Symplectic Embeddings and results in TDA

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Abstract

This thesis is a collection of work under the theme of “applied topology." The linking idea behind seemingly disjoint fields is the existence of a filtration that one uses to study a space. In turn, given the ubiquitous nature of filtrations, applications range from theoretical fields (e.g. symplectic geometry) to applied fields (machine learning).

In paper A, we study when homological information of a simplicial complex can be determined from its components in the following manner: given a data cloud, partition the points in the cloud into two (or more) sets. Form separate simplicial complexes from these sets, and compare the homologies of these simplicial complexes from that of the simplicial complex formed from the point cloud itself. In applied topology, very rarely does a decomposition of a space yield information about the space itself - meaning that it is rare for a Mayer-Vietoris sequence to hold. We study “obstruction complexes" and show that in nice enough cases, there is a relationship between homological information of the space and its decomposition.

In paper B, we study a construction called “realisation" that we apply to posets. This enables the generation of a wealth of examples of posets that might not necessary be the nonnegative reals in topological data analysis. We define various properties of these realizations, and in the end we link these properties to homological properties of the functors that are being studied.

In paper C, we study the classic evasion-path problem. This problem is well-known in robotics and machine learning, and more recently became of interest in the applied topology community through works of Krishnan and Ghrist in addition to work of Adams and Carlsson. The key point is that just studying homology and barcodes could not determine if an evasion path exists. We study a higher invariant, using tools of Goodwillie calculus to yield an obstruction to the existence of an evasion path.

In symplectic geometry, work has been done to try to use the filtration to study symplectic embeddings. The work in this thesis does not get to the direct relationship between the filtration and symplectic embeddings, but it does study the relationship between symplectic embeddings of ellipsoids and polydiscs in dimension four, yielding a rigid-flexible result similar to the one given by the famous nonsqueezing theorem. This is the topic of paper D. There is still much work to be done linking applied topology and symplectic geometry.

Sammanfattning

Den här avhandlingen är en samling av arbeten under temat tillämpad topologi.Den länkande idén bakom till synes osammanhängande fält är förekomsten av en filtrering som man använder för att studera ett utrymme. I sin tur, med tanke på filtreringarnas allestädes närvarande karaktär, varierar tillämpningarna från teoretiska fält (t.ex. symplektisk geometri) till tillämpade fält (maskininlärning).

I artikel A studerar vi när homologisk information om ett förenklat komplex kan bestämmas från dess komponenter på följande sätt: givet ett datamoln, dela upp punkterna i molnet i två (eller flera) uppsättningar. Bilda separata förenklade komplex från dessa uppsättningar och jämför homologierna för dessa förenklade komplex med den för det förenklade komplexet som bildas av själva punktmolnet. I tillämpad topologi ger mycket sällan en nedbrytning av ett utrymme information om själva utrymmet - vilket betyder att det är sällsynt att en Mayer-Vietoris-sekvens håller. Vi studerar öbstruktionskomplexöch visar att det i tillräckligt trevliga fall finns ett samband mellan homologisk information om rummet och dess nedbrytning.

I artikel B studerar vi en konstruktion som kallas förverkligandesom vi tillämpar på posetter. Detta möjliggör generering av en mängd exempel på poseter som kanske inte nödvändigtvis är de icke-negativa realerna i topologisk dataanalys. Vi definierar olika egenskaper hos dessa realiseringar. , och i slutändan kopplar vi dessa egenskaper till homologiska egenskaper hos de funktorer som studeras.

I uppsats C studerar vi det klassiska undanflyktsvägsproblemet. Detta problem är välkänt inom robotik och maskininlärning, och blev mer nyligen av intresse i den tillämpade topologigemenskapen genom verk av Krishnan och Ghrist förutom arbete av Adams och Carlsson. Nyckelpunkten är att bara att studera homologi och streckkoder inte kunde avgöra om en undanflyktsväg existerar. Vi studerar en högre invariant och använder verktyg av Goodwillie-kalkyl för att skapa ett hinder för förekomsten av en undanflyktsväg.

Inom symplektisk geometri har man arbetat med att försöka använda filtreringen för att studera symplektiska inbäddningar. Arbetet i den här avhandlingen kommer inte till det direkta sambandet mellan filtrering och symplektiska inbäddningar, men det studerar sambandet mellan symplektiska inbäddningar av ellipsoider och polyskivor i dimension fyra, vilket ger ett styvt-flexibelt resultat som liknar det som gavs av den berömda icke-klämningssats. Detta är ämnet för papper D. Det finns fortfarande mycket arbete att göra när det gäller att koppla samman tillämpad topologi och symplektisk geometri.

Contents

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1 Introduction & background

This thesis ties together work done under the theme of applied topology, where in general one tries to use a filtration in order to study a space. In particular, the four ideas that were worked on are: symplectic embeddings, the pursuit-evasion problem, a Mayer-Vietoris sequence, and tameness. There is still much work to be done in all areas.

1.1 Symplectic Embeddings

There has been recent work trying to link ideas behind topological data analysis and symplectic geometry. Once one has a filtration, the basic ideas of prevalent in persistence may be applied. We give a brief overview of symplectic geometry and symplectic embeddings in four dimensions.

Definition 1.1. A smooth manifold M is **symplectic** if M is an even-dimensional manifold equipped with a closed, nondegenerate 2-form ω.

We recall that a form ω being *closed* means that $d\omega = 0$, and nondegeneracy means that $\omega^n \neq 0$. We also write this as (M, ω). We will also always assume that M has dimension $2n$ if not explicitly stated.

Example 1.2 (The standard symplectic vector space $\mathbb{R}^{2n}$)**.** Let $M = \mathbb{R}^{2n}$ and let $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

Definition 1.3. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ such that $f^*\omega_2 = \omega_1$ is called a **symplectomorphism**.

Definition 1.4. A smooth embedding $\varphi : M_1 \to M_2$ is called **symplectic** if φ is a symplectomorphism.

Symplectic embeddings are very hard to understand even for simple objects. The definition of a symplectomorphism is not too enlightening, so to provide some intuition, consider a more geometric definition. Let γ be a closed oriented piecewise smooth curve in $\mathbb{R}^2$. If γ is not embedded, decompose γ into embedded pieces. Now, consider the signed area of the disc D bounded by γ and let $A(\gamma)$ be the sum of the signed areas of the embedded pieces. With this, here is a more intuitive explanation of a symplectic structure.

Definition 1.5. The standard symplectic structure on $\mathbb{R}^{2n}$ is the map

$$A(\gamma) = \sum_{i=1}^{n} A(\gamma_i), \quad (\gamma_1, \ldots, \gamma_n) \subset \mathbb{C}^n.$$

Then a symplectomorphism of $\mathbb{R}^{2n}$ is just a diffeomorphism that preserves the signed area of closed curves. These are equivalent formulations because for an oriented smooth disc $D \subset \mathbb{R}^{2n}$ with oriented boundary $\gamma = (\gamma_1, \ldots, \gamma_n)$ wiith projection maps π_i to the ith coordinate,

$$\int_D \omega_0 = \sum_{i=1}^{n} \int_D dx_i \wedge dy_i = \sum_{i=1}^{n} \int_{\pi_i D} dx_i \wedge dy_i = \sum_{i=1}^{n} A(\gamma_i) = A(\gamma).$$

There is a bit of subtlety here, but a symplectic structure on a manifold is equivalent to an atlas where the transition functiions are local symplectomorphisms of $\mathbb{R}^{2n}$ (see Darboux's Theorem).

Now, let us introduce the objects of interest. Let $B^4(r)$ denote an open 4-ball of radius r, thought of as the standard ball centered at the origin in $\mathbb{C}^2$. Similarly we write $E(a, b)$ for the ellipsoid in $\mathbb{C}^2$ with major and minor axes of lengths $2a$ and $2b$ respectively, with standard model

$$E(a, b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi \|z_1\|^2}{a} + \frac{\pi \|z_2\|^2}{b} \leq 1 \right\}$$

Also let $P(a, b)$ denote the polydisc

$$P(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi \|z_1\|^2 \leq a,\ \pi \|z_2\|^2 \leq b\}$$

Since the standard $U(1) \times U(1)$-action on $\mathbb{C}^2$ has moment map

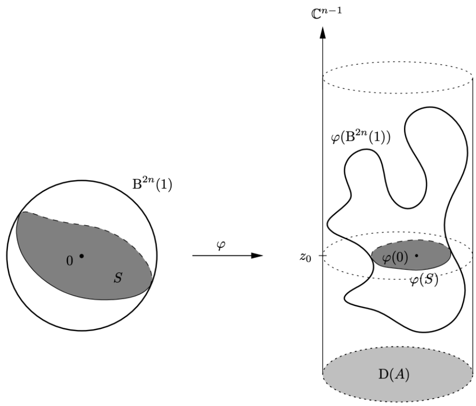
$$\mu(z_1, z_2) = (\|z_1\|, \|z_2\|) \subset \mathfrak{u}(1)^2 \cong \mathbb{R}^2,$$

these regions have moment map images a right triangle with side lengths a, b with right angle at the origin, and a rectangle with side lengths a, b with right angle aligned with the origin, respectively. We will often treat the moment map images, called *moment polytopes*, and their preimages in $\mathbb{C}^2$ interchangeably. This is justified by observing that in the region $\{(x, y) \mid x, y > 0\}$, μ is a fiber bundle, and thus diffeomorphic moment map images correspond to diffeomorphic preimages.

First, let us recall some facts about symplectic embeddings. The Euclidean volume of an open subset $U \subset \mathbb{R}^{2n}$ is given by $\frac{1}{n!} \int_U \omega_0^n$. Since symplectic embeddings preserve ω_0, they preserve ω_0^n so the existence of a symplectic embeddiing $\varphi : U \hookrightarrow V$ means that the $\mathrm{Vol}(U) \leq \mathrm{Vol}(V)$.

Define the cylinder $Z^{2n}(A) = P(A, \infty, \ldots, \infty)$. One of the first results that differentiated symplectic embeddings from embeddings with a volume constraint was Gromov's nonsqueezing theorem:

Figure 11: Idea of the proof of nonsqueezing from [84].



Theorem 1.6 (Gromov-nonsqueezing). *There exists a symplectic embedding* $\varphi : B^{2n}(1) \to Z^{2n}(A)$ *only if* $A \geq 1$.

This means that the identity embedding is the “best" embedding in this case. In the case of Euclidean embeddings, this result would be obvious since a Euclidean embedding is the composition of rotations and translations. Gromov found used the theory of holomorphic curves to find an obstruction for the existence of such an embedding.

However, other invariants have come up for studying symplectic embeddings. We review some methods for detecting symplectic embeddings, which have basis in holomorphic curves and symplectic capacities. Fix $b > 1$. The function

$$c_b(a) := \inf\{\lambda > 0 \,|\, E(1, a) \hookrightarrow P(\lambda, \lambda b)\}$$

is a central function of study. It is continuous in a, so it suffices to compute it for $a > 1$ rational.

Definition 1.7. The **weight expansion** w(a) of such an a is the finite decreasing sequence

$$w(a) = (1^{\times \ell_0}, w_1^{\times \ell_1}, ..., w_n^{\times \ell_n})$$

where $w_1 = a - \ell_0 < 1, w_2 = 1 - \ell_1 w_1 < w_1$, and so on.

Holomorphic curves come up in the ball-packing method. Namely, there is a decomposition of any ellipsoid $E(a, b)$ into a collection of balls

$$B(a, b) := \coprod_i B(w_i)$$

where the w_i are terms in the weight sequence of (a, b).

Theorem 1.8. *Let $a, b, c, d \in \mathbb{Q}$ be positive. There exists a symplectic embedding $E(1, a) \hookrightarrow P(\lambda, \lambda b)$ if and only if there is a symplectic embedding*

$$B(a, b) \coprod B(\lambda) \coprod B(\lambda b) \hookrightarrow B(\lambda(b + 1)).$$

This reduces the polydisc problem to a ball-packing problem of embedding balls of radius e_i into a ball of radius μ:

$$\coprod_i B(e_i) \hookrightarrow B(\mu).$$

Holomorphic curves arise due to the study of blow-ups. Consider a ball $B^{2m}(V + \epsilon)$ of volume $V + \epsilon$. If this ball can be embedded into a symplectic manifold, then there is a symplectic manifold that is the union of $M \backslash B^{2m}(V + \epsilon)$ and a neighborhood of the zero-section in $\mathcal{O}_{\mathbb{P}^{m-1}}(-1)$. This neighborhood has volume ϵ and the zero-section has volume V. Letting $m = 2$, this corresponds to an embedded pseudoholomorphic curve, which is called the exceptional sphere. Let X_n denote the blow-up of $\mathbb{C}P^2$ at n points. If the associated blow-ups of $\mathbb{C}P^2$ carry symplectic forms, then the balls can be symplectically embedded into a symplectic manifold.

Let $\overline{\mathcal{C}}_K(X_n)$ the set of cohomology classes represented by symplectic forms for which the anticanonical class is $K = -3L + \sum_i E_i$, where L is Poincaré dual to a line in $\mathbb{C}P^2$ and each E_i is dual to the ith exceptional sphere. By [73], the symplectic embedding exists when the following cohomology class is in the symplectic cone:

$$\mu L - \sum_i e_i E_i \in \overline{\mathcal{C}}_K(X_n).$$

This lets us check if a cohomology class α lies in $\overline{\mathcal{C}}_K(X_n)$. If a symplectic form is in a given class, it must have non-negative intersection with certain holomorphic curves. [65] give a nicer characterization of $\overline{\mathcal{C}}_K(X_n)$. If $\mathcal{E}_K(X_n) := \{e \in H_2(X_n) \,|\, \langle e, e\rangle = -1,\ \langle K, e\rangle = -1\}$, then we may characterize the above as

$$\overline{\mathcal{C}}_K(X_n) := \{\omega \in H^2(X_n) \,|\, \langle \omega, e\rangle \geq 0\ \forall e \in \mathcal{E}_k(X_n)\}.$$

For the four-dimensional ball-packing problem, we consider the compactification of B^4 with volume V by $\mathbb{C}P^1$ into a $\mathbb{C}P^2$ with volume V. This 4-manifold has rank 1 H_2 with generator the boundary divisor $\mathbb{C}P^1$. We take the dual generator of H^2 to be the form giving $\mathbb{C}P^1$ area $\sqrt{V}$. Subsequent blow-ups of this manifold have homology bases given by this same generator along with the classes of exceptional spheres. The same idea works for any combination of ellipsoids and balls.

For embedding ellipsoids into polydiscs, compactification of $P(a, b)$ yields $S^2 \times S^2$, which is diffeomorphic to the 1-point blow-up of $\mathbb{C}P^2$. In particular, the n-fold blow-up of $S^2 \times S^2$ (denoted Y_n) can be identified with X_{n+1}.

For the problem of embedding ellipsoids into polydiscs, the more natural compactification of $P(a, b)$ adds a single point to each disc, yielding $S^2 \times S^2$. This manifold is in fact diffeomorphic to the 1-point blow-up of $\mathbb{C}P^2$, so then n-fold blow-up of $Y_n := S^2 \times S^2$ can be identified with X_{n+1}. The induced isomorphism on cohomology $\psi : H^2(Y_n) \to H^2(X_{n+1})$ is given by

$$(d, e; m_1, m_2, m_3, ..., m_n) \mapsto (d + e - m_1; d - m_1, e - m_1, m_2, ..., m_n)$$

In particular, we may express any class in H^2 with respect to this basis, so that verifying positivity of intersection with effective classes of spheres reduces to checking positivity of entries of the vector above.

This positivity condition completely characterizes the symplectic cone, so we now require a condition for determining when homology classes are represented by effective spheres. With respect to the basis of $H^2(X_n; \mathbb{R}) \simeq \mathbb{R}^{n+1}$ given above, a *Cremona transform* is the map given by

$$(d; m_1, ...m_n) \mapsto (2d-m_1-m_2-m_3; d-m_2-m_3, d-m_1-m_3, d-m_1-m_2, m_4, ..., m_n)$$

With this, we can state the required condition for a class to be in $\mathcal{E}_K(X_n)$, which is proven in [75] based on work of [64][65].

Theorem 1.9. *A class $(d; m_1, ..., m_n) \in H_2(X_n; \mathbb{Z})$ is in $\mathcal{E}_K(X_n)$ if and only if its entries satisfy the Diophantine equations*

$$3d - 1 = \sum_i m_i$$
$$d^2 + 1 = \sum_i m_i^2$$

and $(d; m_1, ..., m_n)$ reduces to $(0; -1, 0, \ldots, 0)$ after a sequence of Cremona transformations.

Obstructive Classes

Embeddings correspond to cohomology classes $(\lambda b, \lambda; w(a))$ in $H^2(Y_n; \mathbb{R})$ for some n. For the problem $E(1, a) \hookrightarrow P(\lambda, \lambda b)$. If $A := (d, e; \mathbf{m})$ represents the coefficients of a homology class $H_*(S^2 \times S^2; \mathbb{R})$ with the previous basis, then

$$\lambda = \frac{\langle \mathbf{w}(a), \mathbf{m} \rangle}{d + e} > 0$$

means that A has positive symplectic area. Rewriting this as $\lambda(d+e) - \langle \mathbf{w}(a), \mathbf{m} \rangle > 0$,, $\lambda(d + e)$ is the evaluation of A on the ruling lines of $S^2 \times S^2$, and the weight sequence comes from the equivalence of embedding a sequence of balls of decreasing radius into a larger ball [75]. Let w_i denote terms of the weight expansion of a. Then

$$\coprod_i B(w_i) \hookrightarrow B(d + e)$$

symplectically if and only if

$$(d+e)L - \sum_i w_i E_i$$

is in the symplectic cone of the blowup of $\mathbb{CP}^2$. The number a is independent of homology class A.

Theorem 1.10. *An embedding exists iff* $\lambda \geq \sqrt{\frac{a}{2b}}$ *and*

$$\mu_b(d,e;m) := \frac{\sum_i m_i \cdot w_i(a)}{d+be} \leq \lambda$$

for every $(d,e;m_1,...,m_n) \in H_2(Y_n;\mathbb{Z})$ *which satisfies equations (CITE) and reduces to* $(0;-1)$ *after some sequence of Cremona transformations.*

We say that a class A is **obstructive at** $a > 0$ if the infimum

$$c_b(a) = \inf\{\lambda > 0 \,|\, E(1,a) \hookrightarrow P(\lambda, \lambda b)\}$$

is larger than the volume constraint $\sqrt{\frac{a}{2b}}$.

Reduction at a Point

There may be too many obstructive classes to check, even though we get a necessary and sufficient condition for a symplectic embedding to exist. The following method checks existence of an embedding at a single value a.

Definition 1.11. The **defect** δ of an ordered vector $(d;m_1,...,m_n)$ is the sum $d - m_1 - m_2 - m_3$.

The following is established in [75].

Theorem 1.12. *An embedding* $E(1,a) \hookrightarrow P(\lambda,\lambda b)$ *exists if there exists a finite sequence of Cremona moves that transforms the ordered vector*

$$((b+1)\lambda; b\lambda, \lambda, W(1,a))$$

to an ordered vector with non-negative entries and defect $\delta \geq 0$.

ECH Capacities

Capacities are also used in studying embeddings. We use ECH (embedded contact homology) capacities.

Definition 1.13. A **capacity** is a function which assigns to a symplectic manifold a sequence of real numbers

$$0 = c_0(X,\omega) < c_1(X,\omega) \leq c_2(X,\omega) \leq \cdots \leq \infty$$

such that if $(X,\omega) \hookrightarrow (N,\eta)$ symplectically, then $c_k(X,\omega) \leq c_k(N,\eta)$ for all k, and the inequality is strict if $c_k(X,\omega)$ is finite.

Embedded contact homology, which is the homology of a chain complex generated by Reeb orbits in a contact 3-manifold, is used to define these capacities for bounded star-shaped domains in $\mathbb{R}^4$, compact exact symplectic 4-manifolds with boundary with a contact form λ on the boundary such that $d\lambda = \omega|_{\partial X}$. The capacities tell us how much symplectic action is needed to represent a given class. We get a flitration on the generators of ECH given by its action functional. There is a subcomplex of elements of action less than some L. Denote by c_k the smallest L such that the image of the L-subcomplex is k-dimensional. At the kth step, the action is large enough so that a class ni the full complex can be represented by a class in the L-subcomplex. More details can be found in [58].

Let $(a, b)_k$ denote the kth smallest entry in the matrix of real numbers $(am + bn)_{m,n\in\mathbb{N}}$, counted with repetitions so that $c_k = c_{k+1}$ when they repeat. The convention here is that $\mathbb{N}$ includes 0.

Theorem 1.14. *The ECH capacities of an ellipsoid are given by* $c_k(E(a, b)) = (a, b)_{k+1}$*, and the ECH capacities of a polydisc are given by* $c_k(P(a, b)) = \min\{am+bn \,|\, m, n \in \mathbb{N},\ (m + 1)(n + 1) \geq k + 1\}$.

In fact, it is a result of McDuff that these ECH capacities are sharp for ellipsoids; the interior of $E(a, b)$ embeds symplectically in $E(c, d)$ if and only if $(a, b)_k \leq (c, d)_k$ for every k. That is, ECH capacities form a complete list of obstructions, in the sense that if the capacities satisfy the inequality $c_k(E(1, a)) \leq c_k(E(1, 2b))$ for all k then $E(1, a) \hookrightarrow E(1, 2b)$.

Consequently, since the problems $E(1, a) \hookrightarrow P(\lambda, \lambda b)$ and $E(1, a) \hookrightarrow E(\lambda, \lambda 2b)$ are equivalent, checking ECH capacities suffices to guarantee the existence of an embedding. This allows us to reformulate $c_b(a)$ as

$$c_b(a) = \sup_{k\geq 1} \left\{ \frac{c_k(E(1, a))}{c_k(E(1, 2b))} \right\}.$$

We end this introduction with a simple proof of Gromov nonsqueezing when $n = 1$ using the above methods:

Proof. Let $b >> 0$. By Theorem 1.12, the Gromov nonsqueezing theorem can be translated to the problem of finding a sequence of Cremona moves that transforms the ordered vector $((b + 1)M; Mb, M, W(1, 1))$ to an ordered vector with non-negative entries and defect $\delta \geq 0$. Applying a Cremona transformation yields the vector $(2(b+1)M - Mb - M - 1, Mb - 1, M - 1, 0)$ so $M \geq 1$ for this all entries to be positive. The defect is $\delta = Mb - 1$, which is nonnegative for $b > \frac{1}{M}$. □

1.2 The Evasion Path Problem

In the minimal sensor network problem, local information is given to try to solve a global problem. One aspect of this problem is coverage. Consider the following setting: ball-shaped sensors move randomly in a bounded domain. A sensor does not know its location, but it does know when it overlaps with a nearby sensor. The evasion path problem can be described informally as a problem in which there is a bounded domain with multiple sensors searching for intruders, and an intruder moves continuously. For example, imagine a mall with security guards searching for a thief. If the thief remains undetected by the security guards, then an evasion path exists (where in our setting, the security guards have 360 degree vision). The sensors and intruders move continuously in a random manner. We only know the time-varying connectivity data. Topology is a tool that can be used to study this problem.

In this section, we introduce the evasion path problem as a coverage problem as well as some results in the field. To set up the problem, we need a notion of a fibrewise space as the covered and uncovered regions change with time.

Definition 1.15. A **fiberwise space** is a topological space X equipped with a continuous map $p : X \to [0, 1]$. We will write this pair as (X, p).

Thus, up to renormalization of time, for $x \in X$, x represents the snapshot of the space at time $p(x)$. A fibrewise map between two fibrewise spaces (X_1, p_1) and (X_2, p_2) is a continuous map $f : X_1 \to X_2$ such that $p_2 \circ f = p_1$.

Definition 1.16. A **section** for a fibrewise space (X, p) is a continuous map $s : [0, 1] \to X$ such that $p(s(t)) = t$ for all $t \in [0, 1]$.

An evasion path is just a section. Let $D \subset \mathbb{R}^n$ be a bounded domain homeomorphic to a n-dimensional ball for $n \geq 2$. Suppose that a finite number of sensors $\{v_i\}$ move inside D over time $[0, 1]$ in a continuous manner such that they never are in the same location. In particular, the sensors move along a path $\gamma_i : [0, 1] \to D$. Let $B_{v_i(t)}$ be the unit ball covered by the sensor v_i. The connectivity graph at time t has the set of sensors as a vertex set and edge if $B_{v_i(t)} \cap B_{v_j(t)} \neq \varnothing$. The assumption is that the boundary ∂D is contained within the union of the balls of immovable sensors that is homotopy equivalent to ∂D. The union of the balls is the covered region

$$X(t) = \bigcup_{v_i} B_{v_i(t)}.$$

The complement $X(t)^c$ is the uncovered region at time t. Let $X = \cup_{t \in [0,1]} X(t) \times \{t\} \subset D \times I$ be the covered region.

Definition 1.17. An **evasion path** in a sensor network is a section of $p : X^c \to [0, 1]$.

In particular, an evasion path informally exists if an intruder can avoid being detected by the sensors over the interval $[0, 1]$. Connectivity graphs alone cannot detect when an evasion path exist, so we must consider higher connectivity information.

Definition 1.18. The Čech simplicial complex $C_r(X)$ of a finite set X at parameter r is the simplicial complex formed by taking elements of X as the vertex set of $C(X)$, and for each $\sigma \subset X$, let $\sigma \in C_r(X)$ if the set of r-balls centered at points of σ has a nonempty intersection.

Now, we state the evasion path problem:

Evasion Path Problem. Given the time-varying Čech complex $C(t)$ of a sensor network at all times $t \in [0,1]$, can one determine if an evasion path exists?

To review some results in this direction, we define the stacked Čech complex $SC(X)$. It is a cell complex that compiles the information of the Čech simplical complexes for all $t \in [0,1]$. Assume that there are a finite number of times $t_1, \ldots, t_n$ when the (homotopy type of the) Čech complex changes. We assume that simplices are only added or removed at each time $t_i \in [0,1]$, and not both. Choose interleaving times $0 = s_0 < t_1 < s_1 < \ldots < t_n < s_n = 1$.

Definition 1.19. The stacked Čech complex $(SC(X), p)$ is the fibrewise space given by the disjoint union

$$\coprod_{i=0}^{n} C(s_i) \times [t_i, t_{i+1}]$$

where $t_0 = 0$ and $t_{n+1} = 1$ where the identifications $C(s_{i-1}) \times \{t_i\}$ are identified as a subset of $C(s_i) \times \{t_i\}$ if simplicies are added at t_i, and $C(s_i) \times \{t_i\}$ is identified as a subset of $C(s_{i-1}) \times \{t_i\}$ is simplicies are removed at t_i. The map p is projection onto the second coordinate.

De Silva and Ghrist [38] use a slightly different notion of a stacked Vietoris-Rips complex, but the result can be stated as follows. Let $F \times [0,1]$ be the subcomplex of the $SC(X)$ consisting of the sensors covering the boundary of X.

Theorem 1.20 (de Silva and Ghrist)**.** *If there exists $[\alpha] \in H_n(SC(X), F \times [0,1])$ such that $[\partial\alpha] \neq 0$, then there is no evasion path.*

However, the condition that every $[\alpha] \in H_n(SC(X), F \times [0,1])$ has homologically trivial boundary is not sufficient for the existence of an evasion path. Henry Adams and Gunnar Carlsson use zigzag persistence to study this problem [10]]. We will not go into details here, but the result they come up with is the following.

Theorem 1.21 (Adams and Carlsson)**.** *If there is an evasion path, then there is a full-length interval in the zigzag barcode for the codimension* 1 *homology of the stacked zigzag Čech complex.*

However, the converse is false, so more sophisticated tools are needed. There is work done by Ghrist and Krishnan [51] defining a new type of cohomology. In our work, we use functor calculus to create an obstruction toward the existence of a section.

A brief overview of Goodwillie Calculus

More detailed exposition can be found in for example [53][54][55][79][14] among many others. Let $\mathcal{C}$ be the category of spaces, or the category of spaces over a fixed space and let $\mathcal{D}$ be the category of spaces or spectra. We will concern ourselves with homotopy functors between $\mathcal{C}$ and $\mathcal{D}$, which are functors that preserve weak equivalences. We say a map of functors $F \to G$ is a weak equivalence if for every $X \in \mathcal{C}$, $F(X) \to G(X)$ is a weak equivalence. The homotopy functor F is **excisive** if it takes homotopy pushout (cocartesian) squares to homotopy pullback (cartesian) squares. A cubical diagram is called **strongly homotopy cartesian** if all its two-dimensional faces are homotopy pushouts. It is called **homotopy cartesian** if the initial object is weakly equivalent to the homotopy limit of all the others.

Definition 1.22. The homotopy functor F is n**-excisive** if it takes strongly homotopy cartesian $(n+1)$ cubical diagrams to homotopy cartesian diagrams.

We highlight some analogies with standard calculus. Given a homotopy functor F, we can obtain functors P_nF which can be thought of as the nth Taylor polynomials of F. The functors $\{P_nF\}_{n\geq 0}$ fit together to form a tower, and F maps into the limit of the tower. The nth term in the series is analogous to $D_nF := \mathrm{hofib}(P_nF \to P_{n-1}F)$. The constant term $(P_0F)(X)$ is weakly equivalent to the constant functor $F(*)$. The nth term in the series $(D_nF)(X)$ has the form $(C_n \wedge (X^{\wedge n}))_{h\Sigma_n}$, where C_n is a spectrum with a Σ_n action, Σ_n permutes the n copies of $X^{\wedge n}$, and $X_{h\Sigma_n} = E\Sigma_n \wedge_{\Sigma_n} X$ is the homotopy orbit. The spectrum C_n is called the n**th derivative of** F at the one-point space.

We will construct the n-excisive approximation of P_nF by the infinite iteration of another construction T_n.

Definition 1.23. Let X be a space over Y and let U be a space. The *fiberwise join* is the space

$$X *_Y U = \mathrm{hocolim}(X \leftarrow X \times U \to Y \times U).$$

Usually, U is a discrete set. For example, if U has one element, then $X *_Y U \to Y$ is the fiberwise cone of X over Y. Now, consider the poset $\mathcal{P}(\underline{n+1})$ of subsets of $\underline{n+1} = \{1, \ldots, n+1\}$. Let $X \in \mathcal{C}$. Then the map $U \mapsto X *_Y U$ for $U \in \mathcal{P}(\underline{n+1})$ gives an $(n+1)$ cubical diagram in $\mathcal{C}$. Now, consider the composition $U \mapsto F(X *_Y U) : \mathcal{P}(\underline{n+1}) \to \mathcal{D}$. Let $(T_nF)(X)$ denote the homotopy limit of its restriction to $\mathcal{P}_0(\underline{n+1})$, the poset of nonempty subsets of $\underline{n+1}$. This yields a functor $T_nF : \mathcal{C} \to \mathcal{D}$. We also have a natural map $t_n : F \to T_nF$ given by the map from the initial object to the homotopy limit of the others:

$$F(X) = F(X *_Y (\varnothing)) \xrightarrow{t_nF} \mathrm{holim}_{U\in\mathcal{P}_0(n+1)} F(X *_Y U) = (T_nF)(X).$$

Now, let $(P_nF)(X)$ be the sequential homotopy colimit of the diagram

$$F(X) \xrightarrow{(t_nF)(X)} (T_nF)(X) \xrightarrow{(t_nT_nF)(X)} T_n^2F(X) \xrightarrow{(t_nT_n^2F)(X)} \ldots$$

Theorem 1.24. *Let $F : \mathcal{C} \to \mathcal{D}$ be any homotopy functor. The functor P_nF is n-excisive and in the homotopy category $h\mathcal{F}(\mathcal{C}, \mathcal{D})$, p_nF is the universal map from F to an n-excisive functor.*

Example 1.25. We look at $P_1(\mathrm{Id})$ where $\mathrm{Id} : \mathtt{Top}_* \to \mathtt{Top}_*$ is the identity functor from based spaces to based spaces. Then we have the sequence

$$X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \dots$$

with $P_1(X) \simeq \mathrm{hocolim}(\Omega^n\Sigma^n(X)) =: Q(X)$, the stabilization of X. Then the first derivative of the identity functor is $C_1 = \underline{S}$, the sphere spectrum.

Note that the identity functor is typically not 1-excisive, as pushouts are almost never equivalent to pullbacks. In general, describing P_nF is very difficult, even for the identity functor. The second Taylor polynomial of the identity is the homotopy fiber of the James-Hopf map:

$$P_2(I)(X) := \mathrm{hofib}(QX \to Q(X^{\wedge 2}))_{h\Sigma_2}.$$

We classically define the James-Hopf map. To do this, we first define the James splitting. Let X be a connected pointed space. Consider the following diagram:

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r] \arrow[d] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \end{tikzcd}$$

Note that $\mathrm{Cone}(X \times \Omega\Sigma X \to \Omega\Sigma X) \simeq \Sigma\Omega\Sigma X$. Now, we use this fact to deduce the James splitting. Let $X_+ = X \coprod *$, the disjoint union of X and a basepoint. The mapping cone of $X_+ \to S^0$ is ΣX. Now, consider the cofiber sequence

$$X_+ \wedge Y_+ \to Y_+ \to \Sigma X \wedge Y_+.$$

This yields the cofiber sequence

$$(X \times Y)_+ \to Y_+ \to \Sigma X \wedge Y_+.$$

So, the cone of $X \times \Omega\Sigma X \to \Omega\Sigma X$ is equivalent to $\Sigma X \wedge (\Omega\Sigma X)_+$. This yields

$$S^1 \wedge X \wedge (\Omega\Sigma X)_+ \simeq \Omega\Sigma X.$$

In particular, the above gives us $\Sigma X_+ \wedge \Omega\Sigma X \simeq X_+ \wedge \Sigma\Omega\Sigma X$. In particular, note that $\Sigma X \vee (\Sigma X \wedge \Omega\Sigma X) \simeq \Sigma X_+ \wedge \Omega\Sigma X \simeq \Sigma\Omega\Sigma X$ and we iterate the substitution into the wedge sum and group the terms, yielding the James splitting.

Corollary 1.26 (James splitting)**.** $\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{n=1}^{\infty} X^{\wedge n}$.

Remark. In full generality, the statement is that for each integer $n \geq 1$, there is a natural equivalence

$$\Sigma\Omega\Sigma X \simeq \left(\bigvee_{1 \leq i \leq n} \Sigma X^{\wedge i} \right) \vee (X^{\wedge n} \wedge \Sigma\Omega\Sigma X).$$

If X is connected, then the infinite splitting holds since $(X^{\wedge n} \wedge \Sigma\Omega\Sigma X)$ vanishes as $n \to \infty$.

Definition 1.27. Consider the splitting

$$\Sigma\Omega\Sigma X \to \Sigma \bigvee_{n \geq 1} X^{\wedge n} \to \Sigma X^{\wedge n}$$

where the second map is the projection to the nth graded object. The adjoint map $\Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$ is called the **James-Hopf map**.

The constructions above yield an example in Goodwillie calculus.

Example 1.28. The James construction is defined as

$$JX := \coprod_{n \geq 0} X^n / \sim$$

where $(x_1, \ldots, x_n) = (x_1, \ldots, \hat{x}_i, \ldots, x_n)$ with the ith coordinate removed if $x_i = *$ is the basepoint. This is the free associative monoid on X with basepoint as a unit. Let $J_n X := \coprod_{i \leq n} X^n / \sim$. There is a filtration

$$* \subset X \subset J_2 X \subset \ldots \subset JX$$

The graded quotients of JX are given by $X^{\wedge n}$.

Let $\mathcal{O}^n$ be the little n-cubes operad. We can define

$$C_n(X) := \left(\coprod_{j \geq 0} \mathcal{O}^n(j) \times_{\Sigma_j} X^j \right) / \simeq$$

where $\simeq$ is the relation that makes $*$ the identity element. There is a filtration

$$* \subset C_n^1(X) \subset C_n^2(X) \subset \ldots$$

where $C_n^k(X)$ denotes the image of the first $k+1$ parts of the disjoint union in $C_n(X)$. The graded quotients are given by

$$\mathrm{gr}_j C_n(X) = \left(\mathcal{O}^n(j) \times_{\Sigma_j} X^j \right) / (\mathcal{O}^n(j) \times_{\Sigma_j} *)$$

for $j \geq 0$.

Theorem 1.29 (Snaith)**.** *There is an equivalence* $\Sigma^\infty \Omega^n \Sigma^n X = \bigvee_{j \geq 1} \Sigma^\infty gr_j(C_n(X))$.

Now, consider the functor $F(X) = \Sigma^\infty \Omega \Sigma X$. We have $(D_n F)(X) \simeq \Sigma^\infty(X^{\wedge n})$ and the tower splits as a product $P_n F \simeq \prod_{1 \leq k \leq n} D_k F$. The limit of the tower is the product of all the layers and if X is connected, then the tower coverges to $F(X)$ in the following sense: the map $F(X)$ to the homotopy limit of the tower is a weak equivalence.

We studied an *unbased* version of the James-Hopf map to attack the evasion path problem.

1.3 Mayer-Vietoris

A general approach to study problems in both mathematics and data science is to break down a complicated problem into a simpler problem. In algebraic topology, one of the tools used is the Mayer-Vietoris sequence. Let X be a topological space and let $A, B \subset X$ such that X is the union of the interiors of A and B. Then there exists an exact sequence relating the homologies of the components A and B of X and X

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow$$

$$\longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots \qquad \cdots \longrightarrow H_0(X) \longrightarrow 0$$

In topological data analysis, this sequence does not work in general. So, we might ask the question: what information (if any) can we deduce about a space from a decomposition of the space into simpler pieces. We define a couple of notions that we will use to study this idea.

In data science, we can think of data as discrete points in Euclidean space. If these data points are for example sampled from a submanifold as a point cloud, we might ask how to reconstruct the submanifold (or an approximation of the submanifold) from these points. Given a subset $X \subset \mathbb{R}^n$, we create a simplicial complex that we use to give the data some form of shape.

Definition 1.30. Let $r > 0$. The **Vietoris-Rips complex** at scale r on X, denoted $VR_r(X)$ is the simplicial complex whose simplices are finite collections of points in X of pairwise distance less than or equal to r.

Note that this is slightly different than the Čech simplicial complex $C_r(X)$ since we only require nonempty pair-wise intersections.

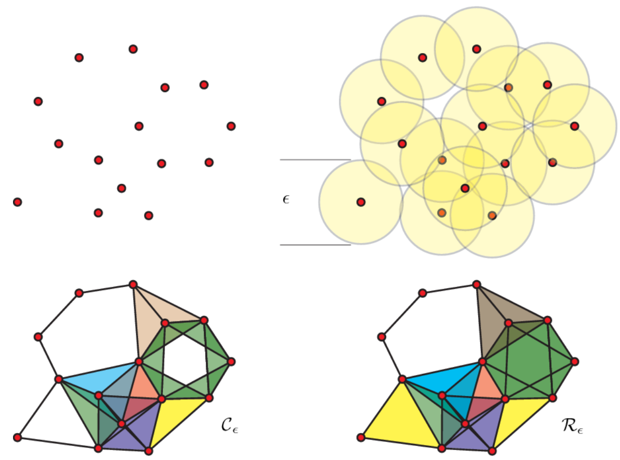


Figure 12: Example of Vietoris-Rips and Čech complexes, from [50].

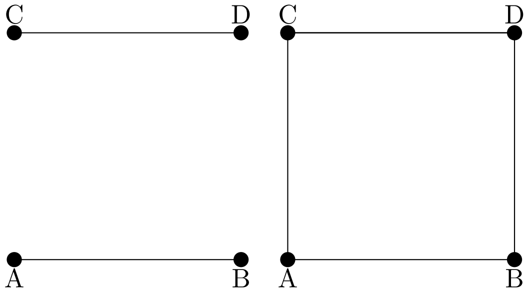


Figure 13: Decompositions are not well-behaved homologically.

Given these complex constructions, we can try studying the homologies that arise from these complexes. Consider the above figure consisting the set of four points $S = \{A, B, C, D\}$ such that $d(A, B) = d(A, C) = d(C, D) = d(D, B) = 1$ and $d(A, D) = d(B, C) > 1$. Now, consider the partition of S into the sets $S_0 = \{A, B\}$ and $S_1 = \{C, D\}$. We may form $VR_1(S_0)$ and $VR_1(S_1)$ to get the line segments in the left-hand side of the figure. Now, consider $VR_1(S)$, which is the square on the right-hand side. Immediately, we see that $H_1(VR_1S) = \mathbb{Z}$, but $H_1(VR_1(S_0)) = H_1(VR_1(S_1)) = 0$ so it's hard to recover information of the homology of the space $VR_1(S)$ from its components $VR_1(S_0)$ and $VR_1(S_1)$. Studying when such a sequence arises is the subject of one of the papers.

We introduce an idea that is used to study this decomposition.

Definition 1.31. Let $f : I \to J$ be a functor between two small categories I and

J. Let $j \uparrow f$ denote the category whose objects are pairs $(i, \alpha : j \to f(i))$ consisting of an object $i \in I$ and a morphism $\alpha : j \to f(i)$ in J.

Let $N(f)$ denote the nerve of $f : I \to J$. Let $f, g : I \to J$ be functors. If the functors $N(f), N(g) : N(I) \to N(J)$ are homotopic, then f and g are homotopic. Furthermore, if there is a natural transformation between f and g, then f and g are homotopic. If I has a terminal object t, then there is a unique natural transformation from the identity functor $\mathrm{id} : I \to I$ to the constant functor $t : I \to I$ with value t. Then the identity functor is homotopic to the constant functor and thus I is contractible. Similarly, a category with an initial object is contractible. This will be a key argument in one of the papers.

1.4 Realizations

In topological data analysis, the poset $[0, \infty)$ is often used as the indexing poset. For example, the homology of the sublevel sets of a function $X \to [0, \infty)$ forms the persistent homology functor, indexed by the nonnegative real numbers. One question that may be asked is: how does one generate examples of posets that might be useful in the context of persistence? We introduce some notions central to posets which will be useful in understanding the corresponding paper. The paper has more detail about computing minimal resolutions, and some more categorical aspects that are present in such a construction.

One example of a discrete poset is the suspension poset. Let S be a finite set. Enlarge S with two elements $\wedge S$ and $\vee S$ to form a disjoint union $S \coprod \{\wedge S, \vee S\}$. Consider the following relation on the enlarged set: elements in S are incomparable and $\wedge S \leq s \leq \vee S$ for all $s \in S$. This poset is an example of a *non-distributive* upper semilattice if $|S| > 2$.

To turn discrete posets into continuous ones, we introduce the idea of a realization. The details behind the construction are in section four of the paper, but the intuition guiding this realization is the one underlying the fact that $\mathbb{Z}$ is "almost" $\mathbb{R}$ in that if we think of an embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$, then adjoining all the points in the image of the embedding between two integers n and $n+1$ for all $n \in \mathbb{Z}$ and gluing everything together yields $\mathbb{R}$. In the figure, the realization of the poset consisting

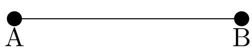


Figure 14: Realization of the poset $\{A, B\}$.

of the points A and B is the line segment with A and B as the vertices of the line segment. Similarly, if we have four vertices of a square, the realization of those four points will be the filled in square with four points on the boundary. Formally, the realization of a poset is built from posets of the form $(-1, 0)^s$, $s \in \mathbb{N}$ by examining every element $a \in I$ and every subset S of parents of a with a common ancestor by taking the disjoint union of all the posets $(-1,0)^{|S|}$. Defining the poset relation on the realization is very nontrivial.

We will also give the definition of left Kan extension, which will play a central role for the study of tame functors.

Definition 1.32. Let $f : I \to J$ be a functor of posets and let C be a category. A functor $F : J \to C$ is called a *left Kan extension* of $G : I \to C$ along f if there is a natural transformation $\alpha : G \to Ff$ satisfying the property that for any other pair $(K : J \to C, \gamma : G \to Kf)$, γ factors uniquely through α.

Definition 1.33. A functor $F : J \to C$ is *tame* if there exists a finite poset I and functors $G : I \to C$ and $f : I \to J$ for which F is isomorphic to the left Kan extension of G along f.

More terminology will be introduced in the paper to study the homological parts of tame functors.

2 Summary of Results

Paper A

In Paper A, we study homotopical decompositions of simplicial and Vietoris-Rips complexes. We define some terms necessary for explaining the main theorem of the paper.

Definition 2.1. A collection C of simpliicial sets is closed if it contains a nonempty simplicial set and is closed under weak equivalences and homotopy colimits indexed by arbitrary small contractible categories.

Two examples of closed simplicial sets are: contractible simplicial sets and n-connected simplicial sets.

Let K be a simplicial complex and let A be a subset in the universe. For a simplex σ in K, we define the obstruction complex as follows:

Definition 2.2. The *obstruction complex* is defined as follows:

$$\mathrm{St}(\sigma, A) := \{\mu \subset A | 0 < |\mu| \text{ and } \mu \cup \sigma \in K\}.$$

Definition 2.3. Let P be the subposet of K given by

$$P := \{\sigma \in K | \sigma \subset X \text{ or } \sigma \subset Y \text{ or } \sigma \cap A \neq \varnothing\}$$

Note that $f : K_X \cup K_Y \hookrightarrow P$ is a weak equivalence (Proposition 7.3).

The main result of the paper is this theorem:

Theorem 2.4. *Let C be a closed collection of simplicial sets. Assume that for every σ in $K \backslash P$, the obstruction complex St(σ, A) also satisfies C. Then the homotopy fibers of $K_X \cup K_Y \subset K$ also satisfy C.*

One important corollary is this:

Corollary 2.5. *If for every σ in $K \backslash P$, the simplicial complex St(σ, A) is contractible, then $K_X \cup K_Y \subset K$ is a weak equivalence.*

This is a form of Mayer-Vietoris in the context of topological data analysis.

Contributions

This paper was developed through regular discussions and my contributions can be found in all aspects of the work. However, I focused more on the categorical proof that the categories involved are contractible and thought more about generalizations.

Paper B

In Paper B, we study realizations of posets. We first begin with two definitions of dimension for posets.

Let

$$\Phi_I(x) := \left\{ U \mid \begin{smallmatrix} U\subset I \text{ is finite, has a proper ancestor, } x \text{ is a sup of it, and} \\ x \text{ is not a sup of any } S \text{ such that } \emptyset \neq S \subsetneq U \end{smallmatrix} \right\}.$$

The following extended (containing ∞) number is called the **dimension** of x:

$$\dim_I(x) := \sup\{|U| \mid U \in \Phi_I(x)\}.$$

Let

$$\Psi_I(x) := \left\{ U \mid \begin{smallmatrix} U\subset(I<x) \text{ is finite, has an ancestor, and} \\ x \text{ is a sup of every two element subset of } U \end{smallmatrix} \right\}.$$

The following extended number is called the **parental dimension** of x:

$$\text{par-dim}_I(x) := \sup\{|U| \mid U \in \Psi_I(x)\}.$$

We also define realization in the following manner. For an element a in I, define $\mathcal{R}_a(I)$ to be the subposet of $\mathcal{G}(I)$ consisting of all the pairs (a, f) satisfying the following conditions:

- $f(x) > -1$, for every x in $\mathcal{P}(a)$;
- $\operatorname{supp}(f)$ is finite;
- $\operatorname{supp}(f)$ has an ancestor in I.

The subposet $\mathcal{R}(I) := \bigcup_{a\in I} \mathcal{R}_a(I) \subset \mathcal{G}(I)$ is called the **realisation** of I. We define a poset relation on the realization.

We study properties of the realization, finding that the realization of a distributive upper semilattice of finite type is also a distributive upper semilattice (Theorem 6.1).

We also put a model structure on $Tame(J, C)$ - the full subcategory of $Fun(J, C)$ whose objects are tame functors.

Let J be an upper semilattice and $\mathcal{M}$ a model category. A natural transformation $\varphi\colon F \to G$ in $\mathrm{Tame}(J, \mathcal{M})$ is called:

- a weak equivalence (resp. fibration) if, for all a in J, the morphism $\varphi_a\colon F(a) \to G(a)$ is a weak equivalence (resp. fibration) in $\mathcal{M}$;

- a cofibration if there is a finite subposet inclusion $f\colon I \subset J$ discretising F and G, and for which $\varphi^f\colon Ff \to Gf$ is a cofibration in $\mathrm{Fun}(I, \mathcal{M})$.

Fibrations are defined objectwise. This is an example of a model structure on $Tame(J, C)$.

Finally, we find a relationship between the Betti diagrams of a functor with the homology of the Koszul complexes of a functor.

Let k be a natural number. Define:

$$(\mathcal{K}_a G)_k := \begin{cases} G(a) & \text{if } k = 0 \\ \displaystyle\bigoplus_{\substack{S \subset \mathcal{P}(a),\ |S|=k \\ S \text{ has an ancestor}}} \mathrm{colim}_{\cap_{s\in S}(I \leq s)} G & \text{if } k > 0 \end{cases}$$

Thus, there is a relationship between homological dimension and the proposed parental dimension.

Theorem 2.6. *Let I be a finite poset and $G\colon I \to vect_K$ a functor.*

1. *For every a in I and $i = 0, 1, 2$, the vector spaces $(\beta^i G)_a$ and $H_i(\mathcal{K}_a G)$ are isomorphic.*

2. *Assume an element a in I has the following property: every subset $S \subset \mathcal{P}(a)$ which has an ancestor has the product $\bigwedge S$ in I. Then, for every i, the vector spaces $(\beta^i G)_a$ and $H_i(\mathcal{K}_a G)$ are isomorphic.*

Contributions This paper was developed through regular discussions and my contributions can be found in all aspects of the work. However, I focused on the model category and part of the paper surrounding the construction of realizations.

Paper C

In Paper C, we begin studying Goodwillie calculus in the unpointed setting. Namely, we take $\mathrm{hofib}(QX_+ \to QS^0)$. In particular, we study the unbased James-Hopf map. So, the first thing we need to do is construct a model of $Q(X \wedge X)_{h\Sigma_2}$ in the unpointed setting.

Proposition 2.7. *For every choice of a basepoint in X, the fiber of the following cube*

$$\begin{tikzcd}
& Q_{\Sigma_2}(X \times X_+)^{\Sigma_2} \arrow[rr, "p"] \arrow[dl, "\rho"] \arrow[dd] & & QX_+ \arrow[dl] \arrow[dd, "c"] \\
QX_+ \arrow[rr, crossing over] \arrow[dd, "c"'] & & * & \\
& Q_{\Sigma_2}(S^0)^{\Sigma_2} \arrow[rr] \arrow[dl] & & QS^0 \arrow[dl] \\
QS^0 \arrow[rr] & & * \arrow[from=uu, crossing over] &
\end{tikzcd}$$

is naturally equivalent to $Q(X \wedge X)_{h\Sigma_2}$. Here the fiber is taken with respect to the standard basepoint, defined by the zero element of all the relevant spaces.

To construct a model of the James-Hopf map, we construct two maps we call `sq` and `id` that map from QX_+ to $Q_{\Sigma_2}(X \times X_+)^{\Sigma_2}$ and show their difference is null-homotopic and thus lifts to the fiber. We also find a combinatorial formula that allows us to study the evasion path problem.

Corollary 2.8. *Let $\pi = \pi_0(X)$, for some unpointed space X. At the level of π_0, the unbased James-Hopf map induces a function between the following groups*

$$\mathbb{Z}[X] \to \mathbb{Z}[(X \times X)_{\Sigma_2}].$$

It is given by the formula

$$\sum_i n_i x_i \mapsto \sum_i \binom{n_i}{2}[x_i, x_i] + \sum_{i<j} n_i n_j [x_i, x_j]$$

with $n_i, n_j \in \mathbb{Z}$.

Applying this to the evasion path problem, we find an obstruction to having a section even if there is a full-length bar.

Contributions This paper was developed through regular discussions and my contributions can be found in all aspects of the work. However, I focused on the combinatorial version of the James-Hopf map and the examples, and some proofs regarding the cubes involved.

Paper D

In Paper D, we study the embedding problem of symplectically embedding four-dimensional ellipsoids into polydiscs. We apply techniques stemming from obstructive classes, reduction at a point, and ECH capacities to reduce the embedding problem to a combinatorial ball-packing problem. We study the problem $E(1,a) \hookrightarrow P(\lambda, \lambda b)$, and examine when the function $c_b(a) := \inf\{\lambda > 0 | E(1,a) \hookrightarrow P(\lambda, \lambda b)\}$ is reduced to the volume constraint. That is, there is no obstruction other than volume for symplectically embedding an ellipsoid into a polydisc. This value, called the 'rigid-flexible' value, is given by the following theorem:

Theorem 2.9. *For all real $b \geq 2$, the rigid-flexible value of $c_b(a)$ is given by*

$$Rf(b) = 2b\left(\frac{2\lfloor b\rfloor + 2\lceil\sqrt{2b} + \{b\}\rceil - 1}{b + \lfloor b\rfloor + \lceil\sqrt{2b} + \{b\}\rceil - 1}\right)^2.$$

In particular, symplectic embeddings are equivalent to a volume constraint problem at this value.

Contributions This paper was developed through regular discussions and my contributions can be found in all aspects of the work. However, I focused on some Cremona computations.

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Part I

Scientific papers