green-Lagrange strain tensor:

\[
\begin{align*}
\mathbf{u}_{,\alpha} &= \mathbf{v}_{,\alpha} + \xi^3(\mathbf{a}^\perp_{3,\alpha} + \mathbf{w}_{,\alpha} - \mathbf{A}_{3,\alpha}) \\
\mathbf{u}_{,3} &= \mathbf{a}^\perp_3 + \mathbf{w} - \mathbf{A}_3
\end{align*} \tag{3.31}
\]
3.2.2 Green-Lagrange Strain

Applying equation 2.37, the geometrically non-linear Green-Lagrange strain coefficients are computed as follows, again neglecting quadratic contributions in $\xi^3$:

\[
\begin{align*}
E_{11}^{RM} &= E_{11}^{KL} + \xi^3 (w_1 \cdot a_1) \\
E_{22}^{RM} &= E_{22}^{KL} + \xi^3 (w_2 \cdot a_2) \\
E_{33}^{RM} &= 0.5w \cdot w \\
E_{12}^{RM} &= E_{12}^{KL} + 0.5\xi^3 (w_1 \cdot a_2 + w_2 \cdot a_1) \\
E_{23}^{RM} &= 0.5w \cdot a_2 + 0.5\xi^3 (w_2 \cdot w) \\
E_{13}^{RM} &= 0.5w \cdot a_1 + 0.5\xi^3 (w_1 \cdot w)
\end{align*}
\]  

(3.32)

The contributions $E_{αβ}^{KL}$ refer to the strains of the Kirchhoff-Love shell which were presented in chapter 3.1.1. A decomposition of the Green-Lagrange strain in a Kirchhoff-Love part and an additional part due to the shear difference vector $w$ can be observed. That is possible due to the assumption of linearised shear rotations as explained in chapter 3.2.1. The terms, which are underlined in equation 3.32, result from the small change of length of the director $a_3$ due to this linearisation. Therefore, they are neglected in the following to be in compliance with the linearisation assumptions.
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with the stated assumption of small shear rotations. The used Green-Lagrange strain for the RM shell is then:

\[
\begin{align*}
E_{11}^{RM} &= E_{11}^{KL} + \xi^3 (w_{,1} \cdot a_1) \\
E_{22}^{RM} &= E_{22}^{KL} + \xi^3 (w_{,2} \cdot a_2) \\
E_{33}^{RM} &= 0 \\
E_{12}^{RM} &= E_{12}^{KL} + 0.5 \xi^3 (w_{,1} \cdot a_2 + w_{,2} \cdot a_1) \\
E_{23}^{RM} &= 0.5 w \cdot a_2 \\
E_{13}^{RM} &= 0.5 w \cdot a_1
\end{align*}
\]

The formulation can be reduced to five strain coefficients since \( E_{33}^{RM} = 0 \). The strain coefficients \( E_{ij} \) may be written in Voigt notation as a vector:

\[
E^{RM}_{ij} = \begin{bmatrix} E_{11}^{RM} \\ E_{22}^{RM} \\ E_{12}^{RM} \\ E_{23}^{RM} \\ E_{13}^{RM} \end{bmatrix}
\]

An important feature of the hierarchic RM shell is to be intrinsically free from transverse shear locking. The constraint of vanishing transverse shear strains in case of pure bending is easily fulfilled by setting the shear difference vector to zero, i.e. \( w = 0 \) yields \( E_{23}^{RM} = 0 \) and \( E_{13}^{RM} = 0 \).

### 3.2.3 B-Operator

The first derivative of the strains with respect to the primal variables \( D_r \) is:

\[
\begin{align*}
E_{11,r}^{RM} &= E_{11,r}^{KL} + \xi^3 (w_{,1r} \cdot a_1 + w_{,1} \cdot a_{1r}) \\
E_{22,r}^{RM} &= E_{22,r}^{KL} + \xi^3 (w_{,2r} \cdot a_2 + w_{,2} \cdot a_{2r}) \\
E_{12,r}^{RM} &= E_{12,r}^{KL} + 0.5 \xi^3 (w_{,1r} \cdot a_2 + w_{,1} \cdot a_{2r} + w_{,2r} \cdot a_1 + w_{,2} \cdot a_{1r}) \\
E_{23,r}^{RM} &= 0.5 (w_{,r} \cdot a_2 + w \cdot a_{2r}) \\
E_{13,r}^{RM} &= 0.5 (w_{,r} \cdot a_1 + w \cdot a_{1r})
\end{align*}
\]

where the primal variables \( D_r \) consist of three translations and the two components of the shear difference vector \( w^1 \) and \( w^2 \) of all control points.

### 3.2.4 Geometric Stiffness Matrix

The geometric stiffness \( K_g \) is the derivative of the B-Operator with respect to the primal variable \( u_s \). As depicted in equation 2.51, the discretized displacement field is lin-
ear in the primal variables. Therefore, certain terms are vanishing in the second derivative:

\[ a_{α,rs} = w_{α,rs} = w_{rs} = 0 \]  \hspace{1cm} (3.36)

The coefficients of the geometric stiffness matrix \( E_{ij,rs}^{RM} \) are then computed as:

\[
E_{11,rs}^{RM} = E_{11,rs}^{KL} + \xi_3^3 (w_{1,1,rs} \cdot a_{1,s} + w_{1,1,rs} \cdot a_{1,r}) \\
E_{22,rs}^{RM} = E_{22,rs}^{KL} + \xi_3^3 (w_{2,2,rs} \cdot a_{2,s} + w_{2,2,rs} \cdot a_{2,r}) \\
E_{12,rs}^{RM} = E_{12,rs}^{KL} + 0.5 \xi_3^3 (w_{1,1,rs} \cdot a_{2,s} + w_{1,1,rs} \cdot a_{2,r} + w_{2,2,rs} \cdot a_{1,s} + w_{2,2,rs} \cdot a_{1,r}) \hspace{1cm} (3.37) \\
E_{23,rs}^{RM} = 0.5(w,_{r} \cdot a_{2,s} + w,_{s} \cdot a_{2,r}) \\
E_{13,rs}^{RM} = 0.5(w,_{r} \cdot a_{1,s} + w,_{s} \cdot a_{1,r})
\]

### 3.2.5 Stress Recovery

The PK2 stresses are obtained in Cartesian space from the constitutive equation 2.44. The transformation from the PK2 stresses to Cauchy stresses was explained in chapter 3.1.5. In the postprocessing, the internal forces and the Cauchy stresses at top and bottom are from special interest. The following computation is valid for an arbitrary number of Gauss points in thickness direction \( n_{GP,t} \). For the postprocessing, the membrane stress distribution is assumed to be linear and the transverse shear stress distribution is assumed to be constant in order to calculate internal forces. These assumptions comply with the distribution of the strains stated in equation 3.33 in combination with a linear elastic material model. Figure 3.5 shows a shell with three Gauss points in thickness direction, which is the number of Gauss points \( n_{GP,t} \) used in the computations within this thesis, and illustrates the mapping from the parametric to the Gaussian space. In thickness direction, the parameter and the physical space are identical. The upper bar (\( \bar{\cdot} \)) in figure 3.5 again denotes that the stresses refer to the local Cartesian basis. The membrane Cauchy stresses at mid-span \( \xi_3 = 0 \) are computed by:

\[
\bar{\sigma}_{αβ}(\xi_3 = 0) = 0.5 \cdot \left( \bar{\sigma}_{αβ}(\xi_3 = \xi_{1}^{3}_{nGP,t}) + \bar{\sigma}_{αβ}(\xi_3 = \xi_{3}^{3}) \right) \]  \hspace{1cm} (3.38)

It has to be made clear that in general there is no Gauss point at mid-span, although this is the case in figure 3.5. Therefore, equation 3.38 is given in a more general form to compute the membrane Cauchy stress at mid-span independent of the number of Gauss points. The Cauchy stresses at top and bottom are computed by a linear extrapolation:

\[
\bar{\sigma}_{ij}(\xi_3 = t/2) = \bar{\sigma}_{ij}(\xi_3 = 0) + \left( \frac{\bar{\sigma}_{ij}(\xi_3 = \xi_{nGP,t}^{3}) - \bar{\sigma}_{ij}(\xi_3 = 0)}{\xi_{nGP,t}^{3}} \right) \\
\bar{\sigma}_{ij}(\xi_3 = -t/2) = \bar{\sigma}_{ij}(\xi_3 = 0) + \left( \frac{\bar{\sigma}_{ij}(\xi_3 = \xi_{nGP,t}^{3}) - \bar{\sigma}_{ij}(\xi_3 = 0)}{\xi_{1}^{3}} \right) \hspace{1cm} (3.39)
\]
The membrane forces are computed by an integration of the stresses over the thickness. For the assumed linear distribution of the membrane stresses $\bar{\sigma}_{\alpha\beta}$, which are equivalent to a combination of membrane forces and internal moments, the computation of the membrane forces $n_{\alpha\beta}$ corresponds to a multiplication of the Cauchy stress at mid-span with the thickness $t$:

$$n_{\alpha\beta} = \bar{\sigma}_{\alpha\beta}(\xi^3 = 0) \cdot t \quad (3.40)$$

For the assumed constant distribution of the transverse shear stresses $\bar{\sigma}_{23}$ and $\bar{\sigma}_{13}$, the computation of the transverse shear forces $q_{23}$ and $q_{13}$ corresponds to a multiplication of the Cauchy stress at mid-span by the thickness $t$:

$$q_{23} = \bar{\sigma}_{23}(\xi^3 = 0) \cdot t$$
$$q_{13} = \bar{\sigma}_{13}(\xi^3 = 0) \cdot t \quad (3.41)$$

The analytical solution of the transverse shear stress determines a quadratic stress distribution [19, pg. 15], whereas the here presented theory indicates a constant one. This could be problematic for simulations with a plastic material model and significant influence of the shear strength. However, this problem is not further investigated in this thesis. The equa-
tion for the calculation of the internal moments is derived by a moment equilibrium at the cross-section:

\[ m_{\alpha\beta} = \left( \bar{\sigma}_{\alpha\beta}(\xi^3 = \xi_{nGP,t}^3) - \bar{\sigma}_{\alpha\beta}(\xi^3 = 0) \right) \cdot t^2 \]

\[ \xi_{nGP,t}^3 \cdot 6 \]

(3.42)

The given formulas and described procedures are only valid for the stated assumption of stress distributions at the cross-section. In more general cases including plastic behaviour, it is not possible to compute internal forces, but the stresses at the Gauss points can be evaluated.

### 3.2.6 Gauss Integration over Thickness

The shell element can be either modeled with a pre-integration or a Gauss integration in thickness direction. The pre-integration requires an assumption about the stress distribution at the cross-section, for example a linear distribution of the membrane stresses. For the new implemented RM shell, a Gauss integration over the thickness with three Gauss points is used. The Gauss integration allows the usage of plastic material laws, where the stress distribution is not clearly defined anymore. However, a pre-integration has a clear advantage concerning the computational efficiency.

The number of Gauss points over thickness \( n_{GP,t} \) should be chosen depending on the complexity of the analysis problem. Suggestions for a proper number of Gauss points can be found in the ABAQUS manual. The number of three Gauss points used in this thesis is appropriate for nonlinear applications such as predicting the response of an elastic-plastic shell up to limit load [18, chapter 15.6.5]. A drawback of the Gauss integration is that no integration points are placed at top and bottom of the shell body. Especially these locations are of special interest since the stresses due to bending are there the highest. In consequence, the computation of an elastic-plastic system may not yield to exact results. The quality of the results should be investigated in further studies in order to assess this error. Another numerical integration rule, namely the Simpson’s rule, would provide integration points at the outer edges. However, the computational time and the required storage space is higher for the Simpson’s rule. Therefore, the Simpson’s integration scheme is not used in this thesis.

The numerical integration of one coefficient of the initial stiffness matrix \( K_{r,s+u} \) is given by:

\[ K_{rs,e+u} = \int_{V_R} \left( B^T C B \right) dV_R = \sum_{i} \sum_{j} w_{i,GP} w_{j,GP,t} B^T C B dV \]

\[ = \sum_{i} \sum_{j} w_{i,GP} w_{j,GP,t} B^T C B dV \]

\[ \frac{t_{ele,\xi^1} t_{ele,\xi^2}}{4} \]

(3.43)

where \( w_{i,GP,t} \) is the Gauss weight of the \( j^{th} \) Gauss point in the thickness direction, \( dV \) is the differential volume responsible for the mapping between physical and parameter space, and \( \frac{t}{2} \)
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considers the mapping from the parameter space to the Gaussian space in thickness direction as illustrated in figure 3.5. The usage of the differential volume instead of the differential area \( dA \) is an important difference to the Gauss integration of the Kirchhoff-Love shell explained in subsection 3.1.4, where the thickness direction is pre-integrated. The differential volume \( dV \) is computed by:

\[
dV = dA \cdot G_3 = \tilde{G}_3 \cdot G_3 = (G_1 \times G_2) \cdot G_3 \quad (3.44)
\]

The differential volume \( dV \) has to be recalculated for each Gauss point in thickness direction because the tangential base vectors of the shell body \( G_\alpha \) vary over the thickness in case of curved structures as it can be seen in equation 2.29.

### 3.2.7 Static Condensation of the Transverse Normal Strain

The transverse normal strain \( \bar{E}^{(e)}_{33} \) should be statically condensed from the system of equations because no deformations in the transverse normal direction occur due to the assumptions for a Reissner-Mindlin shell, i.e. \( \bar{E}^{(e)}_{33} = 0 \). The system of equations including \( \bar{E}^{(e)}_{33} \) is given for each element \( (e) \):

\[
D^{(e)} \bar{E}^{(e)} = \bar{S}^{(e)}
\]

\[
\begin{bmatrix}
\bar{E}^{(e)}_{11} \\
\bar{E}^{(e)}_{22} \\
\bar{E}^{(e)}_{33} \\
2\bar{E}^{(e)}_{12} \\
2\bar{E}^{(e)}_{23} \\
2\bar{E}^{(e)}_{13}
\end{bmatrix}
= \begin{bmatrix}
\bar{S}^{(e)}_{11} \\
\bar{S}^{(e)}_{22} \\
\bar{S}^{(e)}_{33} \\
\bar{S}^{(e)}_{12} \\
\bar{S}^{(e)}_{23} \\
\bar{S}^{(e)}_{13}
\end{bmatrix}
\quad (3.45)
\]

In the following, the condensation is shown for a shorter generic example because the specific position of \( \bar{E}^{(e)}_{33} \) in the middle of the strain vector would yield to increased writing effort. An generic system of equation is considered:

\[
ku = f
\]

\[
\begin{bmatrix}
k_{aa} & k_{ai} \\
k_{ia} & k_{ii}
\end{bmatrix}
\begin{bmatrix}
u_a \\
u_i
\end{bmatrix}
= \begin{bmatrix}
f_a \\
f_i
\end{bmatrix}
\quad (3.46)
\]

where \( u_i \) denotes the variables which should be condensed. The lower part of the system of equations can be reformulated as:

\[
u_i = k_{ii} [f_i - k_{ia} u_a]
\quad (3.47)
\]

In a next step equation 3.47 is inserted in the upper part of the system of equations:

\[
k_{aa} u_a + k_{ai} k_{ii}^{-1} [f_i - k_{ia} u_a] = f_a
\quad (3.48)
The resulting equation is such reformulated that a typical system of stiffness matrix, displacement vector and force vector is again obtained:

$$\left[ k_{aa} - k_{ai}k_{ii}^{-1}k_{ia} \right] u_a = f_a - k_{ai}k_{ii}^{-1}f_i$$  \hfill (3.49)

In our special case $f_i = \bar{S}_{33}^e = 0$ is valid which means that no transverse normal stresses occur. Therefore, the equation can be simplified to:

$$\left[ k_{aa} - k_{ai}k_{ii}^{-1}k_{ia} \right] u_a = f_a$$  \hfill (3.50)

The condensed system of equations is summarized by:

$$\tilde{k}\tilde{u} = \tilde{f}$$

$$\tilde{k} = k_{aa} - k_{ai}k_{ii}^{-1}k_{ia}$$

$$\tilde{u} = u_a$$

$$\tilde{f} = f_a$$  \hfill (3.51)

The shown procedure works analogically for equation [3.45]. The static condensation takes place at element level and, therefore, at each single Gauss point.
### 3.3 3D Shell (7p)

A 3D shell is obtained from the RM shell due to additional consideration of the transverse normal strain $E_{33}$. This strain causes thickness changes. Therefore, two new parameters are engaged in addition to the previously presented RM shell to cover this behaviour. At the Institute for Structural Mechanics of the University of Stuttgart, a 7-parameter 3D shell for IGA was introduced by R. Echter [10] in the context of small rotations. At the same institute, the same shell was extended to large rotations in a master thesis [20]. However, the validation of [20] has to be assessed carefully. Some examples computed with a zero Poisson’s ratio yield different results between the RM and the 3D shell even though no thickness changes are expected for zero Poisson’s ratios. This fact was not at all discussed by [20]. The here presented shell is based on the named works and is implemented with a geometrically non-linear kinematic, which means that it is applicable for large rotations.

3D shells describe the physical behaviour of shells more exactly compared to the KL or RM theory. The additional consideration of thickness changes is of interest in case of high Poisson’s ratios, especially in combination with the application of plastic three-dimensional material laws. R. Echter has examined some geometrically linear examples in [10]. The determined differences of the results between the RM and 3D shell are negligible; the largest difference occurs for the cylindrical shell example with 0.32 % [10, table 3]. However, the computation was geometrically linear and the examples were typical examples of slender shells. It is possible that larger differences are obtained for higher non-linearity or thicker shells.

The following subsections are ordered in the same manner as for the KL and RM shell. However, there is no subsection discussing the stress recovery and the Gauss integration because they work in the same way as for the RM shell.

#### 3.3.1 Geometrically Non-Linear Kinematic

The hierarchic shear difference vector $w$ is equipped with a 3rd parameter $w^3$ in order to capture thickness changes. In contrast to the RM shell, the shear difference vector components $w^i$ refer to a global Cartesian coordinate system. This was suggested by [10, section 2.3] in order to simplify the subsequent computations. The vector $w$ is then defined as:

$$w = w^1 e_1 + w^2 e_2 + w^3 e_3 = w^1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w^2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

(3.52)

However, the model-adaptive concept of the hierarchic shell is lost by this definition of the shear difference vector. In addition to the parameter $w^3$, a 7th parameter $\bar{w}$ is introduced in order to obtain a linear distribution of transverse normal strains in $\xi^3$. This is necessary to avoid Poisson’s thickness locking. The curvature thickness locking is prevented by the usage of a hierarchic difference vector instead of a standard one. Figure 3.6 illustrates the deformed configuration of the mid-surface of the 3D shell.
The director $a_{7p}^p$ is computed as follows:

$$a_{7p}^p = \frac{a_1 \times a_2}{|a_1 \times a_2|} + w + \zeta^3 \bar{w} \left( \frac{a_1 \times a_2}{|a_1 \times a_2|} + w \right) = a_3^\perp + w + \zeta^3 \bar{w} (a_3^\perp + w)$$  \hspace{1cm} (3.53)

In contrast to equation 3.29 which describes the linearised normal director $a_{5p}^p$ of the RM shell element, equation 3.53 constitutes an exact formula of the normal director $a_{7p}^p$ because it is anyway requested that an elongation of the director is possible in case of thickness changes. The formula for the calculation of the position vector of the shell body $x_{7p}$ is also given in figure 3.6. The deformation of a point of the shell body can be thereby computed as:

$$u = x - X = v + \zeta^3 (a_{7p}^p - A_3)$$

$$= v + \zeta^3 (a_3^\perp + w - A_3) + (\zeta^3)^2 \bar{w} (a_3^\perp + w)$$  \hspace{1cm} (3.54)

The derivatives of the displacement field are needed in order to compute the Green-Lagrange strain tensor:

$$u_{\alpha} = v_{,\alpha} + \zeta^3 (a_3^\perp_{,\alpha} + w_{,\alpha} - A_3) + (\zeta^3)^2 \bar{w}_{,\alpha} (a_3^\perp + w)$$

$$u_3 = a_3^\perp + w - A_3 + 2\zeta \bar{w} a_3$$  \hspace{1cm} (3.55)
3.3.2 Green-Lagrange Strain

Applying equation 2.37, the geometrically non-linear Green-Lagrange strain coefficients are computed as follows, again neglecting quadratic contributions in $\xi^3$:

\[
\begin{align*}
E_{11}^{p} &= E_{11}^{KL} + \xi^3 (w_{,1} \cdot a_1) \\
E_{22}^{p} &= E_{22}^{KL} + \xi^3 (w_{,2} \cdot a_2) \\
E_{33}^{p} &= 0.5w \cdot a_1^+ + \xi^3 (a_3^+ \cdot w + 2a_3^+ \cdot w + w \cdot w) \\
E_{12}^{p} &= E_{12}^{KL} + 0.5\xi^3 (w_{,1} \cdot a_2 + w_{,2} \cdot a_1) \\
E_{23}^{p} &= 0.5w \cdot a_2 + 0.5\xi^3 (w_{,2} \cdot w + a_{3,2}^+ \cdot w + a_{3,2}^+ \cdot w + 2\bar{w}a_2 \cdot w) \\
E_{13}^{p} &= 0.5w \cdot a_1 + 0.5\xi^3 (w_{,1} \cdot w + a_{3,1}^+ \cdot w + a_{3,1}^+ \cdot w + 2\bar{w}a_1 \cdot w) \\
\end{align*}
\]

Again a decomposition of the Green-Lagrange strain in a Kirchhoff-Love part and an additional part due to the shear difference vector $w$ and the 7th parameter $\bar{w}$ can be observed. A further subdivision in a Reissner-Mindlin part and a 3D shell part is not possible because the shear difference vector components of the two shell models span in different coordinate systems as discussed in subsection 3.3.1.

3.3.3 B-Operator

The first derivative of the strains with respect to the primal variables $D_r$ is:

\[
\begin{align*}
E_{11,r}^{p} &= E_{11,r}^{KL} + \xi^3 (w_{,1r} \cdot a_1 + w_{,1} \cdot a_{1,r}) \\
E_{22,r}^{p} &= E_{22,r}^{KL} + \xi^3 (w_{,2r} \cdot a_2 + w_{,2} \cdot a_{2,r}) \\
E_{33,r}^{p} &= w \cdot w_{,r} + a_{3,r}^+ \cdot w + a_{3}^+ \cdot w_{,r} + 2\xi^3 \bar{w}_{,r} (a_{3}^+ \cdot a_{3}^+ + 2a_{3}^+ \cdot w + w \cdot w) + \\
&\quad + 4\xi^3 \bar{w} (a_{3,r}^+ \cdot a_{3}^+ + w_{,r} \cdot a_{3}^+ + w \cdot a_{3,r}^+ + w \cdot w_{,r}) \\
E_{12,r}^{p} &= E_{12,r}^{KL} + 0.5\xi^3 (w_{,1r} \cdot a_2 + w_{,1} \cdot a_{2,r} + w_{,2r} \cdot a_1 + w_{,2} \cdot a_{1,r}) \\
E_{23,r}^{p} &= 0.5(w_{,r} \cdot a_2 + w \cdot a_{2,r}) + 0.5\xi^3 (w_{,2r} \cdot w + w_{,2} \cdot w_{,r} + a_{3,2,r}^+ \cdot w + a_{3,2}^+ \cdot w_{,r} + \\
&\quad + a_{3,r}^+ \cdot w_{,r} + a_{3}^+ \cdot w_{,2r} + 2(\bar{w}_{,r} a_2 \cdot w + \bar{w} a_{2,r} \cdot w + \bar{w} a_2 \cdot w_{,r}) \\
E_{13,r}^{p} &= 0.5(w_{,r} \cdot a_1 + w \cdot a_{1,r}) + 0.5\xi^3 (w_{,1r} \cdot w + w_{,1} \cdot w_{,r} + a_{3,1,r}^+ \cdot w + a_{3,1}^+ \cdot w_{,r} + \\
&\quad + a_{3,r}^+ \cdot w_{,1} + a_{3}^+ \cdot w_{,1,r} + 2\bar{w}_{,r} a_1 \cdot w + 2 \bar{w} a_{1,1} \cdot w + 2 \bar{w} a_{1} \cdot w_{,r})
\end{align*}
\]

where the primal variables $D_r$ consist of three translations, the three components of the shear difference vector $w^1$, $w^2$ and $w^3$, and the 7th parameter $\bar{w}$ of all control points.
3.3.4 Geometric Stiffness Matrix

The geometric stiffness $K_g$ is the derivative of the B-Operator with respect to the primal variable $u_s$. As depicted in equation 2.51, the discretized displacement field is linear in the primal variables. Therefore, certain terms are vanishing in the second derivative:

$$a_{rs} = w_{r,s} = w_{rs} = \bar{w}_{r,s} = \bar{w}_{rs} = 0 \quad (3.58)$$

The coefficients of the geometric stiffness matrix $E_{ij,rs}^{T_2}$ are then computed as:

$$E_{11,rs}^{T_2} = E_{11,rs}^{KL} + \xi^3 (w_{1,r} \cdot a_{1,s} + w_{1,s} \cdot a_{1,r})$$

$$E_{22,rs}^{T_2} = E_{22,rs}^{KL} + \xi^3 (w_{2,r} \cdot a_{2,s} + w_{2,s} \cdot a_{2,r})$$

$$E_{33,rs}^{T_2} = w_{,s} \cdot w_{,r} + a_{3,rs}^{\perp} \cdot w + a_{3,rs}^{\perp} \cdot w_{,s} + a_{3,rs}^{\perp} \cdot w_{,r} +$$

$$+ 4 \xi^3 w_{,s} (a_{3,rs}^{\perp} \cdot a_{3,rs}^{\perp} + a_{3,rs}^{\perp} \cdot w + a_{3,rs}^{\perp} \cdot w_{,s} \cdot w) +$$

$$+ 4 \xi^3 w_{,s} (a_{3,rs}^{\perp} \cdot a_{3,rs}^{\perp} + w_{,r} \cdot a_{3,rs}^{\perp} + w \cdot a_{3,rs}^{\perp} + w \cdot w_{,r}) +$$

$$+ 4 \xi^3 w (a_{3,rs}^{\perp} \cdot a_{3,rs}^{\perp} + a_{3,rs}^{\perp} \cdot a_{3,rs}^{\perp} + w_{,r} \cdot a_{3,rs}^{\perp} + w_{,s} \cdot a_{3,rs}^{\perp} + w \cdot a_{3,rs}^{\perp} +$$

$$+ w_{,s} \cdot w_{,r} )$$

$$E_{12,rs}^{T_2} = E_{12,rs}^{KL} + 0.5 \xi^3 (w_{1,r} \cdot a_{2,s} + w_{1,s} \cdot a_{2,r} + w_{2,r} \cdot a_{1,s} + w_{2,s} \cdot a_{1,r})$$

$$E_{23,rs}^{T_2} = 0.5 (w_{,r} \cdot a_{2,s} + w_{,s} \cdot a_{2,r}) + 0.5 \xi^3 (w_{2,r} \cdot w_{,s} + w_{2,s} \cdot w_{,r} + a_{2,rs}^{\perp} \cdot w +$$

$$+ a_{3,rs}^{\perp} \cdot w_{,s} + a_{3,rs}^{\perp} \cdot w_{,r} + a_{3,rs}^{\perp} \cdot w_{,s} + a_{3,rs}^{\perp} \cdot w_{,r} +$$

$$+ 2 w_{,s} a_{2,s} \cdot w + 2 w_{,r} a_{2,s} \cdot w_{,s} + 2 w_{,s} a_{2,r} \cdot w + 2 w_{,r} a_{2,r} \cdot w_{,s} +$$

$$+ 2 w_{,s} a_{2} \cdot w_{,r} + 2 w_{,r} a_{2} \cdot w_{,r} )$$

$$E_{13,rs}^{T_2} = 0.5 (w_{,r} \cdot a_{1,s} + w_{,s} \cdot a_{1,r}) + 0.5 \xi^3 (w_{1,r} \cdot w_{,s} + w_{1,s} \cdot w_{,r} + a_{1,rs}^{\perp} \cdot w +$$

$$+ a_{3,rs}^{\perp} \cdot w_{,s} + a_{3,rs}^{\perp} \cdot w_{,r} + a_{3,rs}^{\perp} \cdot w_{,s} + a_{3,rs}^{\perp} \cdot w_{,r} +$$

$$+ 2 w_{,s} a_{1,s} \cdot w + 2 w_{,r} a_{1,s} \cdot w_{,s} + 2 w_{,s} a_{1,r} \cdot w + 2 w_{,r} a_{1,r} \cdot w_{,s} +$$

$$+ 2 w_{,s} a_{1} \cdot w_{,r} + 2 w_{,r} a_{1} \cdot w_{,r} )$$
4 Numerical Examples

In this chapter, numerical examples are presented in order to validate the hierarchic shell formulations determined in chapter 3. They are tested against analytical solutions and reference solutions given by [13] and [16]. Mainly displacement values are investigated, but also internal and support forces are considered in order to reveal the performance of the shell models. The presented shells are able to capture large deformations. The computed benchmark problems are divided into a set of geometrically linear examples and one geometrically non-linear beam example. In all computations, a linear-elastic St.Venant-Kirchhoff material model is used.

4.1 Geometrically Linear Benchmark Problems

Four geometrically linear benchmark problems are discussed in the present section. At first, a tensile test is performed in order to prove that simple lateral strains can be computed correctly. As a second example, a single-span beam under a constant line load is investigated. The simplicity of the problem makes a comparison to analytical solutions possible. As a third example, a quadratic plate under a constant surface load is discussed with the aim to demonstrate that the presented RM shell is intrinsically free from transverse shear locking. The computation time performance of the implemented shell elements is also examined by means of this example. As the last benchmark problem, a linear computation of the Scordelis-Lo Roof is performed. The hierarchic shell elements’ ability to model more complex geometries is revealed by this example.

Only deformations are investigated within the geometrically linear problems. The shells are all implemented as non-linear. The linear deformations are obtained by performing only one iteration step. In the used open-source software, it is not possible to compute geometrically linear forces within the implemented elements because the values, on which they are based, are directly updated in a non-linear manner, before the forces itself are computed.

4.1.1 Tensile Test

The first example is a tensile test as illustrated in figure 4.1. The structure is only constrained in $x$-direction. Lateral strains occur due to the non-zero Poisson’s ratio in the $y$-direction as well as the thickness direction. The normal strains $\epsilon_{xx}$, $\epsilon_{yy}$ and $\epsilon_{zz}$ can be computed analytically by means of the normal stress $\sigma_{xx}$. No other stresses
Numerical Examples

Figure 4.1: Tensile test, problem setup

occur due to the specific support conditions. The normal stress $\sigma_{xx}$ is computed from the line load $q_x$ as:

$$\sigma_{xx} = \frac{q_x}{t} = \frac{10}{1} = 10$$

(4.1)

The normal strains are computed from $\sigma_{xx}$ applying a linear elastic material law:

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{10}{2.0 \cdot 10^5} = 5.0 \cdot 10^{-5} = \text{const.}$$

$$\epsilon_{yy} = \epsilon_{zz} = -\nu \frac{\sigma_{xx}}{E} = -0.3 \cdot 10 \frac{10}{2.0 \cdot 10^5} = -1.5 \cdot 10^{-5} = \text{const.}$$

(4.2)

The structure is exactly simulated by the three presented hierarchic shells. The KL and the RM shell element are able to compute the in-plane strains $\epsilon_{xx}$ and $\epsilon_{yy}$ correctly, whereas the transverse normal strain $\epsilon_{zz}$ is neglected by definition. The 3D shell additionally yields the transverse normal strain $\epsilon_{zz}$. This simple examples demonstrates the correct consideration of lateral strains for non-zero Poisson’s ratios.

### 4.1.2 Single-Span Beam under Line Load

The second example is a single-span beam under a constant line load $q_z$. The problem setup is presented in figure 4.2.
A geometrically linear computation is performed. Thus, it is possible to compare the numerical results with analytical solutions. The vertical displacement $w$ at mid-span is chosen as target value. The analytical solution for a KL shell, which is corresponding to an Euler-Bernoulli beam in the case of a beam, is computed by:

$$w_{KL} = \frac{qL^4}{76.8EI}$$

The deformation of a Timoshenko beam - here modeled with a RM shell - is obtained by:

$$w_{RM} = w_{KL} + \frac{qL^2}{8GA}$$

The given formulas are simply derived by the principle of virtual work. A discretization with quartic shape functions and one element in each direction is sufficient to obtain exact results since the solution is quartic in the displacements as it can be seen in equation (4.3). This discretization yields to five control points in each direction as shown in figure 4.2.

The structure is investigated for different values of the load and the geometric dimension, especially the thickness. Exact solutions referencing the analytical one are obtained for all cases, where the RM shell is able to capture the solution of the Timoshenko beam and the KL shell the solution of the Euler-Bernoulli beam. The 3D shell presented in section 3.3 yields the same result as the RM shell for a Poisson’s ratio of zero, i.e. $\nu = 0$. The reason is that there is no thickness change without transverse normal strain.

Figure 4.3 illustrates the deformation of the beam. The structure is discretized at the control points. The solution of the system of equations yields the deformation of the control polygon (dotted line). However, the real physical deformation of the structure has to be recovered from the control points by means of the shape functions. The deformation of the control polygon itself holds no physical meaningful values.
Numerical Examples

4.1.3 Quadratic Plate under Surface Load

The aim of this example is to demonstrate that the Reissner-Mindlin hierarchic shell intrinsically avoids transverse shear locking and to examine the computation time performance for the implemented shell elements. A quadratic plate under a surface load \( q_z \) is computed with different slenderness ratios \( \frac{L}{t} \), where the length of the plate \( L \) is kept constant and the thickness \( t \) is variable. The load is such chosen, that the deformation is the same for all thicknesses in the case of the Kirchhoff-Love shell, i.e. \( q_z = t^3 \). The plate has a soft support at all edges, i.e. \( v_z = 0 \). The Poisson’s ratio is unequal zero, i.e. \( \nu = 0.3 \). This means that the 7-parameter shell could yield different results in comparison with the Reissner-Mindlin shell. In case of \( \nu = 0.0 \), the solution of the RM and the 3D 7-parameter shell are identical which was tested as well. The problem setup is illustrated in figure 4.4.

A reference solution for the KL plate is obtained from the analytical Kirchhoff series solution considering the first two terms of the series solution [10]:

\[
\begin{align*}
v_{z,\text{max,KL}} &= \frac{5}{384} \frac{q_z L^4 12(1 - \nu^2)}{Et^3} - \frac{4q_z L^4 12(1 - \nu^2)}{\pi^5 Et^3} (0.68562 + 0.00025) = 0.442892 \quad (4.5)
\end{align*}
\]

where \( v_{z,\text{max,KL}} \) is the maximum displacement of the KL plate. The quadratic plate is simulated by the KL, RM and 3D shell with 100 biquadratic NURBS elements. The results for
$v_{z,max}$ are plotted against different slenderness ratios in figure 4.5 and the analytical values of the solution are given in table 4.1.

![Figure 4.5: Quadratic plate, maximum vertical displacement $v_{z,max}$ in dependency of the slenderness $\frac{L}{t}$](image)

<table>
<thead>
<tr>
<th>Slenderness $\frac{L}{t}$</th>
<th>5</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shell formulation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KL</td>
<td>0.4423</td>
<td>0.4423</td>
<td>0.4423</td>
<td>0.4423</td>
<td>0.4423</td>
<td>0.4423</td>
</tr>
<tr>
<td>RM</td>
<td>0.5839</td>
<td>0.4938</td>
<td>0.4456</td>
<td>0.4431</td>
<td>0.4423</td>
<td>0.4423</td>
</tr>
<tr>
<td>7p</td>
<td>0.5837</td>
<td>0.4936</td>
<td>0.4453</td>
<td>0.4429</td>
<td>0.4421</td>
<td>0.4420</td>
</tr>
</tbody>
</table>

**Table 4.1:** Displacement $v_{max}$ in dependency of the slenderness $\frac{L}{t}$

The results perfectly match the solutions obtained in [10, section 5.1]. The solution for the KL shell is independent of the slenderness and fits well the analytical Kirchhoff series solution. The other two shell formulations are shear deformable. Therefore, larger displacements occur for thicker plates compared to the KL shell. The results for the RM and the 7p shell are slightly different because the 7p shell considers additionally thickness changes. In the thin limit, the same results are expected for all three shell formulations. However, the displacement of the 7p shell differs from the others because the condition of vanishing transverse normal strains is only fulfilled approximately [10, section 5.1].

In [16, subsection 8.1.1], the quadratic plate is simulated by means of further non hierarchic RM shell formulations and the solution is compared to the hierarchic RM shell. The complementary results for the maximum $v_{z,max}$ are illustrated in [16]. The formulation labeled as RM-hr, which means Reissner-Mindlin with hierarchic rotations, is the one implemented in the scope of this thesis. The alternative formulation with hierarchic displacements is called
RM-hd. The standard RM shell with rotations as primal variables instead of a hierarchic shear difference vector is called RM-st. The abbreviation SD refers to a formulation with three mid-surface translations and one shear displacement as parameters. It is observed that the RM-st shell is affected by transverse shear locking. Thus, the discretization is not fine enough to avoid locking in general, but the hierarchic RM shell intrinsically avoids transverse shear locking.

The deformed structure is plotted with contour lines in figure 4.7. The figure presents the solution of the RM shell for a slenderness of five.

As the last part of this subsection, the performance of the implemented shell elements is examined with respect to the computation time. The results are shown in table 4.2. Three
different computation times are specified. The ‘time per element’ refers to the time which one element needs to compute the local element stiffness matrix and the corresponding vector of the internal nodal forces. The ‘build time’ describes the required time to compute the whole system of equations and the ‘system solve time’ refers to the solution time of this system of equations.

<table>
<thead>
<tr>
<th></th>
<th>KL</th>
<th>RM</th>
<th>RM-PreInt</th>
<th>RM-alt</th>
<th>7p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time per element</td>
<td>0.000081</td>
<td>0.001333</td>
<td>0.000754</td>
<td>0.003694</td>
<td>0.008111</td>
</tr>
<tr>
<td>Build time</td>
<td>0.053628</td>
<td>0.413863</td>
<td>0.249025</td>
<td>0.907206</td>
<td>2.880067</td>
</tr>
<tr>
<td>System solve time</td>
<td>0.008720</td>
<td>0.044422</td>
<td>0.044203</td>
<td>0.044845</td>
<td>0.110109</td>
</tr>
</tbody>
</table>

**Table 4.2:** Quadratic plate with slenderness \( \frac{L}{T} = 10 \), computation time in [s]

The three hierarchic shell elements are considered, whereby the Reissner-Mindlin shell is implemented in three different ways. The ‘RM’ element is the one used in all numerical examples. It is an optimized version where all contributions which have the value zero are directly neglected. The ‘RM-alt’ element refers to an alternative implementation where all equations and variables are explicitly stated even though they have the value zero.

The ‘RM-PreInt’ element is computationally optimized as ‘RM’, however, it uses a pre-integration scheme instead of a numerical integration over the thickness. The numerical integration is a Gauss integration with three Gauss points in thickness direction.

A simple example is given to make the difference between ‘RM’ and ‘RM-alt’ more comprehensible. The first derivative of the first strain coefficient with respect to the primal variables \( E_{11}^{RM} \) from equation 3.35 should be considered:

\[
E_{11,r}^{RM} = E_{11,r}^{KL} + \zeta^3 (w_{,1} \cdot a_1 + w_{,1} \cdot a_{1,r})
\]  

(4.6)

According to equation 2.31 in combination with equation 2.53, the first base vector of the mid-surface in the deformed configuration \( a_1 \) is defined as:

\[
a_1 = \left. \frac{\partial \mathbf{x}}{\partial \xi^a} \right|_{\xi^3=0} = \sum_i N_i^1 (\hat{X}^i + \hat{u}^i)
\]  

(4.7)

where \( \hat{u}^i \) is the displacement vector of the control point \( i \). The derivative of equation 4.7 with respect to the primal variables \( D_r \) yields:

\[
a_{1,r} = \sum_i N_i^1 (\hat{X}^i + \hat{u}^i)_{,r} = \sum_i N_i^1 \hat{u}^i_{,r}
\]  

(4.8)

Evaluating this equation with respect to the degrees of freedom \( w^1 \) and \( w^2 \), which are the coefficients of the shear difference vector \( \mathbf{w} \), in general yields \( a_{1,r} = [0, 0, 0]^T \). These vanishing terms are directly neglected in the element ‘RM’, whereas they are considered as zero vectors in ‘RM-alt’.
As expected, the performance of the elements decreases with an increasing number of parameters. The more complex shell formulations require more degrees of freedom. In addition, the number of equations accumulates with the hierarchic order, which is understandable by considering for example the complexity of the equations for the B-operator. The build time for the 'RM' shell is nearly 8-times higher than for the KL shell and that one for the 7p shell is even 54-times higher than for the KL shell. The system solve time is relatively small compared to the build time and its influence on the total computation time decreases with more parameters. The comparison between the alternative implementations of the Reissner-Mindlin shell reveals that the pre-integrated version 'RM-PreInt' is 1.8-times faster than the 'RM' element and this one is then again 2.8-times faster than the 'RM-alt' element. All in all, certain simplifications in the element formulation give rise to significant advantages with respect to the computation.

### 4.1.4 Linear Computation of the Scordelis-Lo Roof

In this subsection, a linear computation of the Scordelis-Lo-Roof is performed by means of the implemented shells. This structure is often used in shell obstacle courses because of its complex state of membrane strains. The problem setup is illustrated in figure 4.8. The structure is a section of a cylindrical shell supported by two rigid diaphragms at the curved edges. The other two edges are free. The structure is loaded by a uniform gravity load $p_z = 90$. The vertical displacement at the midpoint of the free side edge $v_{z,A}$, this point is marked as point A in figure 4.8, is used as reference solution. The computation is performed at the full system without consideration of the symmetry.

![Figure 4.8: Scordelis-Lo-Roof, problem setup](image)

The simulation is performed for different polynomial degrees $p$ of the B-Spline shape functions, namely 2, 3, 4 and 5. A convergence study is carried out by increasing the number of control points (CP) and such the number of elements of each configuration. The simulations are
Numerical Examples

performed for 5, 10, 20, 30 and 40 control points. Only in case of a polynomial degree of five, the coarsest discretization is 6 instead of 5 because 6 control points correspond to the minimum number of one element for this polynomial degree. The results are presented in Figure 4.9. All simulations converge to the same result which is determined as 0.30186. In [16, subsection 8.1.3], the reference displacement $v_{z,A}$ is computed as 0.30192 using the RM shell element with hierarchic displacements. The small deviation of 0.02% between the results is accepted as unknown difference in the implementations.

![Figure 4.9: Scordelis-Lo-Roof, RM shell, maximum vertical displacement at point A $v_{z,A}$ in dependency of the number of control points per side](image)

As expected, the models with higher polynomial degree converge faster. The simulation with $p = 5$ is an exception. At first, the coarsest discretization with 6 control points per side overestimates the reference solution. The solution jumps below the reference value in the next refinement step and stays lower than the results of $p = 4$ until the end of the convergence study. The origin of this behaviour is not yet fully understood. However, an overestimation of the deformation $v_{z,A}$ is also observed in the simulation of the Scordelis-Lo-Roof in [10, section 5.3]. The author of [10] concludes that the reason is connected to the application of a method against membrane locking. However, it is also possible that the origin stems from the hierarchic RM shell itself which was used in the cited paper as well as this thesis. Furthermore, the convergence curve of $p = 4$ also jumps over the reference solution in the second refinement step and back again in the next step. This strange behaviour is also observed in simulations performed with the KL shell element within this thesis even though in both cases the results converge finally to reasonable values. A validated implementation of the KL shell already exists in Carat ++ which is another simulation programme used for research at the Chair of Structural Analysis of the Technical University of Munich, where this thesis is written [7]. Simulations with this software program revealed different results even though the same KL shell formulation is used. The results match exactly the reference solution of $v_{z,A} = 0.3024$ provided by [2]. This gives rise to the presumption that there is a programming error related to the KL shell element in Kratos Multiphysics. Since the KL shell is used as
basis of the hierarchic shell family, it has a high priority to identify and correct the respective mistakes. This can be done by a step-by-step comparison of the two computer programmes. Unfortunately, this was out of the scope of this thesis.

Analytical results of the convergence study are provided for $p = 4$ in table 4.3. The aforementioned jump over the reference result in the second refinement step as well as the convergence to the value 0.30186 are observed.

<table>
<thead>
<tr>
<th>CP per side</th>
<th>5</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>80</th>
<th>100</th>
<th>130</th>
</tr>
</thead>
<tbody>
<tr>
<td>RM, $p = 4$</td>
<td>0.26165</td>
<td>0.30258</td>
<td>0.30136</td>
<td>0.30166</td>
<td>0.30183</td>
<td>0.30186</td>
<td>0.30186</td>
</tr>
</tbody>
</table>

Table 4.3: Scordelis-Lo-Roof, RM shell, $p = 4$, maximum vertical displacement at point A $v_{z,A}$ in dependency of the number of control points per side

Figure 4.10 shows the deformed structure under the uniform gravity load scaled by a factor of 20.

Figure 4.10: Scordelis-Lo-Roof, deformed structure
4.2 Geometrically Non-Linear Single-Span Beam under Line Load

In this example, a geometrically non-linear simulation of the single-span beam under line load from section 4.1.2 is performed. The problem setup is depicted in figure 4.2. Certain properties are further defined for this example: the length $L = 10$, the width $b = 2$ and the load $q_z = 2 \cdot t^3$. To define the load as a function of the thickness $t$ yields that the deformation is independent of the thickness for linear computations. This is also approximately valid for non-linear computations with respect to the scale of the deformations.

The beam example is further subdivided into a problem where small displacements occur and another where large displacements occur. In chapter 4.1, geometrically linear problems are simulated by simply computing only one iteration step of the Newton-Raphson procedure. However, it is not recommended to calculate stresses and internal forces in this way because the elements are implemented with a non-linear kinematic and therefore the stress results are more reliable for a full iterative Newton-Raphson procedure. In subsection 4.2.1, a geometrically non-linear simulation is performed for a beam with small displacements. The Young’s modulus $E = 10^6$ and the thickness $t = 1.0$ are such chosen that small displacements occur. The fact that the displacements are relatively small facilitates the validation of the internal forces against analytical beam solutions.

In subsection 4.2.2, the beam is investigated for large displacements to be able to validate the geometrically non-linear behaviour of the shell elements. Two cases regarding the slenderness of the beam are considered. The Young’s modulus is for both cases the same with $E = 1000$, but the thickness is once $t = 1.0$ and once $t = 0.1$. This refers to a moderately thick beam with a slenderness of $\frac{L}{t} = 10$ and a thin beam with a slenderness of $\frac{L}{t} = 100$ respectively.

4.2.1 Small Displacements: Validation of Distribution of Internal Forces

The problem setup is the same as for the beam example in section 4.1.2 and it is therefore referred to figure 4.2 for a display. The properties of the beam are further defined: $L = 10$, $b = 2$, $t = 1.0$, $E = 10^6$ and $q_z = 2$. The resulting deformations lie in the range of $10^{-3}$ and can be understood as small. Therefore, it is possible to compare the results to well defined analytical beam solutions.

Even though the displacements are small, the analysis procedure is non-linear with two or three iteration steps depending on the refinement. This yields better results for the investigated distribution of the internal forces because the elements are implemented with a non-linear kinematic. The internal moment $m_{11}$ and the shear force $q_{13}$ are investigated for different polynomial degrees $p$, namely 2, 3 and 4. The indices of the moment and the shear force refer to local Cartesian coordinate direction along the deformed beam. The index ’1’ points in the beam direction and ’3’ in the thickness direction. A computation for constant shape functions, i.e. $p = 1$, is not possible because the base vectors are defined by the first derivative of the shape function $N_i$. In the beam direction 80 B-Spline elements
Numerical Examples

are used. The other shell direction is only discretized with 1 B-Spline element because the load carrying direction is clearly defined for the beam problem. The simulation is performed for the KL as well as the RM shell. The results of the 3D shell are identical with those from the RM shell because the Poisson’s ratio is zero. Therefore, they are not presented separately.

The analytical solutions for the moment at mid-span and the shear force at the support are given as:

\[ m_{11} = \frac{q_z L^2}{8} = \frac{2 \cdot 10^2}{8} = 25 \]
\[ q_{13} = \frac{q_z L}{2} = \frac{2 \cdot 10}{2} = 10 \]  \hspace{1cm} (4.9)

Figure 4.11 illustrates the distribution of the internal moment \( m_{11} \) along the \( x \)-axis. The values are computed at the Gauss points and are linear interpolated in the post-processing instead of using the underlying B-spline shape functions for simplicity reasons. The moment is computed with \( p = 2 \) and \( p = 3 \) for the KL and the RM shell. The maximum fits exactly the analytical solution for all configurations. The distribution is stepped for a polynomial degree of 2 because the second derivative of the shape functions \( N_{ij} \), which is not continuous across the element boundaries for a polynomial degree of 2, is required in order to compute the moment (figures 4.11a and 4.11b). Furthermore, it is observed that the moment distribution for the RM shell with \( p = 2 \) illustrated in figure 4.11b looses the stepped shape close to the supports. The reason for this phenomenon lies in the specific formulation of the Reissner-Mindlin shell element by means of hierarchic rotations and is further discussed in subsection 4.2.2. The moment distributions for both shells with \( p = 3 \) are continuous and perfectly match the analytical solution (figures 4.11c and 4.11d).

Figure 4.12 illustrates the distribution of the shear force \( q_{13} \) along the \( x \)-axis. The values are again only computed at the Gauss points and are linear interpolated in the post-processing instead of using the underlying B-spline shape functions for simplicity reasons. The shear force is computed with \( p = 2 \), \( p = 3 \) and \( p = 4 \) for the KL and the RM shell. All distributions are symmetric as expected. The KL shell is not able to compute any reasonable results for \( p = 2 \) as illustrated in figure 4.12a. However, the values obtained are not exactly zero as it seems to be in this figure. The shear force has to be calculated as derivative of the moment for the KL shell, as it was explained in subsection 3.1.5. For this procedure the third derivative of the shape functions \( N_{ijk} \), which is zero for \( p = 2 \), is required. Therefore, a polynomial degree of at least 3 is required to be able to compute reasonable results for the shear forces with the KL shell. For the RM shell, the transverse shear strain is directly parameterized and therefore reliable results are also obtained for \( p = 2 \) as illustrated in figure 4.12b. However, the distribution shows a small oscillation close to the supports. As for the moment, the reason for this phenomenon lies in the specific formulation of the Reissner-Mindlin shell element by means of hierarchic rotations and is further discussed in subsection 4.2.2. The shear force shows a stepped distribution for the KL shell with \( p = 3 \) (figure 4.12c). The stepped shape already occurred for the moment distribution. However, the polynomial degree with a stepped distribution is one order higher in case of the shear force because the shear force is computed as the derivative of the moment. All other distributions are perfectly
The quality of the internal forces significantly depends on the polynomial degree of the shape function. Table 4.4 shows the order of the derivatives of the shape functions needed to compute the particular internal force or moment. This depends also on the used element type due to the different stress recovery procedures discussed in subsection 3.1.5 for the KL shell element and in subsection 3.2.5 for the RM shell element. The stress recovery of the 3D shell works in the same way as of the RM shell and yields therefore the same requirements. The table does not indicate whether the results are exact but only which order of derivative is used in the computation. The accuracy of the results additionally depends on the polynomial order of the loads.
Table 4.4: Requirements of the computation of the internal forces and moments with respect to the derivatives of the shape functions $N^i$

<table>
<thead>
<tr>
<th></th>
<th>internal moments $m_{\alpha\beta}$</th>
<th>membrane forces $n_{\alpha\beta}$</th>
<th>shear forces $q_{\alpha\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KL</td>
<td>2nd derivative $N^i_{\alpha\beta}$</td>
<td>1st derivative $N^i_{\alpha}$</td>
<td>3rd derivative $N^i_{\alpha\beta\gamma}$</td>
</tr>
<tr>
<td>RM</td>
<td>2nd derivative $N^i_{\alpha\beta}$</td>
<td>1st derivative $N^i_{\alpha}$</td>
<td>1st derivative $N^i_{\alpha}$</td>
</tr>
</tbody>
</table>

All discussed effects would be more pronounced for a coarser mesh, i.e., less elements. The resolution of the results can be generally increased by a higher polynomial order or a mesh refinement. However, certain effects, as for example the stepped distribution, would not even occur for a sufficiently high polynomial degree as discussed within this subsection.
Figure 4.12: Comparison of moment distribution of KL and RM shell elements.
4.2.2 Large Displacements: Validation and Convergence Study

The problem setup is the same as for the beam example in section 4.1.2 and it is therefore referred to figure 4.2 for a display. The properties of the beam are further defined: \( L = 10 \), \( b = 2 \) and \( E = 1000 \). The right support is movable in \( x \)-direction. With this support condition, in contrast to the geometrically linear example, displacements in \( x \)-direction really occur. The example is identical to the problem of subsection 8.2.1 in [16] regarding the setup. This facilitates the usage of the respective results as reference solutions. In [16, subsection 8.2.1], the simulations are performed by means of the same hierarchic RM shell, which is implemented in the scope of this thesis, and the SHELL181 from ANSYS 16. The latter one is a four node, shear deformable, geometrically non-linear shell element used in classical FEA, whereby transverse shear locking is eliminated by the ANS-concept [16, pg. 137].

Two different slenderness ratios are investigated, \( \frac{L}{t} = 10 \) and \( \frac{L}{t} = 100 \). The shell in the first configuration is moderately thick and has therefore certain shear deformations, whereas the shell in the second configuration is very thin according to [4, table 2.1]. The thick setup is also used to check the assumption of small shear rotations and the consequence of a linearized director \( \alpha_p^0 \) as described in subsection 3.2.1. This check is possible due to a comparison with the SHELL181 element which is not based on this simplification.

The simulation is performed for a polynomial degree \( p = 2 \) and different numbers of elements \( n_{ele} \). The so-called residual-criterion is applied to decide whether the result is sufficiently exact or whether a further iteration step should be performed. This criterion depends, inter alia, on the number of degrees of freedom in such a way that the tested residual decreases with further mesh refinement. If the number of iteration steps decreases due to refinement, the tolerances in this simulation are always such reduced that the number of iteration steps corresponds to the one of the coarser refinement.

Table 4.5 shows the results of the performed convergence test. For both slenderness ratios, convergence is obtained. The result fits perfectly the one from ANSYS 16 for the thin case. A small difference of 0.20% can be observed in the thick case. This occurs due to the assumption of small shear rotations in the RM model, which is explained in subsection 3.2.1. The fact that the difference is relatively small is used in [16, subsection 8.2.1] to justify the named assumption. The result for the RM shell given in [16, subsection 8.2.1] is with \( v_{z,\text{max}} = 2.3238 \) slightly different from that one obtained in this thesis. It is hard to identify the reason for this difference which can lie in every part of the implementation. However, it is acceptable because it means only a deviation of 0.10%.

The results are also compared to the values obtained by a simulation with the KL shell in order to check whether the classification as thick and thin shell is valid. The maximum displacement in \( z \)-direction is 2.3009 in the thick and 2.2863 in the thin case. In consequence, the difference between the RM and the KL shell is 0.90% in the thick and 0.01% in the thin case. This demonstrates the correctness of the distinction in the two cases. However, the difference is also quite small in the thick case and therefore the advantage of applying the more sophisticated Reissner-Mindlin shell can be questioned. Furthermore, the justification of the assumption of the small rotations can be doubted considering that the difference between the
\begin{table}
\centering
\begin{tabular}{|c|cc|cc|}
\hline
n_{ele} in x-direction & \multicolumn{2}{c|}{\frac{L}{T} = 10} & \multicolumn{2}{c|}{\frac{L}{T} = 100} \\
\hline
10 & 2.3092 & 2.3108 & 2.1098 & 2.2707 \\
20 & 2.3189 & 2.3224 & 2.2711 & 2.2827 \\
40 & 2.3210 & 2.3253 & 2.2851 & 2.2856 \\
80 & 2.3215 & 2.3260 & 2.2864 & 2.2863 \\
120 & 2.3215 & 2.3262 & 2.2865 & 2.2865 \\
160 & 2.3215 & 2.3262 & 2.2865 & 2.2865 \\
\hline
\end{tabular}
\caption{Convergence of the displacement $v_{z,\text{max}}$}
\end{table}

RM and the KL shell is 0.90\% and that one between the RM and the fully non-linear shear deformable SHELL181 0.20\%.

In addition to the displacements, the internal forces, namely the normal force $n_{11}$ and the shear force $q_{13}$, are investigated. The respective results are illustrated in figure 4.13. Three different discretizations are considered to determine the influence of refinement on the quality of the internal forces. The k-refinement is applied which is a combination of the two basic types of refinement for NURBS functions, namely knot insertion and order elevation. The discretization with a polynomial degree of two and 10 B-Spline elements can be understood as starting point. The first refinement refers to a discretization with the same polynomial degree of two but with 80 B-Spline elements and the second refinement refers to discretization with a higher polynomial degree of three but a remaining number of 10 B-Spline elements. The results are plotted against the $x$-position of the deformed configuration. The supports move together and therefore the last value is no longer evaluated at $x = 10$ but somewhere around $x \approx 8.5$.

At first, oscillations are observed in all simulations with a polynomial degree $p = 2$ in figure 4.13. The magnitude of the oscillations depends on the slenderness which means that the oscillations are more pronounced in the thin case which is illustrated in the figures 4.13b and 4.13d. Therefore, it is conjectured that some kind of locking becomes active with increasing slenderness. Two particular reasons are identified by [16]. At first, the structure is affected by membrane locking, which is shown in [16] subsection 8.2.1 by elimination of this locking type by means of a Mixed-Displacement (MD) concept. Furthermore, the formulation of the Reissner-Mindlin shell element by means of hierarchic rotations merely shifts the disbalance from the shear strains to the curvature as shown in [16] section 5.2. In standard formulations, the disbalance in the shear strain leads to transverse shear locking as explained in subsection 2.4.1 The disbalance in the curvature again yields some kind of locking which is described in more detail in [16] section 5.2. However, this locking is especially problematic for thickset structures in case of a pure shear deformation which is from minor interest because it lies beyond the application limits of the Reissner-Mindlin shell. A simulation with continuum elements is anyway recommended for these limit cases. As already mentioned in section 3.2 an alternative formulation of the Reissner-Mindlin shell with hierarchic displacements is also presented by [16] in order to avoid the described disbalance in the curvature. As it can be
observed in figure 4.13, the oscillations are not prevented by refinement, but the magnitude is dramatically decreased. However, the polynomial degree \( p = 3 \) yields smooth curves for the normal force in the thin case and the shear forces in both cases. The results for the shear force match the respective results provided by [16, subsection 8.2.1]. The normal forces are not compared to any reference solutions.

For a simulation with the 3D shell element, the same results are expected as for the RM shell because of the Poisson’s ratio \( \nu = 0.0 \). However, it is not possible to obtain reasonable results for this example with the 3D shell. It was not possible in the scope of this thesis to identify the reason for this problem. Therefore, the implementation of the 3D shell is not validated for geometrically non-linear problems and the ongoing work has to identify and correct
the underlying problem. However, the test results have shown good correspondence to the reference solutions of the linear examples in section 4.1.
5 Conclusion and Prospects

Within this thesis, the Isogeometric Analysis of thin walled shell structures with hierarchic shell elements was discussed. The objective was the implementation of a hierarchic family of isogeometric shell elements in the module IgaApplication of the open-source software Kratos Multiphysics. The validation was an important aim in order to assure the accuracy of the developed software tool. The shell models should be able to simulate geometrically non-linear deformations. The hierarchic shells promised an a priori locking free formulation which was examined in detail. The determined findings are discussed in the subsequent sections and suggestions for further developments are provided.

5.1 Conclusion

The presented hierarchic shell family consists of a 3-parameter Kirchhoff-Love (KL), a 5-parameter Reissner-Mindlin (RM) and a 7-parameter 3D isogeometric shell element. The fundamentals concerning each single formulation were elaborated in this thesis.

The hierarchic shells are intrinsically free from transverse shear locking, curvature thickness locking and Poisson’s thickness locking. This is advantageous with respect to the convergence rate and the stability of the solutions. Furthermore, no additional treatment is required to eliminate locking post hoc. However, not all locking phenomena are fully avoided, as e.g. problems with in-plane shear locking and membrane locking remain, which can be eliminated by additional features. As it was shown in the example of subsection 4.2.2, the disadvantageous effects of locking are reduced by k-refinement which is a combination of order elevation and knot insertion.

It was observed that refinement also plays a crucial role in IGA. Even though a coarse discretization is often sufficient in order to represent the initial geometry, refinement is mostly needed to correctly display the deflections and internal forces (subsections 4.1.4 and 4.2). As a result from the named subsections, a sufficiently high polynomial degree of the NURBS shape functions seems to be more effective than mesh refinement in order to achieve accurate results. Furthermore, a better convergence rate was observed as the polynomial degree increases (subsection 4.1.4).

The consideration of the application limits of the different shell elements is important in order to assess the accuracy of the results. The KL element is limited to thin shells, whereas the RM element can also display thick shells because it additionally captures shear deformations (subsection 4.1.3). The 3D shell element is also shear deformable and, moreover, covers load induced thickness changes. All so far investigated examples determined no significant difference between the results with the RM or the 3D shell. However, the examples did not tackle the expected application field of the 3D shell which is connected to non-linear
Conclusion and Prospects

The analysis of structures with relatively high thickness and high Poisson’s ratio. Besides, the 3D shell facilitates a simpler geometrically non-linear formulation without a linearization of the shear rotation in contrast to the RM shell due to its global definition of the shear difference vector (subsection 3.3.1). In addition, three-dimensional material laws can be applied without further considerations, whereas the RM shell requires a static condensation and the KL shell can only deal with two-dimensional material laws.

More complex material laws considering plasticity require a numerical integration in the thickness direction because the stress distribution at the cross-section is not predefined anymore. The two basic concepts of numerical integration and pre-integration were discussed in this thesis (subsections 3.1.4.2 and 3.2.6). The pre-integration is advantageous with respect to the computation time but is limited to linear elastic material laws. The computation time additionally increased significantly with the number of parameters used for the discretization (subsection 4.1.3). In consequence, it is only reasonable to use element formulations with a higher number of parameters where certain advantages with respect to the accuracy of the results are expected.

5.2 Prospects

In the following, some ideas are proposed for the pursuing software development of the module IgaApplication of Kratos Multiphysics. The highest priority should lie in the collation of the implementation of the KL shell in Kratos Multiphysics with that one in Carat ++. Both programs should yield exactly the same results which is not completely the case at the moment. This was tested for the Scordelis-Lo-Roof in subsection 4.1.4. Furthermore, the 3D shell should be adjusted for non-linear simulations as it is only possible to perform linear simulations at the current state. This work can be seen as a basis for ongoing efforts in order to extend the 3D shell to non-linear problems.

Within this thesis, all structural models were limited to untrimmed single patch objects and simple support conditions. The modeling of more complex boundary and coupling conditions in combination with the hierarchic shell elements requires further development. With this additional features, the elements could be also tested for models which consist of multiple trimmed patches in the sense of an Isogeometric B-Rep Analysis (IBRA). In addition to more complex geometries, the performance of the hierarchic shells should be investigated for non-linear material laws. These deepening studies would reveal the application fields of the three presented shell formulations.
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I hereby declare that the thesis submitted is my own unaided work. All direct or indirect sources used are acknowledged as references. In addition, I declare that I make the present work available to the Chair of Structural Analysis for academic purposes and in this connection also approve of dissemination for academic purposes.

Munich, 30 September 2019

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