Dynamic equations of a transversely isotropic, highly porous, fibrous material including oscillatory heat transfer effects

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The dynamic equations of a transversely isotropic fibrous, highly porous material are presented in terms of microstructure-derived analytical expressions for viscous dissipation, and analytical expressions for the oscillatory heat transfer between the thermal fields of the solid cylindrical glassfibres and the surrounding viscous fluid. This represents the non-equilibrium thermal expansion of the fluid, occurring when waves propagate in the porous material, and results in a frequency-dependent scaling of the fluid dilatation term. A state-space transfer matrix solution of the governing equations has been introduced, allowing the numerical acoustical performance of the fibrous material to be investigated, including the acoustical effects of heat transfer. In order to understand the dissipation mechanisms of the viscous and thermal boundary layers on the surface of the fibres and the validity of the assumptions made in the current model, a thermoviscous acoustic fluid finite element procedure has also been introduced. The results from these simulations illustrate the frequency-dependent interaction of the boundary layers between neighbouring fibres in the porous material. © 2019 Acoustical Society of America. https://doi.org/10.1121/1.5129368

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I. INTRODUCTION

Recent work into the acoustic modelling of fibrous porous materials has increasingly focussed on microstructural aspects, i.e., the fibre diameter distributions, fibre orientation angles, and the distributions of fibre lengths1–3 in order to predict the transport properties of the material. In practice, this means that a combination of numerical and experimental techniques is used to estimate the macroscopic transport properties of the fibrous material, such as static viscous and thermal permeabilities, viscous characteristic lengths, and tortuosity. Using this information, the vibroacoustic performance may then be simulated using established Biot-based poroelastic models.4,5 Through this, the relationship between the physical modelling of these materials and parameters which are controllable in manufacturing processes may be established, thus allowing more efficient virtual material development. To feed such models, a series of intermediate steps are often required: either complex fluid flow and thermal diffusion simulations, and existing material samples for laboratory measurements.

In contrast to these, the approach taken in the present work is to start from analytically-based microstructure models of the viscous and the thermal interaction fields in a highly porous material. It is based on previous work to develop a model for acoustic and structural wave propagation through highly porous, fibrous materials, as originally proposed in Ref. 6, which included the concept of oscillatory heat transfer between the solid fibres and surrounding fluid.9,10 At the time, this was not pursued further due to the perceived theoretical limits of viscous and thermal boundary layer interaction between neighbouring fibres. These limits are now overcome by recent advances in thermoviscous acoustics computational modelling, thus revealing the full potential of the initial ideas. Recently the authors,7,8 have studied the dissipation regions of the viscous and thermal boundary layers, and the interaction with neighbouring fibres, and through this have provided a more solid understanding of the limitations and possibilities of the analytically based modelling approach, and of the importance of fibre diameter and orientation distributions for real materials.

In the current work, the microstructural aspects of the fibrous material are taken into account in the formulation of the governing poroelastic equations, through analytically based dynamic impedances. It is important to note that, contrary to classical approaches, the current modelling neither takes into account the pore structure as such nor does it include effects of irregularities or constrictions in the material. Instead, and under the assumption of high porosity, the contribution to the total viscous and thermal dissipation is computed for each individual fibre in an unbounded fluid at rest, using as inputs known fibre diameter distributions and orientation angles.

The starting points are the coupled fluid and solid momentum equations presented in Ref. 7 for a fibrous porous material. It is assumed that viscous and thermal field interactions between neighbouring fibres may be neglected, meaning the formulation is targeted at higher porosity materials having large spacings between fibres. This allows us to use analytical expressions to describe the dynamic viscous drag forces resulting from the relative motion between an assumed array of longitudinal and transverse cylindrical fibres and the surrounding viscous fluid, as proposed in Ref. 7. Along the same lines, we introduce the analytical oscillatory heat transfer expressions

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into the governing poroelastic equations through a modified fluid dilatation term which is valid for non-equilibrium conditions, allowing for one-way heat flow from the solid to the fluid phases to be considered. Note that in the previous work where non-linear, high intensity sound propagation through assumed rigid, isotropic fibrous materials was studied, only experimentally derived viscous drag effects were considered in the sound propagation modelling. Although expressions for oscillatory heat transfer effects were developed, they were neglected in the analysis since their influence on attenuation was assumed to be much lower than the viscous effects.

To solve the dynamic equations including viscous dissipation and heat transfer, proposed here, a previously published Transfer Matrix Method (TMM) is employed. This particular solution approach, which is based on a state space representation, is completely general in terms of the material symmetry as well as oblique incidence. The method will be briefly reviewed, and the proposed micro-structural modelling approach will be used to solve problems using only as input parameters the geometrical and the constitutive material parameters, and fibre microstructure parameters only. The predicted response is then validated in comparison against impedance tube absorption coefficient measurements.

As part of the validation, we have carried out thermoviscous acoustic fluid finite element modelling in order to investigate the viscous and thermal boundary layer dissipation interaction within a regular array of fibres. This provides a means of confirming the underlying assumptions that viscous and thermal field interactions between neighbouring fibres can be neglected in the analytical modelling of a highly porous fibrous material.

The paper is organised into a section introducing the fibrous material and a section describing the governing equations in their general forms. These are then followed by the derivation of the expressions representing the non-equilibrium heat flow, with some additional details in an Appendix, the introduction of a new way of writing the Biot coupling coefficients, which appear in the governing equations, a brief review of the Transfer Matrix solution approach, and finally validations focused on the predicted plane wave response as well as the proposed thermoviscous finite element modelling approach.

II. MATERIAL DEFINITION

The fibrous material considered is a lightweight and flexible aircraft fuselage acoustical and thermal insulation from the Johns Manville Company. The solid fibre skeleton consists of a distribution of glass fibres, having diameters ranging from 0.125 to 5.0 micrometres, with a mean fibre diameter of 22 micrometres. The orientation of the fibres is assumed to be primarily transversely isotropic, with some fibres being orientated through the thickness of the material, as shown in Fig. 1. The transversely isotropic elastic properties for this material have been previously determined from in vacuo static measurements. A complete list of the fluid properties (air at 20°C), the assumed glass fibre properties, and the measured transversely isotropic macroscopic elastic properties are provided in Table I.

III. GOVERNING EQUATIONS WITHOUT HEAT TRANSFER

The following notations will be used in the constitutive and momentum equations presented in this section: $u'$ is the solid frame displacement, $u$ is the fluid displacement, $e'$ is the solid frame Cauchy strain tensor, $e$ is the solid dilatation, $c$ is the fluid dilatation and $\phi$ is the open porosity. Furthermore, $I$ is the 3 by 3 diagonal identity matrix.

A. Momentum equations

Following Ref. 7, the momentum equation for a continuous porous-solid model for the thermal insulation is given as

$$\nabla \cdot \sigma^e = -\omega^2 \rho_s u' + i \omega Z(u' - u'), \tag{1}$$

where $\rho_s = (1 - \phi)\rho I$ and $\rho$ is the solid frame density, $\sigma^e$ the solid frame stress tensor, and $Z$ is the 3 by 3 spatially-orientated dynamic drag impedance tensor per unit volume of homogenised material, which the skeleton exerts on the fluid. Note that this is analogous to Biot’s permeability, and is described further in the Appendix.

Similarly for the fluid phase, the momentum equation was given as

**TABLE I.** Fibre microstructure model material properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean fibre diameter</td>
<td>$2r_s$</td>
<td>$1.28 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>Mean fibre spacing</td>
<td>$b$</td>
<td>$18.052 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>Bulk density</td>
<td>$\rho_b$</td>
<td>10.88 kg/m$^3$</td>
</tr>
<tr>
<td>Fluid density</td>
<td>$\rho_f$</td>
<td>1.2044 kg/m$^3$</td>
</tr>
<tr>
<td>Fluid density</td>
<td>$\rho_s$</td>
<td>2450 kg/m$^3$</td>
</tr>
<tr>
<td>Porosity</td>
<td>$\phi$</td>
<td>0.9961</td>
</tr>
<tr>
<td>Fluid dynamic viscosity</td>
<td>$\mu_f$</td>
<td>1.8140 $\times 10^{-5}$ kg/ms</td>
</tr>
<tr>
<td>Fluid bulk modulus</td>
<td>$K_f$</td>
<td>101325 Pa</td>
</tr>
<tr>
<td>Fibre bulk modulus</td>
<td>$K_s$</td>
<td>35 GPa</td>
</tr>
<tr>
<td>Fluid thermal conductivity</td>
<td>$\kappa_f$</td>
<td>0.02577 W/mK</td>
</tr>
<tr>
<td>Fibre thermal conductivity</td>
<td>$\kappa_s$</td>
<td>0.775 W/mK</td>
</tr>
<tr>
<td>Fluid specific heat</td>
<td>$C_{pf}$</td>
<td>1005 J/kgK</td>
</tr>
<tr>
<td>Fibre specific heat</td>
<td>$C_{ps}$</td>
<td>775 J/kgK</td>
</tr>
<tr>
<td>Fluid expansion coeff.</td>
<td>$\eta$</td>
<td>3.411 $\times 10^{-3}$ 1/K</td>
</tr>
<tr>
<td>Young’s modulus, z</td>
<td>$E_1$</td>
<td>135 Pa</td>
</tr>
<tr>
<td>Young’s modulus, x, y</td>
<td>$E_2$</td>
<td>10300 Pa</td>
</tr>
<tr>
<td>Shear modulus, z</td>
<td>$G_1$</td>
<td>700 Pa</td>
</tr>
<tr>
<td>Shear modulus, x, y</td>
<td>$G_2$</td>
<td>8200 Pa</td>
</tr>
<tr>
<td>Poisson ratio, z</td>
<td>$\nu_1$</td>
<td>0</td>
</tr>
<tr>
<td>Poisson ratio, x, y</td>
<td>$\nu_2$</td>
<td>0</td>
</tr>
<tr>
<td>Loss factor, z</td>
<td>$\eta_1$</td>
<td>0.1</td>
</tr>
<tr>
<td>Loss factor, x, y</td>
<td>$\eta_2$</td>
<td>0.1</td>
</tr>
</tbody>
</table>
\[ \nabla \cdot \sigma' = -\omega^2 \rho_z \mathbf{u}' + i\omega Z (\mathbf{u}' - \mathbf{u}), \] (2)

where \( \rho_z = \phi \rho_1, \rho_j \) is the ambient fluid density, and \( \sigma' \) the fluid stress tensor.

### B. Constitutive equations

The constitutive laws for the solid phase of the fibrous material are written under the assumption that the bulk modulus of the fibres is much higher than the bulk modulus of the porous material itself. The Cauchy stress tensor \( \sigma' \) may then be written as

\[ \sigma' = (\hat{C} + D) \varepsilon' + Ee, \] (3)

where \( D = (1 - \phi)^2 \tilde{K}_{eq} \mathbf{I}, E = \phi (1 - \phi) \tilde{K}_{eq} \mathbf{I}, \tilde{K}_{eq} \) is the scalar fluid compressibility modulus, here introduced in the following form:

\[ \tilde{K}_{eq} = \frac{K_f}{\phi}, \] (4)

and \( \hat{C} \) corresponds to the \textit{in vacuo} Hooke tensor of the solid phase, such that the \textit{in vacuo} stress tensor of the solid phase, in the absence of fluid, is given by Hooke’s law

\[ \hat{\sigma}' = \hat{C} \varepsilon'. \] (5)

The fluid stress tensor \( \sigma' \) is given as

\[ \sigma' = (\phi (1 - \phi) \varepsilon + \phi^2 \varepsilon') \tilde{K}_{eq} \mathbf{I}. \] (6)

In addition to the above defined stress tensors, it is also useful to consider the total stress tensor \( \sigma \), which is linked to the other stress tensors by the following relations:

\[ \sigma' = \sigma' + \sigma' = \sigma' - p \mathbf{I}, \] (7)

which may be used to express the link between the solid and \textit{in vacuo} stress tensors

\[ \sigma' = \hat{\sigma}' - (1 - \phi) p \mathbf{I}. \] (8)

### IV. CONSTITUTIVE RELATIONS INCLUDING HEAT TRANSFER

In the following, the constitutive relations linking the solid stresses and the fluid pressure to the fluid dilatation as well as the solid strains will be reformulated to include effects of heat transfer. The modelling will be restricted to a one-way heat transfer, i.e., from the solid to the fluid. This restriction implies that the following set of hypotheses are assumed to be valid:

- (A) Thermoelastic coefficients may be neglected. This means that the solid phase is treated as having zero coefficient of thermal expansion (or negligible as compared with the thermal expansion of the fluid phase);
- (B) Thermal expansion of the fluid does not alter \textit{Eq. (3)} for the skeleton stresses. In other words the skeleton stresses are determined uniquely, at a given frequency, by the skeleton strains and the fluid pressure.

This assumed one-way heat transfer between the solid and fluid phases leads to thermal expansion of the fluid, which is reflected in an altered fluid dilatation term as will be shown in the following.

#### A. Fluid dilatation with heat transfer

To introduce the heat transfer from the solid to the fluid phase, the linearised entropy equation for the two-phase porous material needs to be established. If we consider a unit control volume of the fibrous porous material, it is occupied partly by the fluid of volume \( \phi \), and partly by the solid fibres having volume fraction \( (1 - \phi) \). The mass conservation equation for the fluid contents of the unit cube is also written as

\[ \frac{\partial}{\partial t} \left( \rho_f \phi \right) = -\nabla \cdot \left( \rho_f \phi \mathbf{u}' \right), \] (9)

where \( \partial / \partial t (\rho_f \phi) \) refers to its rate of change with time, and \(-\nabla \cdot (\rho_f \phi \mathbf{u}') \) is the rate of mass inflow through the boundaries. The rate of heat input to the fluid from the solid fibres, per unit total volume, is introduced as \( Y \), and the rate of entropy input to the control volume, per unit total volume, is defined by a thermodynamic equation of state as

\[ \dot{S} = \frac{Y}{T_f}, \] (10)

where \( T_f \) is the absolute temperature of the fluid, and the entropy input from the solid fibres is \( \dot{S} \). If we define \( s \) as the specific entropy of the fluid, then the rate of specific entropy input per unit total volume is governed by a macroscopic entropy balance equation which relates the rate of entropy change within a fixed region to the entropy input from the solid fibres and the net inflow across the fluid boundaries

\[ \frac{\partial}{\partial t} (\rho_f \phi s) = \dot{S} - \nabla \cdot \left( \rho_f \phi \mathbf{u}' \right). \] (11)

Here, the rate of entropy production within the fluid region of the control volume is second order and is therefore neglected in a linear approximation. Subtracting \( s \) times Eq. (9) from Eq. (11) then gives

\[ \rho_f \phi \frac{\partial s}{\partial t} = \dot{S} - \rho_f \phi (\nabla \cdot \mathbf{u}^s ) s. \] (12)

Neglecting the higher order term on the right hand side (RHS) of Eq. (12), and combining Eqs. (10) and (12) results in

\[ \frac{\partial s}{\partial t} \approx \frac{Y}{\rho_f \phi T_f}. \] (13)

For oscillations at a single frequency, \( Y \) is written in terms of the fluid temperature perturbation \( T_f' \) as

\[ Y = -\bar{Y} e T_f', \] (14)
where $Y_r$ is called the unit volume effective thermal impedance function, written in terms of the respective fluid and fibre thermal impedances as outlined in the Appendix by Eq. (A14), since it relates the heat transfer rate to the fluctuating fluid temperature, and $T_f'$ is given in Ref. 15 as

$$T_f' = \left( \frac{T_f \eta}{\rho_f C_p} \right) s + \left( \frac{T_f}{C_p} \right).$$

From Eqs. (13) and (14), the following relation between $Y$ and the fluid stress may be derived:

$$Y = \alpha' = -\alpha \phi p,$$

where

$$\alpha = \eta T_f \left( \frac{\rho_f C_p \phi}{Y_f} + \frac{1}{j \omega} \right)^{-1}.$$

Combining Eqs. (9) and (16) with the general relationship for fluid density in terms of pressure and specific entropy

$$d \rho_f = \frac{1}{\rho_f c^2} dp - \frac{\rho_f \eta T_f}{C_p} ds,$$

we arrive at a pressure-density relation for the fluid medium which allows interphase heat transfer

$$\frac{1}{\rho_f} \frac{\partial \rho_f}{\partial t} = \frac{1}{\rho_f c^2} \frac{\partial p}{\partial t} + \frac{\alpha \eta}{\rho_f C_p} \phi,$$

where the second term on the RHS of Eq. (19), models the assumed heat transfer.

**B. General continuity equation**

The general continuity equation for a two-phase composite, in which both the fluid and solid fibre skeleton phases are regarded as compressible, may be written as

$$(1 - \phi) \dot{\varepsilon} + \phi \dot{\varepsilon}' + (1 - \phi) \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial t} + \phi \frac{1}{\rho_f} \frac{\partial \rho_f}{\partial t} = 0.$$  (20)

This is a kinematic relation and is not affected by the heat transfer. For given values of $\varepsilon$ and $\varepsilon'$, we can use Eqs. (13) and (14) to determine how the fluid dilatation is affected by heat transfer and departs from equilibrium. Note that by hypothesis (B) posed earlier, $\sigma'$ is the same function of $\varepsilon$ and $\phi$, whether or not heat transfer is present; the same therefore applies to the density of the solid phase $\rho_s$. The skeleton dilatation is then determined by $\dot{\varepsilon}$ alone without regard to heat transfer effects. Thus, without loss of generality, the solid phase may, in the derivation of the corrections required for the fluid dilatation with heat transfer effects included, be considered to be incompressible. This implies that $\dot{\varepsilon}$ and also $\dot{\rho}_s / \partial t$ are both zero, reducing Eq. (20) to

$$\dot{\phi} \dot{\varepsilon} + \frac{1}{\rho_f} \frac{\partial \rho_f}{\partial t} = 0.$$  (21)

which introducing Eq. (19) and the relation $\sigma' = -\phi p I$ together with the fluid stress tensor with heat transfer taken into account, $\sigma'$, gives

$$\sigma' = \left( 1 + \frac{\alpha \eta \phi K_{eq}}{j \omega \rho_f C_p} \right) \sigma'.$$

The substitution of Eq. (22) into Eq. (6), using the relation $\sigma' = -\phi p I$ and introducing also Eq. (4) then allows the fluid constitutive equation with heat transfer to be written as

$$\chi \sigma' = \phi K_{eq} ((1 - \phi) \varepsilon + \phi \varepsilon') I,$$

where $\chi$ is defined by

$$\chi = \left[ 1 - \frac{\alpha \eta \phi K_{eq}}{j \omega \rho_f C_p} \right].$$  (24)

The role of heat transfer from the solid fibres to the surrounding fluid can then be considered as a frequency-dependent scaling by $\chi$ of the fluid continuity expression to represent non-equilibrium conditions.

It is important to highlight that these relations are valid for single temperatures only; the dynamic viscous drag force and heat transfer terms are presented here as being uncoupled, i.e., an increase of viscous forces does not increase the fluid temperature, which would be a non-linear effect. This means that the long-term increase of temperature in the porous material due to the continued presence of dynamic viscous forces is not accounted for.

The isothermal to adiabatic transition occurs at higher frequencies for fibres of nanoscale diameter, and shifts to increasingly lower frequencies for larger diameter fibres, as shown in Fig. 2. The transition frequency occurs as the thermal impedance, i.e., resistance of, the solid fibres becomes dominant, as compared to the surrounding fluid or gas (see Fig. 3), meaning that heat transfer through the fibres becomes diminished.

**V. PLANE WAVE PROPAGATION THROUGH A LAYER OF INSULATION MATERIAL**

As a means of verification of the theory presented here, the absorption coefficient of a prescribed thickness of...
rigidly-backed insulation material is calculated and compared with standard impedance tube experimental results. In this case, the fibrous insulation sample considered is assumed to be composed of stacked layers of fibres having a normalised diameter and are distributed at random orientations within an isotropic plane (transversely isotropic). Alternatively, a material composed of a distribution of fibre diameters at completely random spatial orientations (fully anisotropic) can be considered as well.

As shown in Ref. 8, the proposed model of a fibrous material may be written in terms of the complex densities introduced by Biot as

\[
\rho^{*}_{11} = \frac{1}{\omega \rho_0} (1 - \phi) \rho_s I - \frac{Z}{\omega}, \tag{25}
\]

\[
\rho^{*}_{12} = \frac{Z}{\omega}, \tag{26}
\]

\[
\rho^{*}_{22} = \frac{\phi \rho_f I - \frac{Z}{\omega}}{\omega}. \tag{27}
\]

These may be used in a Transfer Matrix solution as have been presented for the case of transverse isotropy, see, e.g., Ref. 8, where a normal incidence absorption case was presented.

A. \(u^s\)–\(u^t\)–\(p\)-formulation

To address also more complex materials, an alternative formulation, well suited for poroelastic materials with general anisotropy, will be proposed here. We introduce

\[
\tilde{\rho} = \frac{Z}{\rho_f \phi}, \tag{28}
\]

where \(\tilde{Z} = Z/\rho_f \phi\), which allows us to write the complex densities appearing in Eqs. (1) and (2) in a slightly different form as

\[
\tilde{\rho}^{22} = \phi \rho_f \tilde{z}, \tag{29}
\]

\[
\tilde{\rho}^{12} = \phi \rho_f (I - \tilde{z}), \tag{30}
\]

\[
\tilde{\rho}^{11} = (1 - \phi) \rho_f I + \phi \rho_f (\tilde{z} - I). \tag{31}
\]

Using these relations, the momentum equations may be rewritten as

\[
\sigma^t + \nabla p = -\omega^2 \rho u^t, \tag{32}
\]

with the solid apparent density given as

\[
\tilde{\rho} = (1 - \phi) \rho_s I + \phi \rho_f (I - \tilde{z}^{-1}). \tag{33}
\]

Similarly, the fluid momentum equation is given as

\[
-\phi \nabla p = -\rho_0 \omega^2 \tilde{u}^t - \rho_0 \omega^2 \tilde{u}^t, \tag{34}
\]

and from Eq. (23), the pressure \(p\) may be related to the divergence of the total displacement \(u^t\)

\[
\nabla p = -\tilde{K}_eq \nabla \cdot u^t, \tag{35}
\]

where

\[
u^t = \phi u^t + (1 - \phi) u^t, \tag{36}
\]

where the assumption that the solid frame modulus is much higher than the bulk modulus of the porous material, has been used.

B. Transfer matrix solution

To solve the governing system of equations, i.e., Eqs. (32), (34), (35), (36), the plane wave expansion in Eq. (37) is introduced. We will assume a plane wave solution of the form, see Fig. 4

\[
(\cdot)(x, y, z, t) = (\cdot)(z)e^{i(\omega t - k_x x - k_y y)}, \tag{37}
\]

where \((\cdot)(z)\) is the complex amplitude of the corresponding physical field across the layer, and \(k_x\) and \(k_y\) are the wavenumbers of the arbitrarily oriented incident plane wave as determined by the incident wave. Only a subset of the field amplitudes representing the poroelastic medium are needed in the solution of the problem. This is due to a linear dependence which is partially originating from the spatial dependence prescribed by the wavenumbers \(k_x, k_y\), as well as being required to establish the coupling relations at the
the evolution of $s(z)$ can be rewritten in the form of a State-Space representation

$$\frac{\partial}{\partial z} s(z) = -\gamma s(z),$$

where $\gamma$ is referred to as the State Matrix. To assemble $\gamma$, the system of equations are collected in a general system of first-order equations in a Cartesian coordinate system $xyz$, and the transfer matrix representation is obtained from

$$M(d) = e^{-\gamma t},$$

where $d$ is the thickness of the layer and $e^{(\cdot)}$ is the matrix exponential. We will not detail the different ways of solving the governing equations derived here, for, e.g., the absorption coefficient. The reader is referred to either of the approaches discussed in Refs. 6, 8, and 19, or using the transfer matrix in Eq. (40), as discussed in Refs. 11 and 12.

**VI. MODEL VALIDATION AND RESULTS**

The dynamic equations of motion of a transversely isotropic porous fibrous insulation material developed in this work are assumed to represent an array of axial and transverse fibres, having measured fibre diameter distributions, orientations and spacing between fibres. They have been formulated entirely based on microstructure-derived viscous and thermal dissipation mechanism assumptions which are suitable for high-porosity lightweight fibrous porous materials. For validation purposes, the aforementioned model has been applied in the transfer matrix form discussed above to calculate the normal incidence absorption coefficient of a 50 mm thickness of rigidly-backed sample of fibrous thermal insulation material, refer to Table I, and is compared to impedance tube measurements. A mean fibre diameter of 1.28 micrometres was used in this analysis, and a representative fibre inclination angle of 50 degrees (from the $xy$ plane) was chosen, which was sufficient for the validation shown here. We will present results using an expanded set of fibre distributions and orientation angles in future work.

The comparison between the numerical simulation and measured absorption coefficient is shown in Fig. 5. Good agreement exists in the comparison throughout the medium and higher frequency ranges, but as expected, the one-dimensional numerical TMM used here does not capture low-frequency resonances present in the measurement due to sample edge effects. Material inertial and dissipative effects have been correctly represented in the simulation through the modified stress-strain relations, and the analytical expressions used to describe microstructural viscous and thermal dissipation and coupling in the model. Further improvements in the results are expected as more detailed fibre diameter and orientation distributions are considered.

The acoustical influence of heat transfer from the solid fibres to the surrounding air is also demonstrated. The thermal expansion of the air surrounding the fibres has effectively stiffened the fibrous porous material, resulting in the absorption behaviour being shifted to higher frequencies. This is due to the expected increase in phasespeed of the dilational fluid wave, refer to Fig. 6, and a decrease in attenuation, refer to Fig. 7, as compared to the case when heat transfer effects are not considered. Here, the phase velocity of the fluid dilatational wave has been estimated as defined in Ref. 11. The propagation of the dilatational solid skeleton wave is not influenced by heat transfer effects.

The analytical expressions for dynamic viscous drag impedance $Z$ through the thickness of the material sample, as defined by Eq. (A16), may also be scaled according to fibre diameter distributions and orientations to provide an estimate of the airflow resistance of the material. This is shown in Fig. 8 in the form of the low frequency asymptote of the dissipative (real part) of the $(3, 3)$ component of $Z$. The through-thickness measured value of airflow resistivity for this material was 23 400 Ns $m^{-4}$, while the low frequency asymptotic estimate provided by Eq. (A18) was 22 470 Ns $m^{-4}$, which was again in very good agreement.
VII. THERMOVISCOUS ACOUSTIC FLUID FINITE ELEMENT MODELLING

In the formulation of the microstructure-based equations for acoustic-structural wave propagation through a fibrous porous material presented here, the underlying assumption is that the dynamic viscous and thermal fields surrounding individual fibres do not interact significantly with the fields of neighbouring fibres. For a given frequency, the boundary layer thicknesses for these fields are $l_{\text{visc}} = \left( \frac{2\mu_f}{\text{op}_f} \right)^{1/2}$ and $l_{\text{ent}} = \left( \frac{2k_f}{\text{op}_f C_p f} \right)^{1/2}$, such that for the case of having air as the fluid surrounding the cylindrical glass fibres, the typical viscous and thermal boundary layer thicknesses are $l_{\text{visc}} \approx 2.2 \times 10^{-3}$ m, and $l_{\text{ent}} \approx 2.6 \times 10^{-3}$ m for a fibre excitation frequency of 1 Hz. Therefore, in the low frequency range, these boundary layer thicknesses are very much greater than the average spacing between fibres (18.052 micrometres) in the fibrous material.

This would theoretically invalidate the assumptions inherent to the analytically derived viscous and thermal expressions in our formulation. To address this, we have utilized the thermoviscous acoustic fluid modelling capabilities in COMSOL, implemented in the form of the linearised Navier-Stokes equations, as a finite element based virtual laboratory, in order to fully understand the behaviour of the viscous and thermal boundary layers on the surface of the fibres, and also to reasonably assess the interaction of the boundary layers within a representative statistical array of fibres. In this approach, the fluid surrounding the solid glass fibres is then represented as a superposition of coupled acoustic, vorticity, and entropy modal fields.

A. Viscous and thermal boundary layers of a single cylindrical fibre

The first case of a sinusoidally oscillating elastic cylinder, embedded in an infinite thermoviscous acoustic fluid was considered in order to establish a modelling criteria for the finite element (FE) numerical procedure, using as a reference the analytical formulation for transverse dynamic drag force as defined by Eq. (A18). The main challenges with this approach are realising an exact representation of the viscous and thermal boundary layers in the vicinity of the fibre surface, respecting the elastic coupling of the viscous fluid to the solid glass fibre, and the definition of non-reflecting outer boundary conditions to allow sufficient decay of the boundary layers with increasing radial distance from the fibre surface.

The fluid region surrounding the fibre of mean diameter was divided into two separate domains, an inner thermoviscous acoustic fluid one having a radius of 15 mm, which was then encased by a 5 mm thickness acoustic pressure field having a non-reflecting radiation boundary on the outer surface, as shown in Fig. 9. The combined depth of the fluid surrounding the fibre is then 20 mm. For consistency with the discussion in Ref. 7 where computational limitations dictated the choice, the excitation frequency was chosen as 1 Hz, at which the depths of the viscous and thermal boundary layers are large, and the fibre oscillation amplitude was chosen to be 0.01 micrometres.

In the single fibre model, a combination of linear pressure, and quadratic velocity and thermal elements were used for the thermoviscous fluid and acoustic pressure regions, respectively. Quadratic elastic stress elements were used for...
the stress field internal to the fibre, which was then coupled to the surrounding fluid regions. This newly refined single-fibre model, as compared to our previous work, contained a total of approximately 420,000 degrees-of-freedom. Integrating over the fibre surface to estimate the reaction forces then allows the transverse dynamic viscous drag impedance per unit length values to be estimated. From the single fibre FE model, a value of 87,613 Nsm$^{-4}$ was found, which is within 0.06% of the analytical result of 87,665 Nsm$^{-4}$ derived using Eq. (A18) and is a very satisfactory validation of the numerical FE modelling procedure.

Examination of the viscous and thermal power dissipation density fields, as shown in Figs. 10 and 11, indicates the concentration of the dissipative region of the viscous and thermal boundary layers in the vicinity of the fibre surface, at micro and nano-scale levels. Considering these fields radially in the x direction from the fibre surface, as shown in Figs. 12 and 13, further demonstrates that even for the very low excitation frequency of 1 Hz, there is a concentration of dissipation within the boundary layers to the immediate vicinity of the fibre surface. The dissipation region of the viscous boundary layer, which is responsible for most of the losses in the analytical model, has decayed approximately 98% at the position of the next neighbouring fibre, according to the known fibre spacing parameter $b$.

This is significantly less than the viscous boundary layer penetration depth value of 2.2 mm at 1 Hz. As frequencies increase, the concentration of the boundary layers to the immediate vicinity of the fibre surface will be even more pronounced, meaning that the interaction of viscous boundary layers between fibres is not expected to be an important limitation of the theory for practical frequency ranges.

The dissipation region of the thermal boundary layer has not decayed to the same extent at the position of the next neighbouring fibre. Since the level of dissipation in the thermal boundary layer is orders of magnitude less than that of the viscous one, a reasonable approximation then is to neglect this interaction. This may not be true for the interaction of the temperature fields between neighbouring fibres, which then influences the aforementioned thermal expansion of the fluid, the significance of which needs to be better understood.

B. Boundary layer interaction within a multi-fibre array

To further investigate the significance of viscous and thermal boundary layer interaction between fibres, a more realistic case with a large array of 225 fibre diameters based on the known diameter distribution was modelled, using the mean fibre spacing $b$ of 18.052 micrometres, as schematically shown in Fig. 14. This model adds not only increased complexity due to a larger number of interacting fibres, it also incorporates the effects related to a variation of the fibre diameters in the array. The array was centered around a fibre with the mean diameter. Using the verified single-fibre modelling procedure, the resulting FE model consisted of approximately $35 \times 10^6$ degrees-of-freedom.

All fibres in the array were excited in-phase at a frequency of 1 Hz, corresponding to an in-phase displacement of all fibres in the array, and the fibre oscillation amplitude was chosen to be 0.01 micrometres. The resulting viscous and thermal power dissipation density fields are shown in Figs. 15–18, zoomed to the immediate vicinity of the fibres. As a comparison, the arithmetic mean of the analytical viscous drag force impedance function, using the measured statistical distribution of fibre diameters, gives a value of 79,527 Nsm$^{-4}$. Integrating over the surfaces of all 225 fibres in the FE simulation, the arithmetic mean of the fibre reaction forces allowed the dynamic drag force impedance to be
FIG. 13. (Color online) Single fibre thermal power dissipation along the \(x\)-direction of the fibre array for 1 Hz excitation frequency assuming a linear excitation amplitude of 0.01 micrometres in the \(x\)-direction.

FIG. 14. The 225 fibre array, with diameters according to the measured distribution, and the estimated fibre spacing \(b\). The center fibre in the array has the mean fibre diameter of 1.28 micrometres.

FIG. 15. (Color online) The 225 random fibre distribution viscous power dissipation density field (W/m\(^3\)) for 1 Hz excitation frequency assuming a linear excitation amplitude of 0.01 micrometres in the \(x\)-direction for all fibres in the array.

FIG. 16. (Color online) Close up of Fig. 15, fibres immediately surrounding the central mean diameter fibre from the 225 random fibre distribution viscous power dissipation density field (W/m\(^3\)).

FIG. 17. (Color online) The 225 random fibre distribution thermal power dissipation density field (W/m\(^3\)) for 1 Hz excitation frequency assuming a linear excitation amplitude of 0.01 micrometres in the \(x\)-direction for all fibres in the array.

FIG. 18. (Color online) Close up of Fig. 17, fibres immediately surrounding the central mean diameter fibre from the 225 random fibre distribution thermal power dissipation density field (W/m\(^3\)).
estimated as $78,403 \text{ Nm}^{-1}$, which is accurate to approximately 1.1%. Considering these fields radially in the $x$ direction from the fibre surface, as shown in Figs. 19 and 20, demonstrates that the viscous dissipation fields of individual fibres remain independent of those from neighbouring fibres as one moves across the fibre array. This suggests that for the glass fibre thermal insulation material, there is not a significant level of viscous dissipation interaction between the fibres. This is not the case for thermal dissipation in the fibre array, as there is an accumulation of thermal dissipation with each subsequent fibre as we move across the fibre array, indicating a certain level of interaction between the thermal dissipation fields of neighbouring fibres.

VIII. CONCLUSIONS

In this work, we have considered a control volume representation of a lightweight fibrous acoustic and thermal aircraft fuselage insulation material in order to develop a general set of dynamic equations for fibrous porous materials based entirely on geometrical and physical microstructure considerations, and constitutive material parameters. Viscous losses are included in the model through analytical expressions for the dynamic drag impedance of cylindrical fibres in a form similar to Biot’s permeability tensor. Non-equilibrium conditions are included in the fluid dilatation expressions to characterise heat transfer effects between the solid fibres and surrounding viscous fluid. Fibre spacing (i.e., porosity) and distributions of diameter and fibre orientation angles are easily considered in the modelling approach.

A state-space transversely isotropic transfer matrix representation of the model has been developed with the subsequent simulation of an impedance tube experiment providing very good agreement with measurements. The proposed modelling approach has a direct relationship to geometrical and constitutive parameters which are controllable in fibrous material manufacturing processes, and thus lends itself to the physically-based numerical optimisation of lightweight fibrous porous materials for acoustics.

Dynamic thermoviscous acoustic fluid finite element simulations have also been used to investigate the interaction of viscous and thermal dissipation between neighbouring fibres in the material. It was determined that the dissipative effect of the viscous boundary layers was significantly more dominant as compared to dissipation in the thermal boundary layer, and is concentrated in the immediate vicinity of the fibre surface. This suggests that for highly porous fibre insulation materials, the interaction of viscous dissipation between neighbouring fibres can be considered to be negligible, which supports the assumptions considered in the development of the aforementioned analytical dynamic viscous drag force expressions.

We have also observed that there is interaction between the thermal dissipation of neighbouring fibres, the implications of which are not yet fully understood regarding the thermal field in a fibre array, and the resulting influence upon the phase speed of acoustic wave propagation through the material.

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APPENDIX: MICRO-STRUCTURAL MODEL SUMMARY

1. Oscillatory heat transfer effects for a single fibre in an infinite fluid

The modelling of the heat transfer effects is a local treatment, valid under the assumptions of high porosity and a sparse network of contact points in between the individual fibres. Inspired by the approach of Refs. 9 and 10, the expressions for the effective thermal impedance of the fibrous insulation are presented here based upon the oscillatory thermal fields of both the solid fibres and surrounding fluid, which are assumed to be infinite in this analysis. The
heat transfer is assumed to be through conduction under isothermal conditions only, which is appropriate for low intensity sounds propagating through the interstitial fluid of the porous material.

2. Thermal impedance of a fluid external to a cylindrical fibre

The temperature perturbations for a fluid surrounding a cylindrical fibre are assumed to have the form

\[ T' = \Psi H_0^{(2)}(k_f r)e^{i\omega t}, \tag{A1} \]

where \( \Psi \) is a potential and is based on a temperature distribution governed by a Helmholtz-type equation where the thermal wavenumber is

\[ k_f = \sqrt{-\frac{\omega}{\alpha_f}}, \tag{A2} \]

and \( \alpha_f = \frac{k_f}{(\rho_f C_p)} \) is the thermal diffusivity of the fluid. If the temperature perturbation has the harmonic form \( T'(r = r_s) = T_0 e^{i\omega t} \) at the fluid/cylindrical fibre interface, then the heat transfer rate from the fluid to the surrounding fluid at the boundary is

\[ q' = -k_f \frac{\partial T'}{\partial r}|_{r=r_s} = T_0 k_f k_g \frac{H_1^{(2)}(k_f r_s)}{H_0^{(2)}(k_f r_s)} e^{i\omega t}. \tag{A3} \]

Integrating around the circumference of the cylinder yields the heat transfer per unit cylinder length

\[ q = \int_0^{2\pi} q' r_s d\theta = 2\pi r_s q', \tag{A4} \]

and, subsequently, the thermal impedance per unit cylinder length for the fluid external to the cylinder can be defined as

\[ Y_f = \frac{q}{T'(r = r_s)} = 2\pi \alpha_f k_f \frac{H_1^{(2)}(k_f r_s)}{H_0^{(2)}(k_f r_s)}. \tag{A5} \]

3. Thermal impedance of a cylindrical fibre

Assuming that the temperature distribution within the fibre is in the form of solution to the unsteady heat conduction equation

\[ T'(r < a) = \Psi J_0(k_o r) e^{i\omega t}, \tag{A6} \]

with corresponding wavenumber

\[ k_o = \sqrt{-\frac{\omega}{\alpha_o}}, \tag{A7} \]

where \( \alpha_o = \kappa_o/(\rho_o C_{po}) \) is the thermal diffusivity. The heat flux into the fibre at the cylindrical boundary is

\[ Y_s = \frac{-q}{T'(r = r_s)} = -2\pi r_s \alpha_o k_o \frac{J_1(k_o r_s)}{J_0(k_o r_s)}. \tag{A8} \]

4. Effective thermal impedance function

Let \( v \) be the rate of heat input to the fluid from the solid fibre skeleton per unit fibre length, given by

\[ v = -Y_f \left( T'_f - T'_f \right), \tag{A9} \]

where \( Y_f \) is the fluid thermal impedance function. Similarly, the rate of heat input to the fibre per unit fibre length, neglecting the heat transfer from fluid to fibre, is given by

\[ -v = Y_s T'_s, \tag{A10} \]

where \( Y_s \) is the fibre thermal impedance function. Equating Eqs. (A9) and (A10) gives, after some manipulation

\[ \frac{T'_f}{T'_s} = \frac{Y_f}{Y_f + Y_s}. \tag{A11} \]

With the aim to eliminate the temperature perturbation in the solid, we introduce \( Y_c \) as the effective thermal impedance function, and the heat transfer rate per unit length from an individual fibre to the surrounding fluid may be written as

\[ v = -Y_c T'_f. \tag{A12} \]

This must, however, equal the heat transfer as expressed in Eq. (A9), and from these two relations, \( Y_c \) is given as

\[ Y_c = \left[ \frac{1}{Y_f} + \frac{1}{Y_s} \right]^{-1}, \tag{A13} \]

and likewise, per unit volume

\[ \bar{Y}_c = \frac{(1 - \phi)}{A} \left[ \frac{1}{Y_f} + \frac{1}{Y_s} \right]^{-1}, \tag{A14} \]

where \( A = b^2 \) is the insulation control volume cross-sectional area. Here, \( Y_f(\zeta), Y_s(\zeta) \) are known thermal impedance functions of the fibre diameter. If the thermal insulation blanket includes a distribution of fibre diameters \( \zeta \), with specified length per unit volume, then the total heat transfer rate from all the fibres in the unit volume is

\[ \bar{Y} = -T'_f \int_0^\infty g(\zeta) \bar{Y}_c d\zeta. \tag{A15} \]

5. Viscous drag impedance expressions

In general, the fibres in a unit volume will have a distribution of both fibre orientation angles and fibre diameters. In Ref. 7, it was described in detail how to obtain the viscous drag force impedance matrix for such a case. Here it is assumed that the fibrous material is transversely isotropic with the fibres in the plane randomly distributed. In such a material, the spatially-averaged dynamic drag impedance tensor is given as

\[ Z = \left( \frac{1 - \phi}{2A} \right) \begin{bmatrix} Z_t + Z_s & 0 & 0 & 0 \\ 0 & Z_t + Z_s & 0 & 0 \\ 0 & 0 & 2Z_t \end{bmatrix}. \tag{A16} \]
where the longitudinal drag force impedance $Z_l$ and the transverse $Z_t$ viscous drag force impedance, per unit fibre length, for an individual fibre are

$$Z_l = 2\pi r_s k_B \frac{H_1^{(2)}(k_B r_s)}{H_0^{(2)}(k_B r_s)},$$  \hspace{1cm} (A17)

and

$$Z_t = \frac{j \pi \rho_f \omega r_s^2}{4} \left[ 1 - \frac{4H_1^{(2)}(k_B r_s)}{k_B r_s H_0^{(2)}(k_B r_s)} \right],$$  \hspace{1cm} (A18)

and $(1 - \phi)/A$ is the total fibre length per unit volume of the thermal insulation material. The shear wavenumber of the infinite viscous fluid medium surrounding the fibre is given by

$$k_B = \left[ -\left( j \omega \rho_f \right)/\mu_f \right]^{1/2}.$$