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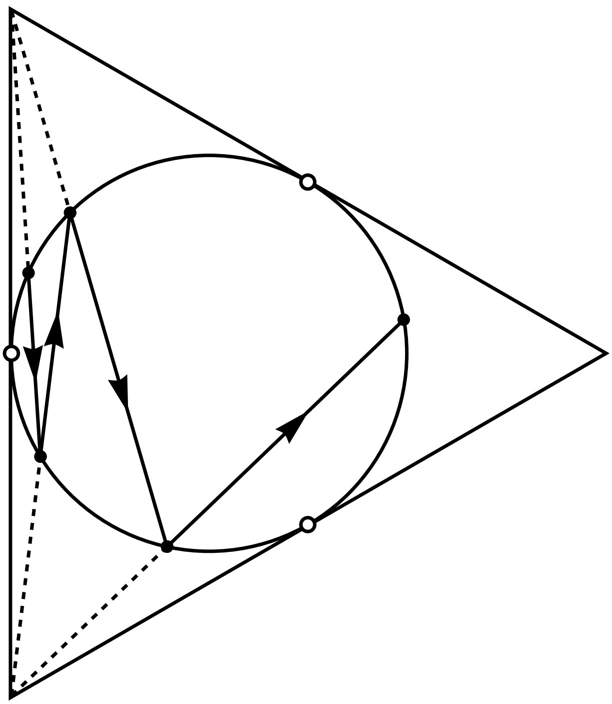
Doctoral Thesis in Mathematics

Quiescent regimes in cosmology

HANS OUDE GROENIGER



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Abstract

This thesis is about cosmological solutions to Einstein's equations of general relativity, in particular spacetimes whose mean curvature diverges. Moreover, we consider anisotropic spacetimes with big bang singularities. In this setting the singularity is expected to generically be oscillatory if no matter is present. However, complementary to an oscillatory singularity is the notion of quiescence, i.e. the convergence of the eigenvalues of the expansion-normalized Weingarten map $\mathcal{K}$. This thesis contains results related to two regimes in which quiescence is expected to occur, namely the presence of certain geometrical features or the satisfaction of an algebraic condition on the eigenvalues of $\mathcal{K}$.

Paper A is concerned with Bianchi type VI_0 spacetimes with an orthogonal perfect fluid, and we show that generically their initial singularity is anisotropic and quiescent. The quiescence that occurs may be understood as a consequence of the Abelian subgroup G_2 of the isometry group acting orthogonally-transitively. These results are then used to obtain asymptotics for solutions to the Klein-Gordon equation on backgrounds of this type.

Paper B is about Bianchi type $\mathrm{VI}_{-1/9}$ spacetimes with an orthogonal stiff fluid. Bianchi type $\mathrm{VI}_{-1/9}$ is known as exceptional, for the fact that the dynamics of vacuum and orthogonal perfect fluid cosmologies of this type have the same degrees of freedom as those of Bianchi type VIII or IX. This is due to the G_2 not necessarily acting orthogonally-transitively for type $\mathrm{VI}_{-1/9}$. The main result is that, generically, the initial singularity of such solutions is anisotropic and quiescent, and the eigenvalues of $\mathcal{K}$ converge to strictly positive values. Here quiescence is a result of the stiff fluid matter, which allows for the algebraic condition on the eigenvalues of $\mathcal{K}$ to be satisfied. Complementary to this generic behaviour are the spacetimes with special geometrical features, in particular those in which the G_2 does act orthogonally-transitively, and those that (asymptotically) satisfy a polarization condition. In these cases it occurs that the smallest limit of the eigenvalues of $\mathcal{K}$ is negative. This is in contrast with type VIII or IX cosmologies with an orthogonal stiff fluid, for which the eigenvalues of $\mathcal{K}$ always converge to strictly positive limits. As a secondary result we obtain a concise way to represent the dynamics.

In paper C, which is joint work with Oliver Petersen and Hans Ringström, we consider CMC initial data to the Einstein-nonlinear scalar field equations for a certain class of potentials. The main result is that if a certain bound on expansion-normalized quantities holds, if an algebraic condition on the eigenvalues of $\mathcal{K}$ is satisfied, and if the eigenvalues of $\mathcal{K}$ remain separated over the manifold, then there exists a threshold for the initial mean curvature, which, if surpassed, guarantees that the development has a quiescent big bang singularity. By this we mean past global existence of the development until the blowup of the Kretschmann scalar, and convergence of the eigenvalues of $\mathcal{K}$. We also obtain asymptotics for the eigenvalues of $\mathcal{K}$ and expansion-normalized quantities relating to the scalar field. Combining the main result with results by Ringström concerning Bianchi class A solutions leads to a proof of the future and past global non-linear stability of a large class of spatially locally homogeneous solutions.

Sammanfattning

Den här avhandlingen handlar om kosmologiska lösningar till Einsteins ekvationer i allmän relativitetsteori, särskilt om rumtider där medelkrökning divergerar. Specifikt studerar vi anisotropa rumtider med en Big Bang-singularitet. Under dessa omständigheter förväntar man sig i allmänhet en oscillerande singularitet i frånvaro av materia. Men, som komplement till oscillerande singulariteter finns begreppet ”quiescence”, det vill säga, konvergens av egenvärdena av den expansionsnormaliserade Weingartenavbildningen $\mathcal{K}$. Denna avhandling innehåller resultat beträffande två regimer där quiescence sker, nämligen då vissa geometriska villkor är uppfyllda, eller då ett algebraiskt villkor på egenvärdena av $\mathcal{K}$ är uppfyllt.

Artikel A behandlar Bianchi typ VI_0-rumtider med en ortogonal ideal vätska, och vi visar att deras ursprungssingulariteter i allmänhet är anisotropa och quiescenta. Den quiescence som sker kan förstås som en konsekvens av att den Abelska undergruppen G_2 av isometrigruppen agerar ortogonalt transitivt. Resultaten används sedan för att erhålla asymptotik för lösningar av Klein–Gordon-ekvationen på bakgrunder av denna typ.

Artikel B handlar om Bianchi typ $\mathrm{VI}_{-1/9}$-rumtider med en ortogonal stel vätska. Bianchi typ $\mathrm{VI}_{-1/9}$ kallas för exceptionell, på grund av att vakuums eller ortogonala ideala vätskekosmologier av denna typ har samma antal frihetsgrader som Bianchi typ VIII och IX. Det är på grund av att G_2 inte nödvandigtvis agerar ortogonalt transitivt för typ $\mathrm{VI}_{-1/9}$. Huvudresultatet är att singulariteten är, generellt sett, anisotrop och quiescent, och egenvärdena av $\mathcal{K}$ konvergerar till strikt positiva gränsvärden. Här är quiescence en följd av den stela västkan, vilken gör att det algebraiska villkoret på egenvärdena av $\mathcal{K}$ uppfylls. Likväl finns det ett komplement till det generiska beteendet, nämligen rumtider med vissa geometriska egenskaper, särskilt de i vilket G_2 agerar ortogonalt transitivt, och de som (asymptotiskt) uppfyller ett polariseringsvillkor. I dessa faller händer det att det minsta av gränsvärdena av egenvärdena av $\mathcal{K}$ är negativt. Detta står i kontrast med kosmologier av typ VIII eller IX med en ortogonal stel vätska, för vilka egenvärdena av $\mathcal{K}$ alltid konvergerar till strikt positiva gränsvärden. Som ett sekundärt resultat får vi ett får vi ett koncist sätt för att representera dynamiken.

I papper C betraktar vi CMC begynnelse data till Einsteins ekvationer kopplat till ett icke-linjärt skälarfält för en viss klass av potentialer. Huvudresultatet är att, om en uppskattning för vissa expansionsnormaliserade storheter gäller, om ett algebraiska villkor på egenvärdena av $\mathcal{K}$ är uppfyllt, och om egenvärdena av $\mathcal{K}$ förblir distinkta över mångfalden, finns det en undre gräns för den initiala medelkrökning som, om den överskrids, garanterar att utvecklingen har en quiescent Big Bang singularitet. I synnerhet menar vi global existens bakåt i tiden av utvecklingen fram till att Kretschmann-skalären exploderar, och konvergensen av egenvärdena av $\mathcal{K}$. Vi erhåller också asymptotik för egenvärdena av $\mathcal{K}$ och expansionsnormaliserade storheter relaterade till skalärfältet. Kombinationen av detta med resultater av Ringström beträffende Bianchi klass A lösningar leder till ett bevis av global icke-linjär stabilitet för en stor klass av rumsligt lokalt homogena lösningar.

小知不及大知，
小年不及大年。
奚以知其然也？
朝菌不知晦朔，
蟪蛄不知春秋，
此小年也。

Small knowledge is not comparable to great wisdom,
a short life not to a prolonged one.
How do we know that it is so?
The morning mushroom sees not the night and dawn,
the cicada of summer knows aught of seasons' change.
They live too short a while.

— Zhuangzhi - Wandering Freely

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Contents

Part II: Scientific Papers

Paper A

On Bianchi type VI_0 *spacetimes with orthogonal perfect fluid matter*
Preprint: arXiv:1908.02677
A similar version of this article was published in:
Annales Henri Poincaré **21** (2020), 3069-3094

Paper B

Quiescence for the exceptional Bianchi cosmologies
Preprint: arXiv:2311.05522

Paper C

Formation of quiescent big bang singularities
(joint with Oliver Petersen and Hans Ringström)
Preprint: arXiv:2309.11370

Introduction

This thesis is a synthesis of three studies of quiescent regimes in cosmology. The adjective *quiescent* means the absence of change, of volatility or, in this context, of chaotic oscillations, and could equally well be replaced by the word convergent. By *cosmology* we mean the study of spacetimes that solve Einstein's equations of general relativity and that undergo expansion. This expansion may potentially culminates in some kind of singularity. What is meant by a singularity depends on the context. It can be as weak as geodesic incompleteness, which means as much as the timeline of a certain observer ending at some finite time. Or it can be as strong as the curvature of spacetime becoming infinite for all observers.

The three studies each concern a *regime* or a mathematical setting in which quiescence occurs. The notion that our universe is expanding, supported by the measurements of redshift, has a history virtually as long as the theory of general relativity itself. Within general relativity, it is very well possible for the universe to display anisotropy relative to a given slicing of spacetime into space and time. In other words, it is possible for the rate of expansion to be different in different directions. These rates relative to the overall expansion, i.e. relative to the mean curvature, are captured by the expansion-normalized Weingarten map $\mathcal{K}$, a notion derived from the extrinsic curvature. The map $\mathcal{K}$, which we introduce formally in Definition 1.3, plays a major role in this story. Through its eigenvalues and eigenspaces it informs us about the rates and directions in which spacetime is expanding or contracting relative to the overall expansion. And it is the behaviour of the eigenvalues of the expansion-normalized Weingarten map, as a cosmological singularity is approached, that the adjective quiescent applies to. Thus by quiescence we mean the convergence of these rates, of the eigenvalues of $\mathcal{K}$.

Quiescence is not expected to be a generic feature of a cosmological singularity in an *empty* universes, i.e. those solving Einstein's equations without matter. Instead, it is expected that generically an empty universe undergoes an infinite series of chaotic oscillations of the different rates of expansion in different directions toward an initial singularity. This picture is the outcome of the work of Belinski, Khalatnikov, Lifschitz and their coauthors. There is a growing body of analytical and numerical evidence in support of this picture, but rigorous mathematical results in full generality are lacking. On the other hand, there are examples of settings in which this scenario does not play out, when specific types of matter or when special geometric conditions related to symmetry are present in the spacetime, and the results in this thesis contribute to that literature. In this thesis we wish to further the understanding of cosmological settings wherein this chaotic behaviour does not occur, i.e. to understand quiescent regimes in cosmology.

It should perhaps be noted at this early stage that this is a thesis in mathematics, not in physics. In particular, the objects of study are not necessarily guided or restricted by observational data. Observational cosmology is a field in development, and the contribution of mathematical relativity in general, and this thesis in particular, is not necessarily what is physically relevant, but rather what is mathematically possible within the theory of general relativity, under reasonable assumptions. However, I hope that this scientific contribution might help shed some light on the relevance of taking into account inhomogeneity but most of all anisotropy in our models of the universe.

1.1 Informal introduction

The topic of this thesis lies within the subfield of mathematics known as *mathematical relativity*. Broadly speaking, this constitutes studying the implications of Einstein's general theory of relativity, as introduced in [18, 17] in 1915, using a mathematician's toolbox as opposed to a physicist's toolbox. Hence it also features different perspectives and different focal points of research.

The theory of general relativity is a way to model *gravity*. Before Einstein introduced this theory of general relativity, gravity was understood as the force that the mass of an object has on other massive objects, and in particular pulls us to earth and lets the earth rotate the sun. In contrast, in the theory of general relativity there is no such force, but gravitation as a phenomenon is explained by the curvature of spacetime. Instead, spacetime is curved by matter, and free-falling objects follow geodesic paths in this curved spacetime. Matter and curvature are thus coupled, and in particular the curvature determines the paths of matter and the movement of matter in turn determines how the curvature changes.

A solution to Einstein's equations of general relativity, for some given matter, is a spacetime (M, g), i.e. a manifold M and a Lorentzian metric g with a time-orientation, along with functions describing the matter, such that the equation

$$G + \Lambda g = T \tag{1.1}$$

is satisfied.[1] Not every spacetime is a solution, and in the context of cosmology we frequently refer to spacetimes solving Equation (1.1) as *cosmologies* or sometimes also as *universes*.

A manifold is a mathematician's way to speak of an object consisting of various points, for which one can moreover, at every point, define various directions of travel. The number of independent directions is the dimension of the manifold, which in the current setting is typically four and sometimes written as 3+1. The Lorentzian metric allows one to discriminate between various types of these directions, and describes what is known as the *causal structure*. It allows one to talk about lightlike directions, which are the only possible directions for light (or photons) to travel along in spacetime; lightlike directions are precisely the directions tangent to the *light cones*. There are also timelike directions, which are the possible directions of travel for massive objects (such as ourselves), and spacelike directions, but are the directions orthogonal to the directions of travel in spacetime for massive objects. Spacelike directions are crucial to speak about an initial data formulation for Equation (1.1), as we do below in Section 1.4. In mathematical relativity, spacetime is often sliced into spacelike hypersurfaces, which can be thought of as snapshots of space at some (galactic) time.

Einstein's equations of general relativity are *tensorial*, which means that although we can describe the manifold locally in various ways, the objects that appear in the equation are independent of the chosen description. (Note

[1] Here the mathematician's convention of choosing units in such a way that the constant $8\pi K$ equals unity is adopted.

that although there is only one equation displayed, a tensorial equation is referred to in plural as it has many different components.) The tensor G on the left-hand side is the *Einstein tensor*, constructed using the Lorentzian metric g and its derivatives, and it is determined by the curvature of the spacetime. In particular it reads

$$G = \mathrm{Ric}_g - \tfrac{1}{2}\,\mathrm{Scal}_g\, g, \tag{1.2}$$

where Ric_g is the Ricci tensor of g and Scal_g the scalar curvature. The tensor T on the right-hand side of the equation is the *energy-momentum tensor*, and describes physical properties of the matter contents of the universe, for example its energy density. In the term Λg, the constant Λ is known as the *cosmological constant*, and its contribution to the equation is also known as "dark energy". The inclusion of this term in the equation has an interesting history, but it does not play a major role in this thesis. From the above we gather that, if Equation (1.1) holds for a given spacetime, then the curvature of spacetime and matter content are intricately linked.

Example 1.1. Let us consider now an example of a spacetime solving Einstein's equations in vacuum, i.e. for $T = 0$, namely four-dimensional *Minkowski space*. This is the setting of the theory of special relativity. The manifold is simply $\mathbb{R}^4$, and the Lorentz metric, in this case usually denoted η, is given by

$$\eta = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i, \tag{1.3}$$

where x^1, x^2, x^3 are the usual Cartesian coordinates on $\mathbb{R}^3$. This is a flat spacetime, meaning it has no curvature so that also $G = 0$.

Mathematical relativity is primarily concerned with the mathematical aspects of solutions to Equation (1.1). Questions that are of interest in this research community are:

- ◇ The sensitivity of (special) solutions to small changes in the initial data, i.e. the nonlinear stability or instability of a solution.
- ◇ The breakdown of determinism, which features prominently in the cosmic censorship conjectures.
- ◇ Defining quantities that we know from classical mechanics such as mass or angular momentum in special settings, such as for black holes.

◇ The mathematical and physical natures of singularities, which can typically be understood as the breakdown of some classical aspect of the theory in some region of spacetime.

It is the lattermost that is of most interest for our purposes.

Two classical results in cosmology

There are two classical results in cosmology that remain a driving force for much contemporary work. On the one hand, there are the Friedmann-Lemaître-Robertson-Walker (FLRW) models of cosmology. On the other hand, there is Hawking's singularity theorem. A mathematical derivation of the statements informally presented here may be found in many introductory textbooks on general relativity, see e.g. Sections 5 and 9 of [60] (though perhaps a more fitting reference concerning the singularity theorems is [24]).

The FLRW spacetimes are four-dimensional, spatially homogeneous and isotropic spacetimes filled with fluid matter, typically a radiation fluid or dust, and which solve Einstein's equations. The notion of spatial homogeneity can be understood, for the purposes of this informal discussion, as there not being a preferred point in space, while isotropy on the other hand can be understood as there not being a preferred direction. The combination of these two assumptions is also known as the cosmological principle. The FLRW spacetimes still form the basis of many standard models in cosmology used by physicists by virtue of their mathematical tractability. The assumptions of spatial homogeneity and isotropy pose severe restrictions on the Lorentz metric g. In particular, the surfaces of homogeneity must be spaces of constant curvature, which means that, locally, they must be either spheres, or copies of flat space or hyperbolic space. If moreover Einstein's equations hold with the energy-momentum tensor of a perfect fluid, then one obtains a set of coupled ordinary differential equations describing the evolution of the energy density of the fluid and of the length scale of the universe. An immediate consequence of these equations is that if the length scale at any point in time is expanding, then there must have been a moment in the past when the length scale of the spacetime became zero, and the density of the matter became infinite. In other words, the distance between any two points in space shrank to zero. This moment is then what is known as the *big bang*.

Another important result is Hawking's singularity theorem. It does not rely on such heavy symmetry assumptions, but its conclusions are, naturally, also not as strong as the properties that FLRW universes have. It presents the geometrical ingredients required for a cosmological singularity, which may be rather weak in principle, as it concern only geodesic incompleteness. Hawking's singularity theorem roughly states that if the matter in the universe is such that gravity attracts and if the universe is expanding with a universal lower bound on a certain kind of spacelike hypersurface, then there must have been some singularity a finite time ago, The nature of this singularity is such that the timelines of particles going back into the past can only exist finitely long. However, whether the curvature of spacetime or the energy density of matter becomes infinite we cannot say.

Between these two results there are a lot of possibilities. We know for example that the universe is neither exactly spatially homogeneous nor isotropic. What happens if we relax one of these two symmetry assumptions? And what is the nature of the singularities that Hawking's theorem predicts? Such questions guide us in this story and in the back of our mind as we proceed.

1.2 Mathematical preliminaries and conventions

The natural language of the general theory of relativity is the one of differential geometry, in particular Lorentzian geometry. From here on forward, we shall take several concepts associated to the Lorentzian manifold (M, g) for granted, such as:

- ◇ Fundamental notions for smooth manifolds such as tensors and vector fields, the space of which we denote using $\mathcal{X}$, the Lie derivative $\mathcal{L}$, the Lie bracket on vector fields $[\cdot,\cdot]$ and the exterior derivative d.

- ◇ Geometrical notions such as covariant differentiation and the Levi-Civita connection ∇ of the Lorentzian metric g.

- ◇ Basic concepts from causality theory, in particular spacelike and timelike vector fields and hypersurfaces, global hyperbolicity and Cauchy surfaces.

- ◇ The first fundamental form h and the second fundamental form k, for a given spacelike hypersurface Σ with future pointing normal ν, as well as the mean curvature $\theta := \mathrm{tr}_h(k)$.

- ◇ The Riemann, Ricci and scalar curvature tensors $\mathrm{Riem}, \mathrm{Ric}$ and Scal, where a subscript shall denote the relevant metric if the need arises.

- ◇ The notion of an isometry of a Lorentzian or Riemannian manifold, and that of Killing vector fields.

- ◇ The musical isomorphisms $\flat$ and $\#$ used to transform a vector field to its metric equivalent one-form and vice versa, respectively.

We refer the interested reader to e.g. [38], [60] and [50] for various aspects of the above. We moreover use the following notational conventions: Latin indices refer to spatial indices $1, 2, 3$, while Greek indices refer to spacetime indices $0, 1, 2, 3$; our convention for the Riemann tensor is $\mathrm{Ric}_{\alpha\beta} = \mathrm{Riem}^{\gamma}{}_{\alpha\beta\gamma} = g^{\gamma\delta}\,\mathrm{Riem}_{\delta\alpha\beta\gamma}$ which is the opposite of [38], and the same as that of [60] and [50]. We use the sign convention $k(X, Y) = g(\nabla_X \nu, Y)$. We moreover use the mathematician's convention to choose geometrical units so that $8\pi G/c^4 = 1$.

1.3 The energy-momentum tensor

Einstein's equations model gravity by coupling the curvature of spacetime to the matter contents inside it. The effect of the matter on the curvature is described by the energy-momentum tensor T. One immediate consequence of Einstein's equations is that as the Einstein tensor G is divergence-free, also the energy-momentum tensor must be divergence-free (or vice versa depending on your viewpoint). In particular

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.4}$$

which is interpreted as local conservation of energy-momentum.

There are two types of matter which appear in this thesis. Orthogonal perfect fluids appear in Papers A and B, while nonlinear scalar fields appear in Paper C. We refer any interested reader to Section 1.1 of [58] or Section 3 of [42] for an overview of other matter models and their energy-momentum tensors, and to [48] for a more comprehensive discussion on non-linear scalar fields.

1.3.1 Perfect fluids

One possible form of matter is a fluid. Fluids are widely used to model physical matter, in particular in cosmology. A *perfect fluid*, which appears in both Papers A and B, is a fluid that is completely characterized by its energy density ρ, pressure p, entropy s and its four-velocity u. Note that the former three are scalar quantities while the latter is a future directed unit timelike vector field in the spacetime. In particular, effects due to viscosity, heat conduction or shear stresses are not taken into account. We shall also consider only the isentropic case, for which the entropy is constant so that s does not play a role in the equations.

In this case, the contribution to the energy momentum tensor due to the fluid may be written as

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p(g_{\alpha\beta} + u_\alpha u_\beta). \tag{1.5}$$

On top of the local conservation of energy-momentum $\nabla_\beta T^{\alpha\beta} = 0$, we require an additional equation, the equation of state, relating the pressure with the energy density, and we consider in both cases a linear equation of state, written as

$$p = (\gamma - 1)\rho \tag{1.6}$$

where $\gamma \in [0, 2]$ is a fixed parameter; for this range and for $\rho \geq 0$ the dominant energy condition is satisfied, though for $\gamma < 2/3$ and $\rho > 0$ the strong energy condition is violated. Notable examples are $\gamma = 1$, corresponding to dust, $\gamma = 4/3$, corresponding to a radiation fluid, and $\gamma = 2$, corresponding to a stiff fluid, i.e. when the pressure equals the energy density, or in other words when the speed of sound in the fluid equals the speed of light. The value $\gamma = 0$ may be used to effectively obtain a cosmological constant. Heuristically, due to the analysis in FLRW spacetimes, one may think of fluids with larger γ being of larger relative importance at an earlier stage of the universe. This is evidenced by studies of Bianchi cosmologies containing two non-interacting fluids, cf. Section 8.2 of [58]. In Paper A the range $\gamma \in (\frac{2}{3}, 2)$ is considered, while in Paper B we consider the stiff fluid case, i.e. $\gamma = 2$.

Tilted and orthogonal perfect fluids

Given a foliation of spacetime into spacelike hypersurfaces with future pointing unit normal vector field ν (as discussed in Section 1.4 below), we may consider $u = \nu$ or $u \neq \nu$. In the former case we speak of an *orthogonal* perfect fluid, while in the latter case we speak of a *tilted* perfect fluid. In Papers A and B we consider only orthogonal perfect fluids. Nevertheless, Bianchi spacetimes with tilted perfect fluids are certainly of interest, in particular in relation to the notion of orthogonal-transitivity cf. in particular [29] on Bianchi type II as well as [27] and [14] on Bianchi type VI_0.

1.3.2 Scalar fields

A *scalar field* is often not studied for its physical relevance but for its mathematical tractability. As the name suggests, it is a scalar quantity, here denoted by ϕ, and its contribution to the energy-momentum tensor is given by

$$T_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \left(\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right) g_{\alpha\beta}. \tag{1.7}$$

The equation of motion of the scalar field is given by

$$\Box_g\phi = V'(\phi) \tag{1.8}$$

where $\Box_g = \nabla_\mu\nabla^\mu$ denotes the d'Alembertian associated to g.

Two important choices for the potential V are $V \equiv 0$, which is known as the massless linear scalar field and $V(x) = m^2x^2$, $m > 0$, know as the massive linear scalar field. (The Klein–Gordon equation is then either a massive linear scalar field, or by setting m to zero it may also mean a massless linear scalar field.) If V is not one of the two above cases, then we refer to the source term as a non-linear scalar field. In cosmology exponential potentials are often studied, and they are among the class of potentials allowed in Paper C. We also note that adding a constant to V has a similar effect as introducing a cosmological constant Λ in Equation (1.1), thus allowing us to omit Λ from the equation.

Equivalence between scalar fields and fluids

If the gradient $\nabla\phi$ of a scalar field is timelike, we can define a unit timelike vector field u by

$$u = \frac{\nabla\phi}{\sqrt{-\nabla_\mu\phi\nabla^\mu\phi}}. \tag{1.9}$$

We then recover the algebraic form of a perfect fluid from

$$\rho = -\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi), \tag{1.10}$$

$$p = -\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi), \tag{1.11}$$

cf. Section 1.1.2 of [58]. Vice versa, if the four-velocity of a perfect fluid u is hypersurface-orthogonal, i.e. the planes orthogonal to it are surface-forming, which is the case for an orthogonal perfect fluid, then the four-velocity is proportional to the gradient of a scalar function, so we may write $u = \nabla\phi/\sqrt{-\nabla_\mu\phi\nabla^\mu\phi}$, and we recover the algebraic form of a non-linear scalar field. In particular, a massless scalar field with a timelike gradient is effectively the same as a stiff fluid, and we observe that the four-velocity is orthogonal to the level sets of the scalar field. It is thus no surprise that Einstein's equations for one of these types of matter has a similar effect as when it coupled to the other.

1.4 Gauge and the Cauchy problem

A somewhat abused word in the research community studying general relativity is the word *gauge*. It is often said that the theory of general relativity is gauge-invariant, or alternatively, that it possesses the property of *general covariance*, which simply means that the equations are invariant under diffeomorphisms, i.e. that they are tensorial, as they manifestly are. In principle this is a feature of the theory, as it thus is perfectly geometric and not dependent on a choice of coordinates.

However, one major consequence of general covariance is that, for any given energy-momentum tensor, Einstein's equations per se do not belong to one of the families of partial differential equation for which well-posedness has been established. (We recall that a partial differential equation with some given data is well-posed in the sense of Hadamard if solutions exist, are unique and depend continuously on either initial data or boundary data.) The classical problem of interest for general relativity is the Cauchy problem, where one gives certain initial data on a spacelike hypersurface and asks for the future and past development of this data, which is then a spacetime. Existence is this case then refers to short-time existence. If one now obtains such a development, then due to the property of general covariance, applying a diffeomorphism will yield a new development.

On the other hand, there are proofs of geometric well-posedness of the Cauchy problem for Einstein's equations for various matter models, for geometric initial data that satisfy certain compatibility conditions. In practice, these have been established by fixing the gauge, or in other words, by choosing a specific way to represent the equations, and afterward ensuring that any other development is related to the constructed one by an isometry. Most famously, existence of a spacetime, or more precisely, the existence of a maximally globally hyperbolic development, wherein the initial data embeds, was proven first for Einstein's equations in vacuum in 1952 by Yvonne Choquet-Bruhat in [21], and completed with a proof of geometric uniqueness by Yvonne Choquet-Bruhat and Robert Geroch in 1969 in [13].

1.4.1 Initial data for Einstein's equations

For an evolution problem, such as the massless linear scalar field on Minkowski space, i.e. $\Box_\eta \phi = 0$ for a function ϕ on $\mathbb{R}^4$, typically one is interested in the future or past development from a set of initial data, in this case for example data prescribing $\phi(0,\cdot)$ and $\partial_t\phi(0,\cdot)$. However, depending on the precise problem not any initial data might suffices, as there might be compatibility conditions.

For Einstein's equations these compatibility conditions are the so-called *constraint equations.* As explained for example in Section 2 of [5], these arise as follows. Given a spacetime (M, g), we may consider a spacelike hypersurface Σ embedded in M, with future pointing unit normal vector field ν. The Lorentzian metric on the spacetime then induces a Riemannian metric h, the first fundamental form, which comes with its Levi-Civita connection, and a symmetric covariant two-tensor k, the second fundamental form on Σ. Moreover, these tensors satisfy the Gauss, Codazzi and Mainardi equations, which are equations relating the spacetime metric and curvature tensors to the induced metric and the second fundamental form and curvature tensors constructed from the metric. Using these, we may obtain expressions for the components of the Einstein tensor G, namely $G_{00} := G(\nu,\nu)$ and G_{0i}, defined by $G_{0i}X^i = G(\nu, X)$ for a vector field $X \in \mathcal{X}(\Sigma)$.

If the spacetime is a solution to Einstein's equation for a given matter field, then we may replace G_{00} and G_{0i} by $\rho := T_{00}$ and $J_i := T_{0i}$ to obtain the Hamiltonian constraint

$$\mathrm{Scal}_h + \mathrm{tr}_h(k)^2 - \mathrm{tr}_h(k^2) = 2\rho \tag{1.12}$$

and the momentum constraints, which we write as equation of one-forms

$$\mathrm{div}_h(k) - d\,\mathrm{tr}_h(k) = J, \tag{1.13}$$

where $J = T(\nu, \cdot)^{\#}$. These equations are called constraint equations as they express parts of the Einstein tensor without referring to second derivatives in time of the metric. Instead, we can use information present on the hypersurface, namely the first and second fundamental forms, as well as ρ and J to express G_{00} and G_{0i}.

For the two matter models discussed in Section 1.3, there are corresponding notions of initial data. Here we only introduce the relevant notion of initial data for the Einstein-scalar field equations. The relevant notion for orthogonal perfect fluid data we introduce in Section 1.5, as we only consider spatially homogeneous initial data for that type of matter.

Definition 1.2. Let Σ denote a manifold and $V \in C^\infty(\mathbb{R})$. *Smooth initial data* $\mathfrak{I} := (\Sigma, \bar{h}, \bar{k}, \bar{\phi}_1, \bar{\phi}_0)$ *on* Σ *to the Einstein-nonlinear scalar field equations with potential* V consists of a smooth Riemannian metric $\bar{h}$ on the manifold Σ, a symmetric covariant two-tensor $\bar{k}$ on Σ, and two smooth functions $\bar{\phi}_1$, $\bar{\phi}_0$ on Σ such that

$$\mathrm{Scal}_{\bar{h}} + \mathrm{tr}_{\bar{h}}(\bar{k})^2 - \mathrm{tr}_{\bar{h}}(\bar{k}^2) = \bar{\phi}_1^2 + \bar{h}(\mathrm{d}\bar{\phi}_0, \mathrm{d}\bar{\phi}_0) + 2V(\bar{\phi}_0), \tag{1.14a}$$

$$\mathrm{div}_{\bar{h}}(\bar{k}) - d\,\mathrm{tr}_{\bar{h}}(\bar{k}) = \bar{\phi}_0 \mathrm{d}\bar{\phi}_1. \tag{1.14b}$$

1.4.2 Gauge and foliations

The act of *fixing the gauge* refers to choosing a preferred foliation of spacetime by spacelike hypersurfaces, as well as relating how local descriptions on these hypersurfaces, be it by coordinates or by frame vector fields, relate to one another. Making the right (or wrong) choice of gauge may significantly simplify (or complexify) the analysis of the same physical system. Probably the most famous choice of gauge, employed in the groundbreaking work in [21] is that of *wave coordinates*, also known as *De Donder* gauge, which was used to prove geometric well-posedness of the Einstein-vacuum equations. This is a propagating condition on local coordinates that transforms the Einstein-vacuum equations into a quasi-linear hyperbolic system for the components of the metric, Ultimately this procedure allows for the conclusion of well-posedness of the Cauchy problem for the Einstein-vacuum equations, meaning for initial data satisfying Equations (1.12) and (1.13) with vanishing right-hand sides.

There is a particular choice of foliation that often features in the cosmological setting, namely that of a constant-mean-curvature (CMC) foliation.

In order to discuss this, it is useful to recall the notions of the lapse function and the shift vector field, or simply *lapse* and *shift*. Given a spacetime (M, g) with a time-function t, i.e. a function whose gradient is timelike everywhere, we obtain a foliation by the level sets of t, say Σ_t with future pointing unit normal ν. The decomposition $\partial_t = N\nu + \chi$ then defines the lapse N and the shift χ. The shift vector field plays no major role in this thesis and is set to 0, but the lapse does appear in Paper C.

A CMC foliation is a foliation of spacetime by spacelike hypersurfaces such that the mean curvature $\theta = \mathrm{tr}_h(k)$ is constant on each hypersurface. In Paper C, we moreover use $t := \theta^{-1}$ as a time-function. In order for this condition to be satisfied, we obtain an elliptic partial differential equation for the lapse (which can be understood either as a consequence of the Raychaudhuri equation, or by tracing the evolution equation for the second fundamental form with respect to a frame). Note that not any initial data necessarily gives rise to a development with a CMC foliation, but if the initial data has constant mean curvature, then there is a choice of gauge propagating this condition. However, in conjunction with introducing a notion of cosmological spacetime, Bartnik conjectured in [4] that a CMC foliation always exists in a globally hyperbolic spacetime that admits a compact Cauchy surface and satisfies the timelike convergence condition.

Another relevant choice of foliation is a Gaussian foliation, for which the time-function is a normal coordinate, and may be obtained locally around a spacelike hypersurface as follows. Given some spacelike hypersurface Σ with future pointing unit normal ν, one may locally define a time-function by the distance travelled by geodesics tangent to the unit normal ν. We note that in Bianchi spacetimes, which feature in Papers A and B and are discussed below in Section 1.5, the surfaces of spatial homogeneity are both CMC foliations and Gaussian foliations.

1.4.3 The expansion-normalized quantities

We shall here introduce the geometric expansion-normalized quantities $\mathcal{H}$, $\mathcal{K}$, Φ_1 and Φ_0. The setting here is a spacetime (M, g) which has a foliation by spacelike hypersurfaces $I \times \Sigma$ with a first fundamental form h and second fundamental form k, and we assume that the hypersurfaces have a constant, strictly positive mean curvature $\mathrm{tr}_h(k) = \theta > 0$.[2] Also, let $\phi \in C^\infty(M)$ be a

[2]In order to define $\mathcal{H}$, $\mathcal{K}$, Φ_1 and Φ_0 it actually suffices if the mean curvature is positive everywhere on a single given spacelike hypersurface Σ with unit normal ν.

function representing the scalar field, for example in the case that (M, g, ϕ) solves the Einstein-scalar field equations.

Let us consider first the Weingarten map K, which is the $(1,1)$-tensor metrically equivalent to the second fundamental form k of the foliation, i.e. $K(X) = k(X, \cdot)^{\#}$ for any $X \in \mathcal{X}(\Sigma)$. Then we can define the expansion-normalized Weingarten map as follows:

Definition 1.3 (Paper C, Definition 4)**.** Assume that $\theta > 0$. The *expansion-normalized Weingarten map* is the endomorphism

$$\mathcal{K} := \frac{K}{\theta} : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma). \tag{1.15}$$

As K and thus also $\mathcal{K}$ are symmetric with respect to h, they are diagonalizable and have real eigenvalues. If $\theta > 0$, then $\theta^{\mathcal{K}}$ is a well defined smooth endomorphism on vector fields, defined by

$$\theta^{\mathcal{K}}(X) := e^{\ln(\theta)\mathcal{K}}(X) = \textstyle\sum_{m=0}^{\infty} \frac{(\ln(\theta)\mathcal{K})^m}{m!}(X), \tag{1.16}$$

for any $X \in \mathcal{X}(\Sigma)$. This leads us to the definition of $\mathcal{H}$.

Definition 1.4 (Paper C, Definition 5)**.** Assume that $\theta > 0$. The *expansion-normalized first fundamental form* is the covariant two-tensor field $\mathcal{H}$ given by

$$\mathcal{H}(X, Y) = h(\theta^{\mathcal{K}}(X), \theta^{\mathcal{K}}(Y)) \tag{1.17}$$

for any $X, Y \in \mathcal{X}(\Sigma)$.

Next, there are the expansion-normalized quantities related to the scalar field, see also Definition 6 of Paper C.

Definition 1.5 (Paper C, Definition 6)**.** Assume that $\theta > 0$. The *expansion-normalized normal derivative of the scalar field* is given by

$$\Phi_1 := \theta^{-1}\nu(\phi), \tag{1.18}$$

where ν is the future pointing unit normal vector field on Σ. The *expansion-normalized induced scalar field* is given, on Σ, by

$$\Phi_0 := \phi + \Phi_1 \ln(\theta). \tag{1.19}$$

We already encountered $\mathcal{K}$ various times as it is central to the notion of quiescence, but let us also briefly discuss the other expansion-normalized quantities. In a favourable setting one may expect not just the eigenvalues of $\mathcal{K}$, but all of $\mathcal{K}$, $\mathcal{H}$, Φ_1 and Φ_1 to converge for a spacetime with a quiescent singularity. Furthermore one could hope to uniquely reconstruct a spacetime metric using only the information contained in these quantities. One such favourable setting is for families of solutions that induce initial data on the singularity following the framework of [46], which happens for example for Bianchi class A solutions, cf. in particular [45]. This latter fact is also of importance to construct examples to apply the main result of Paper C.

More concretely, the expansion-normalized quantities give information about the quantities k, h, ϕ_0 and ϕ_1 relating to the expansion but with appropriate scaling factors. Take for example a spacelike hypersurface Σ with unit normal ν and consider a basis of vector fields $\{X_I\}_{I=1}^n$ which are orthogonal to ν and which are eigenvectors of $\mathcal{K}$, with say eigenvalues p_I. We may then normalize X_I with respect to $\mathcal{H}$. As $\theta^{\mathcal{K}}(X_I) = \theta^{p_I} X_I$ (no summation), the normalized vector fields are thus precisely $\theta^{-p_I} X_I / |X_I|_h$ (no summation).

In order to get a feeling for these expansion-normalized quantities, it is useful to discuss an example, namely the scalar field-analogs of the well-known Kasner vacuum solutions. These are solutions to the Einstein-scalar field equations. They have gotten a variety of names in the literature, but we shall simply refer to them as the *Kasner scalar field solutions*.

Example 1.6. A *Kasner scalar field solution* is a spacetime of the form $((0,\infty) \times \Sigma, g)$, along with a function $\phi : \Sigma \to \mathbb{R}$ for $\Sigma = \mathbb{R}^n$ or $\Sigma = \mathbb{T}^n$ and $n \geq 3$, and

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{n} t^{2p_i} \mathrm{d}x^i \otimes \mathrm{d}x^i, \quad \phi = a \ln(t) + b, \tag{1.20}$$

where $t \in (0, \infty)$ and $a, b, p_i, i = 1, ..., n$ are constants satisfying the relations

$$\sum_{i=1}^{n} p_i = 1 = a^2 + \sum_{i=1}^{n} p_i^2. \tag{1.21}$$

Now we may compute the expansion-normalized quantities for these spacetimes relative to the given foliation, cf. Example 8 in Paper C. Indeed, we may compute that

$$\mathcal{K} = \sum_{i=1}^{n} p_i \, \partial_{x^i} \otimes \mathrm{d}x^i,$$
$$\mathcal{H} = \sum_{i=1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^i,$$
$$\Phi_1 = a, \quad \Phi_0 = b.$$

In particular, the expansion-normalized quantities $\mathcal{K}$, $\mathcal{H}$, Φ_1 and Φ_0 are independent of the time-function t for these spacetimes.

Remark 1.7. In the case that $a = b = 0$, then (M, g) satisfies $\mathrm{Ric}_g = 0$, and we thus have a solution to the Einstein-vacuum equations. These solutions are known as a *Kasner vacuum solution.* The constants $p_i, i = 1, ..., n$ are known as the *Kasner exponents*, which in that case satisfy the so-called *Kasner relations*, namely

$$\sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} p_i^2. \tag{1.22}$$

The case where one Kasner exponent equals one, and the others vanish, is known as the *flat Kasner solution*, which can be shown to be isometric to a quotient of a part of Minkowski space.

Note that if $n = 3$, i.e. four-dimensional spacetime, then precisely one of the Kasner exponents must be negative for a Kasner vacuum solution aside from the special case of the flat Kasner solution. In contrast with the Kasner vacuum solutions, for a Kasner scalar field solution it is possible that all the Kasner exponents are strictly positive also in the case $n = 3$.

We may associate expansion-normalized initial data to initial data of the Einstein-nonlinear scalar field equations. We give here a definition required for the summary of the results.

Definition 1.8 (Paper C, Definition 9)**.** Let $\mathfrak{I} := (\Sigma, \bar{h}, \bar{k}, \bar{\phi}_0, \bar{\phi}_1)$ be initial data for the Einstein-nonlinear scalar field equations with potential $V \in C^\infty(\mathbb{R})$ as in Definition 1.2. If $\bar{\theta}$ is constant on Σ, $\mathfrak{I}$ are said to be *constant mean curvature (CMC)* initial data for the Einstein-nonlinear scalar field equations. In case $\bar{\theta} > 0$ on Σ, we define the associated expansion-normalized Weingarten map $\bar{\mathcal{K}}$, expansion-normalized first fundamental form $\bar{\mathcal{H}}$, expansion-normalized normal derivative of the scalar field $\bar{\Phi}_1$ and expansion-normalized induced scalar field $\bar{\Phi}_0$ by appealing to Definitions 1.3, 1.4, and 1.5. Then $(\Sigma, \bar{\mathcal{H}}, \bar{\mathcal{K}}, \bar{\Phi}_0, \bar{\Phi}_1)$ are said to be the *expansion-normalized initial data associated to* $\mathfrak{I}$.

1.4.4 The orthonormal frame approach

Instead of using local coordinates to describe the metric (as is the more usual approach in general relativity), one may choose to locally describe the metric and its derivatives using an orthonormal frame. Note that this is essentially dual to the Cartan formalism, where one works with a basis of one-forms instead of vector fields.

Consider a foliation of spacetime by spacelike hypersurfaces $I \times \Sigma$ given by a time-function t, and assume that the shift χ vanishes. Next consider locally an orthonormal frame $\{e_0, e_1, e_2, e_3\}$ (in the four-dimensional case), where $e_0(t)$ is normal to the hypersurfaces $\{t\} \times \Sigma$. If Σ is parallellizable, which is the case in Papers A and B, and with some exceptions the case in Paper C, then one may in fact use the frames to describe the spacetime globally. If we denote the (local) one-forms dual to the frame by $\{\omega^0, \omega^1, \omega^2, \omega^3\}$, then the spacetime metric g takes the simple form

$$g = -\omega^0 \otimes \omega^0 + \sum_{i=1}^{3} \omega^i \otimes \omega^i, \tag{1.23}$$

and the induced metric h on the hypersurface is simply

$$h = \sum_{i=1}^{3} \omega^i \otimes \omega^i. \tag{1.24}$$

Instead of the components of the metric and the Christoffel symbols, the structure coefficients $\gamma^{\mu}_{\alpha\beta}$, given by

$$[e_\alpha, e_\beta] = \gamma^{\mu}_{\alpha\beta} e_\mu, \tag{1.25}$$

are used to describe the Levi-Civita connection and the curvature. We may equivalently use the connection coefficients $\Gamma^{\mu}_{\alpha\beta}$, which are defined through

$$\nabla_{e_\alpha} e_\beta = \Gamma^{\mu}_{\alpha\beta} e_\mu, \tag{1.26}$$

as these can be recovered from the structure coefficients and vice versa through the Koszul formula and the fact that the Levi-Civita connection is free of torsion. Moreover, the components of the second fundamental form k with respect to this frame may be recovered from

$$k_{ij} = \frac{1}{2}\left(\gamma^j_{i0} + \gamma^i_{j0}\right), \tag{1.27}$$

where we have $k = k_{ij}\omega^i \otimes \omega^j$. There is a certain freedom left in choosing the antisymmetric part of γ^j_{i0} (similar to a gauge freedom), which is related to the propagation of the orthonormal frame.

In Paper C we use the orthonormal frame approach to obtain a system of equations in the variables k_{ij}, γ^i_{jk}, in the components of e_i with respect to a fixed frame $\{E_j\}_{j=1}^n$, as well as in similar variables related to the scalar field. The frame is chosen to be *Fermi-Walker propagated*, which simply means that the anti-symmetric part is chosen to vanish. We refer to Section 2 of Paper C for more details.

In Papers A and B we adapt the orthonormal frame to the spatial homogeneity, and we obtain that on the surfaces of homogeneity the components k_{ij} and the structure coefficients γ^i_{jk} are functions of time only. Furthermore, in Paper A there is a very favourable choice of frame which simultaneously diagonalizes the second fundamental form as well as a symmetric matrix n constructed from the structure coefficients, and which does not require a rotation. However, in Paper B due to the momentum constraints the situation is more complicated, as k and n cannot be diagonalized simultaneously. There we choose to only diagonalize the symmetric matrix n related to the intrinsic geometry.

1.5 Symmetry and Bianchi spacetimes

The spacetimes which are the subject of study in Papers A and B are *Bianchi spacetimes*. These spacetimes possess a high degree of symmetry, as they are spatially homogeneous, but are typically anisotropic at any stage of their development.

Definition 1.9. A *Bianchi spacetime* is a Lorentz manifold (M, g) of the form $M = I \times G$, where I is an open interval and G a connected, three-dimensional Lie group and the metric may be written as

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a_{ij}(t)\xi^i \otimes \xi^j; \tag{1.28}$$

here the ξ^j are dual to a basis e_i of the Lie algebra of G, and the functions a_{ij} are smooth and form the components of a symmetric, positive definite matrix at any $t \in I$.

The Kasner scalar field spacetimes of Example 1.6 are examples of Bianchi spacetimes where the underlying Lie group is simply $\mathbb{R}^3$ or $\mathbb{T}^3$. The name Bianchi spacetimes is due to the Bianchi-Behr or simply Bianchi classification of three-dimensional Lie algebras. To understand what follows in this section, it is important that we understand this classification.

1.5.1 The Bianchi-Behr classification

A simply connected, three-dimensional Lie group G may be classified by its Lie algebra $\mathfrak{g}$, according to the so-called Bianchi-Behr classification. The interested reader is referred to [34] for a recount of its history and the many people involved in its development.

We first recall some basic definitions from Lie theory. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a three-dimensional Lie algebra, i.e. a vector space $\mathfrak{g}$ of dimension three equipped with a Lie bracket $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, which is an anti-commutative bilinear operator on $\mathfrak{g}$, satisfying the Jacobi identity. The bracket induces for any element $x \in \mathfrak{g}$ an automorphism of $\mathfrak{g}$ by $\mathrm{ad}_x(y) = [x, y]$. We can define a linear functional $\hat{a} \in \mathfrak{g}^*$ by

$$\hat{a}: \ \mathfrak{g} \to \mathbb{R}, \qquad x \mapsto \tfrac{1}{2}\,\mathrm{tr}(\mathrm{ad}_x). \tag{1.29}$$

This map immediately leads to the following two classes of Lie algebras. If the map $\hat{a}$ is trivial, then we say that $\mathfrak{g}$ is *unimodular*, and in dimension 3 we say that $\mathfrak{g}$ is of Bianchi class A. If $\hat{a}$ is non-trivial we say that it is of Bianchi class B. See e.g. [37] for more details.

Let now $\mathfrak{g}$ be of class B, and consider $\mathfrak{h} := \ker(\hat{a})$. This is a Lie subalgebra and an ideal, as $\mathrm{tr}\left(\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}\right) = 0$ by the Jacobi identity, hence $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{h}$. This subalgebra is known as the *unimodular kernel*, which is itself unimodular, and it is an easy computation to show that any unimodular, two-dimensional Lie algebra is actually Abelian, a fact that becomes relevant later on when discussing symmetries in spacetime.

Given a basis of $\mathfrak{g}$, say $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, we may define structure constants $\hat{\gamma}_{ij}^k$ of $\mathfrak{g}$ by

$$[\hat{e}_i, \hat{e}_j] = \hat{\gamma}_{ij}^k \hat{e}_k. \tag{1.30}$$

Class	Type	$\hat{n}^{11}$	$\hat{n}^{22}$	$\hat{n}^{33}$
A	I	0	0	0
	II	0	0	+
	VI_0	0	+	-
	VII_0	0	+	+
	VIII	+	+	-
	IX	+	+	+

(A) The various Bianchi types of class A.

Class	Type	$\hat{n}^{22}$	$\hat{n}^{33}$
B	IV	0	+
	V	0	0
	VI_η	+	-
	VII_η	+	+

(B) The various Bianchi types of class B; observe that type III is the same as type VI_{-1}.

Table 1.1: The Bianchi types of class A (left), for which $\hat{a} = 0$, and class B (right), for which $\hat{a} \neq 0$, typified by the different signs of the eigenvalues of the matrix $\hat{n}$. For type VI_η and VII_η there is also the relation (1.32).

We may then define the quantities

$$\hat{n}^{kl} := \epsilon^{ij(k}\hat{\gamma}^{l)}_{ij}, \quad \hat{a}_i := \tfrac{1}{2}\hat{\gamma}^k_{ik}. \tag{1.31}$$

The Lie algebra being of class B thus means that not all $\hat{a}_i$ vanish. The Jacobi identity applied to the basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is on the other hand equivalent to the equality $\hat{n}^{ij}\hat{a}_j = 0$. Therefore for a Lie algebra of class B it holds that $\hat{a}^{\#} = \sum_{j=1}^{3} \hat{a}_j\hat{e}_j$ is an eigenvector of the matrix $\hat{n}$ with components $\hat{n}^{ij}$, and the associated eigenvalue is 0. For types VI and VII, one has to take into account an additional parameter $\eta \in \mathbb{R}$ given by the relation

$$\hat{a}_i\hat{a}_j = \tfrac{\eta}{2}\epsilon_{ikl}\epsilon_{mnq}\hat{n}^{km}\hat{n}^{lq}, \tag{1.32}$$

an observation going back to [15].

Using an appropriate rotation in the left-invariant basis we may diagonalize the matrix $\hat{n}$. In the case of class B, we may write $\hat{a}^{\#} = \hat{a}_1\hat{e}_1$ so that $\hat{n}^{11} = 0$ and the algebraic relation (1.32) becomes $\hat{a}_1^2 = \eta\hat{n}^{22}\hat{n}^{33}$ for types VI_η and VII_η. We may then classify $\mathfrak{g}$ according to the relative signs of the eigenvalues of the matrix $\hat{n}$, as shown in Table 1.1. (Of course the indices of the basis may be permuted, and a change of the orientation of the basis leads to all signs becoming opposite.)

The cases of primary interest here are the ones of Bianchi type VI_0, which features in Paper A, and type $\mathrm{VI}_{-1/9}$, which features in Paper B. For both types of the three eigenvalues of the matrix $\hat{n}$ one is positive and one is negative. For type $\mathrm{VI}_{-1/9}$, moreover there is the relation $\hat{a}_1^2 = -\frac{1}{9}\hat{n}^{22}\hat{n}^{33}$, after we choose a basis diagonalizing $\hat{n}$ and such that $\hat{n}^{11} = 0$.

Let us note that all three-dimensional Lie algebras except those of type VIII or IX have a two-dimensional Abelian subalgebra, which gives rise to an Abelian two-dimensional Lie subgroup of the three-dimensional Lie simply connected Lie group integrated from the original Lie algebra. For the class B algebras, the two-dimensional Abelian subalgebra is simply the unimodular kernel mentioned above. For the class A algebras, say that $\hat{n}^{11} = 0$ after diagonalizing $\hat{n}$. Then it follows that $[\hat{e}_2, \hat{e}_3] = \hat{n}^{11}e_1 = 0$, so that in particular the span of $\hat{e}_2$ and $\hat{e}_3$ is an Abelian Lie subalgebra.

1.5.2 Orthonormal frames for a Bianchi spacetime

As a result of the Bianchi-Behr classification, describing a Bianchi spacetime using an orthonormal frame (as opposed to coordinates) works well the underlying structure. The orthonormal frame approach has been used by many authors, but we wish in particular to mention the foundational work [19] by Ellis and MacCallum, though we refer to Section 1.5 of [58] for a more comprehensive overview.

Similar to the Bianchi classification, we may decompose the structure coefficients γ_{ij}^k (with purely spatial indices) by

$$\gamma_{ij}^k = \varepsilon_{lij}n^{kl} + a_i\delta_i^k - a_j\delta_j^k. \tag{1.33}$$

On the other hand, for one temporal index, we may decompose as

$$\gamma_{0i}^j = -k_{ij} - \varepsilon_{ijk}\Omega^k. \tag{1.34}$$

Here $k_{ij} = k(e_i, e_j)$ denote the components of the second fundamental form with respect to the frame e_1, e_2, e_3, and Ω^k, $k \in \{1, 2, 3\}$ prescribes the rotation of the frame as discussed in Subsection 1.4.4 above. Note that $\gamma_{ij}^0 = 0$ in the case of a foliation with zero shift. The matrix n and the co-vector $a = a_i\omega^i$ must be of the same type as the Lie group G underlying the Bianchi spacetime, though they are not constants but depend on time t.

If we assume now that a Bianchi spacetime solves Einstein's equations, say for convenience either in vacuum or coupled to an orthogonal perfect fluid, then we may obtain evolution equations for n and a from the Jacobi identities applied to the vectors e_0, e_i, e_j, $i \neq j$; evolution equations for the second fundamental form k may be obtained by considering the difference between the spatial and spacetime Ricci curvatures. In practice one does not work with the second fundamental form, but rather with the its trace-less version σ known as the shear. One can obtain an equation for the evolution of the mean curvature from the Raychaudhuri equation. We refrain here from writing down the equations, but instead refer to Equations (1.90 - 1.99) of [58].

1.5.3 Wainwright-Hsu equations

A very fruitful approach to analyze the equations when the Lie algebra of the Lie group is of class A is that of [59], and yields the so-called Wainwright-Hsu equations. The equations are obtained by first dividing the variables by the mean curvature, and then redefining the time-coordinate $\tau(t)$ by $d\tau/dt = \theta/3$. (Note that is a different expansion-normalization than we used to define the expansion-normalized quantities!) Here we use a slightly different expansion-normalization, following the conventions of [49], We obtain the following set of equations, cf. e.g. [49], which we require for the summary of the results in the next chapter:

$$N_{11}' = (q - 4\Sigma_+)N_{11}, \tag{1.35a}$$
$$N_{22}' = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_{22}, \tag{1.35b}$$
$$N_{33}' = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_{33}, \tag{1.35c}$$
$$\Sigma_+' = -(2-q)\Sigma_+ - 3S_+, \tag{1.35d}$$
$$\Sigma_-' = -(2-q)\Sigma_- - 3S_-, \tag{1.35e}$$
$$\Omega' = 2(q - q^*)\Omega, \tag{1.35f}$$

as well as the Hamiltonian constraint

$$1 = \Omega + \Sigma_+^2 + \Sigma_-^2 + K. \tag{1.35g}$$

Here $q^* = \frac{1}{2}(3\gamma - 2)$ while the functions q, Σ_+, Σ_- and K are given by

$$q = q^*\Omega + 2\left(\Sigma_+^2 + \Sigma_-^2\right), \tag{1.35h}$$
$$S_+ = \tfrac{1}{2}\left(N_{22} - N_{33}\right)^2 - \tfrac{1}{2}N_{11}\left(2N_{11} - N_{22} - N_{33}\right), \tag{1.35i}$$
$$S_- = \tfrac{\sqrt{3}}{2}\left(N_{22} - N_{33}\right)\left(N_{22} + N_{33} - N_{11}\right), \tag{1.35j}$$
$$K = \tfrac{3}{4}\left(N_{11}^2 + N_{22}^2 + N_{33}^2 - 2\left(N_{11}N_{22} + N_{22}N_{33} + N_{33}N_{11}\right)\right). \tag{1.35k}$$

In the above, Σ_+ and Σ_- describe the expansion-normalized shear, and similarly S_+ and S_- describe the expansion-normalized trace-less spatial Ricci curvature. Ω is the expansion-normalized energy density, and K relates to the expansion-normalized scalar curvature. Note that we do not give any momentum constraint equations, as they are implicitly used in the choice of variables, Indeed, for Bianchi class A in vacuum or with an orthogonal perfect fluid the momentum constraints imply precisely that n and k commute, and can thus be simultaneously diagonalized. If the Lie algebra is of class B, then the description becomes more complicated. One can find the corresponding equations in Appendix A of Paper B, where we choose a frame adapted to the intrinsic geometry.

1.5.4 Bianchi perfect fluid data and developments

For the summary of the results we need appropriate notions of initial data for Bianchi spacetimes; we refer the reader to [49, 43, 58] for more details.

Definition 1.10. *Bianchi orthogonal perfect fluid data* (G, h, k, ρ) consist of a connected three-dimensional Lie group G, a left-invariant metric h on G, a left-invariant, symmetric, covariant two-tensor field k on G and a non-negative constant ρ, satisfying

$$\mathrm{Scal}_h + \mathrm{tr}(k)^2 - \mathrm{tr}_h(k^2) = 2\rho, \tag{1.36a}$$

$$\mathrm{div}_h(k) - d\,\mathrm{tr}_h(k) = 0. \tag{1.36b}$$

From fluid data in the sense above for which G is of class A one may obtain initial conditions to the system of equations (1.35). There is then a corresponding development which is Bianchi spacetime. Conversely, given an orbit of the system of equations (1.35), and given the mean curvature θ, one may obtain perfect fluid data as above, cf. [59].

1.5.5 The Abelian G_2 and polarization conditions

Algebras of the lower Bianchi types, i.e. except types VIII and IX, have an Abelian subalgebra. As a result, the isometry group of a Bianchi solution, for which the Bianchi type is neither of type VIII or IX, has an Abelian subgroup, a so-called G_2. In this sense, the Bianchi cosmologies of lower type are special cases of G_2 *cosmologies*, which are solutions to Einstein's equations that admit an Abelian group of isometries whose orbits are two-dimensional spacelike hypersurfaces, cf. e.g. [58, 55]. We note that besides the lower type Bianchi solutions, there are important classes of G_2 cosmologies, namely the so-called $\mathbb{T}^2$-symmetric solutions and the Gowdy solutions. Gowdy solutions and $\mathbb{T}^2$-symmetric solutions have been studied extensively, in particular as tractable inhomogeneous solutions to Einstein's equations for various matter models, cf. the review article [44] and the references therein.

Following the classification of G_2 cosmologies by Wainwright in [56], the Abelian G_2 may have a property that is relevant to quiescence, and which plays an important role in the results of Papers A and B. Namely it may or may not act *orthogonally-transitively*, which means that the planes orthogonal to the group orbits are the tangent planes of some two-surface in spacetime. We note that in Bianchi vacuum or orthogonal perfect fluid cosmologies of the lower types the G_2 acts orthogonally-transitively, unless the Bianchi type is the exceptional type $\mathrm{VI}_{-1/9}$ which we consider in Paper B. The reason type $\mathrm{VI}_{-1/9}$ does not have this property is related to a certain degeneracy of the momentum constraints. Orthogonal-transitivity of the action of the G_2 is also what characterizes the subclass of $\mathbb{T}^3$-Gowdy spacetimes within the class of $\mathbb{T}^2$-symmetric spacetimes. [3] If the G_2 acts orthogonally-transitively, then it follows that the shear σ and therefore also the expansion-normalized Weingarten map $\mathcal{K}$ both have an eigenvector orthogonal to the G_2, cf. Theorem 3.1 of [56]. But the G_2 being Abelian then implies the vanishing of a structure coefficient with respect to a frame diagonalizing $\mathcal{K}$. As we discuss in Section 1.6 below, this property is very relevant to the notion of quiescence.

If a solution to Einstein's equations has a spacelike Killing vector field ξ, like Bianchi cosmologies and in general G_2 cosmologies do, then ξ may be *hypersurface-orthogonal*. This means that the planes orthogonal to ξ are hypersurface-forming. This is equivalent to the condition $d\xi^\flat \wedge \xi^\flat = 0$ and is also known as *polarization*.

It can be shown that, given a foliation induced by a time function t with zero shift and such that ξ is tangent to the foliation, ξ being a hypersurface-orthogonal Killing vector field implies that it must be an eigenvector of $\mathcal{K}$ of this foliation. Moreover, with respect to an eigenframe of $\mathcal{K}$ with say $\xi = \xi^2 e_2$ the structure coefficient γ^2_{31} must vanish. Many of these computations are known if one specializes to the case of a G_2 cosmology, see in particular

[3]The twist constants vanishing for Gowdy spacetimes are precisely what is required for orthogonal-transitivity of the G_2, cf. e.g. Section 1.6.1 of [58]

the Appendix of [57], but we show them here for completeness. Indeed, consider a local orthonormal frame e_0, e_1, e_2, e_3 and assume that $\xi = \xi^i e_i$ is a hypersurface-orthogonal Killing vector field. Now firstly the (00)-component of the equation $\mathcal{L}_\xi g = 0$ implies that $\xi^i \gamma^0_{i0} = 0$. Secondly, the $(0i)$-component implies that

$$\langle [\xi, e_0], e_j \rangle = -e_0(\xi^j) + \xi^m \gamma^j_{m0} = 0, \tag{1.37}$$

as $\langle [\xi, e_i], e_0 \rangle = 0$ follows from from the assumption of zero shift and the fact that ξ is tangent to the foliation. We gather that in particular $[\xi, e_0] = 0$. The fact that ξ is hypersurface-orthogonal means that $d\xi^\flat \wedge \xi^\flat = 0$, where $\xi^\flat = \xi_m \omega^m$ (and $\xi^m = \xi_m$, i.e. we raise indices with δ) denotes the one-form metrically related to ξ. On the other hand, observe that

$$\begin{aligned} d\xi^\flat(e_0, e_i) &= e_0(\xi^\flat(e_i)) - e_i(\xi^\flat(e_0)) - \xi^\flat(\mathcal{L}_{e_0} e_i) \\ &= e_0(\xi_i) - \xi^\flat(\nabla_{e_0} e_i - \nabla_{e_i} e_0) \\ &= e_0(\xi_i) - \langle \xi, \nabla_{e_0} e_i \rangle + k_i^m \xi_m \\ &= \langle \nabla_{e_0} \xi, e_i \rangle + k_i^m \xi_m \\ &= 2k_i^m \xi_m, \end{aligned} \tag{1.38}$$

where the last equation follows precisely because $[e_0, \xi] = 0$, and $k_{im} = k(e_i, e_m)$ denotes the (im)-component of the second fundamental form. The $(0ij)$-component of the equation $d\xi^\flat \wedge \xi^\flat = 0$ (for $i \neq j$) is then equivalent to $(k(\xi) \wedge \xi)_{ij} = 0$, which in turn is equivalent to the statement that ξ is an eigenvector of the second fundamental form.

On the other hand

$$\begin{aligned} d\xi^\flat(e_i, e_j) &= e_i(\xi_j) - e_j(\xi_i) - \xi_m \gamma^m_{ij} \\ &= 2e_i(\xi_j) - 2\xi_m \gamma^m_{ij}, \end{aligned} \tag{1.39}$$

where for the second equation we rewrite $e_j(\xi_i) = -e_i(\xi_j) + \xi^m(\gamma^j_{mi} + \gamma^i_{mj})$ using the (ij)-component of the equation $\mathcal{L}_\xi g = 0$. Then the (ijk)-component of the hypersurface-orthogonality condition may be written in a perhaps more familiar form, namely

$$e_{[i}(\xi_j)\xi_{k]} = \xi_{[k}\Gamma^m_{ij]}\xi_m, \tag{1.40}$$

with the square brackets denoting complete anti-symmetrization. If we then choose a frame diagonalizing the second fundamental form such that

$\xi = \xi^2 e_2$, then Equation (1.40) becomes $0 = (\xi^2)^2 \gamma^2_{31}$ with respect to this frame, implying that $\gamma^2_{31} = 0$.

In particular, we find that with respect to a frame diagonalizing the expansion-normalized Weingarten map, one of the structure coefficients vanish. Again, as discussed below this observation is relevant to quiescence.

Remark 1.11. There is an interplay between the orthogonality of a G_2 and polarization conditions, as shown in Theorem 2.1 of [57]. Namely, assume a G_2 cosmology has a spacelike, hypersurface-orthogonal Killing vector field tangent to the G_2. Then there exists another spacelike, hypersurface-orthogonal Killing vector field that is both tangent to the G_2 and orthogonal to the first one if and only if the G_2 acts orthogonally transitively. This happens precisely what happens for polarized Gowdy cosmologies, and also for the cosmologies corresponding to orbits in $S_1^+(\mathrm{VI}_0)$ in Paper A and the cosmologies corresponding to orbits in $\mathrm{S}_+ \cap \mathrm{OT}_+$ in Paper B.

1.6 Regimes of quiescence

Before we continue with the summary of the results, we wish to discuss in some more detail the two regimes of quiescence appearing in this thesis, and to mention some of the relevant results. We note of course that this overview of results is far from complete.

Though this thesis is not about oscillatory cosmological singularities, the work by Belinski, Khalatnikov, Lifschitz and their coauthors (BKL) is still important to this story. In a series of articles, cf. e.g. [9, 8] and references therein, BKL set out a framework, known as the BKL conjecture, characterizing the nature of a cosmological singularity in an empty universe. (We refer the reader also to the more recent article [7] by Belinski for a recount of the history of this development.) The framework can be understood effectively in terms of the expansion-normalized Weingarten map $\mathcal{K}$. In particular, the nature of a generic cosmological singularity in vacuum is characterized by the eigenvalues of $\mathcal{K}$ being governed by a specific chaotic one-dimensional dynamical system. The dynamics moreover decouple for different causal curves going into the singularity. This framework has been refined by many other authors, see for example [16, 25, 36]. In the spatially homogeneous setting, in particular for Bianchi type VIII and IX cosmologies, rigorous results have been established concerning this oscillatory nature of the initial singularity/ In particular we note [49, 26] and more recently [6].

Complementary to these oscillatory singularities are quiescent singularities, i.e. a cosmological singularity for which the eigenvalues of the expansion-normalized Weingarten map of a preferred foliation converge along a preferred foliation. Some of the first results on quiescent singularities were obtained by using so-called Fuchsian methods, most notably in [3]. In that article, the authors construct analytic solutions to the Einstein-scalar field and Einstein-stiff fluid equations essentially by prescribing the behaviour of the solution at the singularity. There are many results concerning quiescent solutions following a similar script, cf. e.g. [33], and more recently [22, 1] in the smooth setting. A framework for understanding the solution by its properties at the singularity has been formalized in [46] in the notion of initial data on the singularity, with special cases of solutions with Bianchi class A symmetry or Gowdy symmetry considered in [45].

Quiescence has also been observed in the presence of certain symmetries, even in vacuum or for non-stiff perfect fluids. These solutions tend to have special geometric features which are unstable under perturbations outside of the symmetry class. There are two such conditions that are of particular interest to us, namely the presence of an orthogonally-transitive Abelian G_2 and polarization conditions, as discussed in Subsection 1.5.5. Recall that the Bianchi solutions of lower type, but also the $\mathbb{T}^2$-symmetric solutions, are special cases of G_2 cosmologies. Quiescence has been observed in many cosmologies with an orthogonally-transitive G_2. Concerning the lower Bianchi class A solutions we note in particular that [59, 41, 49] and also the results of Paper A are in this setting. For the Bianchi class B solutions we refer to the results of [31, 40, 39]. Some of the results of Paper B are also relevant here. As mentioned above, the action of the G_2 being orthogonally transitive is also what sets the class of $\mathbb{T}^3$-Gowdy solutions apart within the class of $\mathbb{T}^2$-symmetric solutions; for a comprehensive discussion of the Gowdy solutions we refer to [44]. There exist also various results concerning quiescence in polarized cosmologies and, using Fuchsian methods, solutions have also been constructed that only satisfy such a condition asymptotically. These latter have been called half-polarized in the literature. We note in particular [32, 33, 28, 1, 2], and some of the results of [23] and of Paper B are also relevant in this regard.

On the other hand, a new class of results on quiescence the past decade, namely those proving stable formation of quiescent big bang singularities, by which we mean the past global existence of the development until the blowup of the Kretschmann scalar, and convergence of the eigenvalues of

$\mathcal{K}$. Crucially, these results all require the presence of either a scalar field, a stiff fluid or high dimension. In the symmetric setting there have long been results on quiescence given such matter contents, cf. in particular [49, 39], but not outside of symmetry. We note [53, 54, 20, 11] in the Einstein-scalar field or the Einstein-stiff fluid setting, and [12] in the Einstein-Euler-scalar field setting. In these results the stable formation of quiescent big bang singularities is demonstrated for developments of initial data which is both nearly isotropic and nearly spatially homogeneous. On a similar note we note [52] concerning developments of mildly anisotropic initial data to the Einstein-vacuum equations in high dimension. However, the more relevant result to us, is the result [23] by Fournodavlos, Rodnianski and Speck, concerning the stable formation of quiescent big bang singularities for initial data which is close to one of the Kasner scalar field solutions of Example 1.6. These are in particular spatially homogeneous solutions to the Einstein-scalar field equations that may be highly anisotropic. For the main result of [23] there is moreover an algebraic condition on the eigenvalues of the map $\mathcal{K}$ that needs to be satisfied, though alternatively a polarization condition may be satisfied. Paper C fall in this category of results on stable formation of quiescent big bang singularities, and in fact the same algebraic condition is required there as well, but we postpone a more detailed discussion of the setting to the summary of the results below.

To sum up, two settings in which quiescence has been demonstrated to occur are either in the presence of certain geometrical features due to symmetry, or in the presence of stiff fluid or scalar field matter (or similarly due to a high dimension of spacetime). The heuristics of why quiescence may occur in these regimes were already brought forward by BKL, but may formally be understood using the framework set out in [47]. The results in [47] rely on certain a priori assumptions concerning the behaviour of $\mathcal{K}$ as well as the behaviour of the expansion-normalized normal derivative of $\mathcal{K}$, but notwithstanding those assumptions the framework is still helpful to understand the regimes of quiescence.

For simplicity we shall restrict ourselves to the 3+1-dimensional setting. We consider a frame e_1, e_2, e_3, diagonalizing the expansion-normalized Weingarten map $\mathcal{K}$ of some preferred foliation, with corresponding eigenvalues which, say, satisfy $p_1 \leq p_2 \leq p_3$. Then two sufficient causes for quiescence are identified in [47], cf. Section 3 of [47] in particular. Both of these causes can be understood as turning off an unstable mode related to the growth of a structure coefficient γ^1_{23}. Either all the p_i are positive along the foliation all the way from the initial hypersurface up to the initial singularity, or if p_1 is negative then the structure coefficient γ^1_{23} (associated to the frame diagonalizing $\mathcal{K}$) must vanish identically along the foliation.

The former case is precisely what happens in the results on stable big bang formation, and the presence of a scalar field or stiff fluid matter is required for reasons related to the expansion-normalized Hamiltonian constraint. (Of course we should note that there are examples of Bianchi cosmologies for which that the p_i's are only positive asymptotically.) We discuss more details regarding the expanion-normalized Hamiltonian constraint, in the summary of the results of Paper C.

The latter case is, on the other hand, non-generic and caused by geometrical features due to symmetries. As discussed in Subsection 1.5.5, an orthogonally-transitively acting G_2 is one of these geometrical features, and a hypersurface-orthogonal spacelike Killing vector field is another one. There are moreover examples of cosmologies where the structure coefficient does not vanish identically but only asymptotically, and the presence of the geometrical feature may be understood to be only at the initial singularity, for example the so-called half-polarized solutions which appear e.g. in [33] and this type of solutions also appears in Paper B.

Summary of Results

In this chapter we summarize the results of Papers A, B and C and we provide some insights into the proofs of these results For each paper there is also a discussion section wherein we informally discuss some ideas which might prove relevant for future research.

2.1 Summary of Paper A

In Paper A we study the initial singularity of Bianchi type VI_0 spacetimes solving Einstein's equations with orthogonal perfect fluid matter. We consider a linear equation of state $p = (\gamma - 1)\rho$ where $\gamma \in (2/3, 2)$. Bianchi type VI_0 orthogonal perfect fluid cosmologies are of interest, for example, because they form an important building block to the understanding of the dynamics of type VIII cosmologies. But they are also of interest because they have geometrical features of interest, such as an orthogonally-transitive G_2 and may potentially have hypersurface-orthogonal Killing vector fields, cf.Subsection 1.5.5.

The results in Paper A are twofold. On the one hand, we obtain information concerning the asymptotics of expansion-normalized quantities toward the initial singularity of Bianchi type VI_0 cosmologies with orthogonal perfect fluids, in particular concerning the expansion-normalized Weingarten map. The main result is not unexpected but in fact resolve the conjecture in Section 6.3.3 of [58], and is stated as Theorem 2.1 below. In words, the result may be stated as follows: *The initial singularity of a Bianchi* *type* VI_0 *orthogonal perfect fluid cosmology is generically vacuum-dominated, anisotropic, silent and quiescent.* The following conclusions, which we state here for their contrast with the results of Paper B, also hold: *The past limits of the eigenvalues of the expansion-normalized Weingarten map exist. One of the limits is negative, and it is the limit of the eigenvalue whose eigenspace coincides with the direction orthogonal to the Abelian G_2.*

On the other hand, we apply these results to obtain information concerning the Klein-Gordon equation in case that these type VI_0 cosmologies form the background. The resulting statement may be phrased as follows: *If u is a solution to the Klein-Gordon equation on a generic Bianchi type* VI_0 *orthogonal perfect fluid cosmology background, then the time derivative u_τ with respect to the expansion-normalized time coordinate converges exponentially to a smooth function on the underlying group towards the initial singularity.* This completes results of [43], cf. Example 29 and Section 15.2, where Bianchi type VI_0 in particular was excluded due to a lack of results on the initial singularity for the type VI_0 orthogonal perfect fluid cosmologies.

2.1.1 The initial singularity of a generic solution

We recall from Section 1.5 of the introduction that for a Bianchi type VI_0 Lie group the matrix $\hat{n}$ has two non-zero eigenvalues with opposite signs. Hence, in the system of equations (1.35) we may specialize to the case in which $N_{11} = 0, N_{22} > 0, N_{33} < 0$. Then, by letting

$$N_\pm = \tfrac{\sqrt{3}}{2}\left(N_{22} \pm N_{33}\right), \tag{2.1}$$

we obtain the following system of equations:

$$\begin{aligned}
N_+' &= (q + 2\Sigma_+)N_+ + 2\sqrt{3}\Sigma_- N_-,\\
N_-' &= (q + 2\Sigma_+)N_- + 2\sqrt{3}\Sigma_- N_+,\\
\Sigma_+' &= -(2-q)\Sigma_+ - 2N_-^2,\\
\Sigma_-' &= -(2-q)\Sigma_- - 2\sqrt{3}N_+N_-,\\
\Omega' &= 2(q - q^*)\Omega,
\end{aligned} \tag{2.2}$$

and the Hamiltonian constraint becomes

$$\Omega + \Sigma_+^2 + \Sigma_-^2 + N_-^2 = 1. \tag{2.3}$$

Moreover, the *deceleration parameter* q is now given by

$$q := q^*\Omega + 2(\Sigma_+^2 + \Sigma_-^2), \tag{2.4}$$

where $q^* = q^*(\gamma) = \frac{1}{2}(3\gamma - 2)$. The flow of the system of equations (2.2) is denoted by $\tau \mapsto \varphi^\tau$ and we define the *phase space* for this system, now a hypersurface in $\mathbb{R}^5$, to be the set

$$B_1^+(\mathrm{VI}_0) := \left\{ (N_+, N_-, \Sigma_+, \Sigma_-, \Omega) \in \mathbb{R}^5 : \begin{array}{l} \Omega + \Sigma_+^2 + \Sigma_-^2 + N_-^2 = 1, \\ N_- > |N_+|,\ \Omega \geq 0 \end{array} \right\}. \tag{2.5}$$

The closure of the phase space contains several isolated fixed points, of which all but one reside in the boundary. The boundary consists of the Bianchi type I perfect fluid and vacuum orbits, the Bianchi type II perfect fluid and vacuum orbits with either $N_{22} > 0$ or $N_{33} < 0$, as well as Bianchi type VI_0 vacuum orbits with $N_{22} > 0, N_{33} < 0$. We list the fixed points in Table 2.1; a corresponding table can be found in Section 6.2 of [58]. Note that all of these isolated fixed points have non-zero matter density. Here $p^* = p^*(\gamma)$ stands for

$$p^*(\gamma) := \sqrt{q^*(2 - q^*)} \in (0, 1).$$

The column corresponding to u denotes the dimension of the unstable manifold relative to the phase space; note that all these fixed points are isolated and hyperbolic for $\gamma \in (2/3, 2)$.

The invariant set $S_1^+(\mathrm{VI}_0)$ is the shear-invariant set (which is similar to the set of polarized orbits encountered in Paper B) and in terms of the variables above corresponds to $\Sigma_- = 0, N_+ = 0$. The invariant sets $S_2^+(\mathrm{II})$

Symbol	$(N_+, N_-, \Sigma_+, \Sigma_-, \Omega)$	Invariant set	u
F	$(0, 0, 0, 0, 1)$	$B(\mathrm{I})$	2
$P_2^+(\mathrm{II})$	$(\frac{1}{4}p^*, \frac{1}{4}p^*, -\frac{1}{8}q^*, -\frac{\sqrt{3}}{8}q^*, 1 - \frac{1}{8}q^*)$	$S_2^+(\mathrm{II})$	1
$P_3^-(\mathrm{II})$	$(\frac{1}{4}p^*, -\frac{1}{4}p^*, -\frac{1}{8}q^*, \frac{\sqrt{3}}{8}q^*, 1 - \frac{1}{8}q^*)$	$S_3^-(\mathrm{II})$	1
$P_1^+(\mathrm{VI}_0)$	$(0, \frac{1}{2}p^*, -\frac{1}{2}q^*, 0, \frac{1}{2}(2 - q^*))$	$S_1^+(\mathrm{VI}_0)$	0

Table 2.1: (Paper A, Table 2). An overview of the isolated fixed points in the closure of the phase space $\mathrm{cl}\, B_1^+(\mathrm{VI}_0)$.

and $S_3^-(\mathrm{II})$ are the shear-invariant sets for the Bianchi type II orbits, given respectively by $\Sigma_+ = \pm\frac{1}{\sqrt{3}}\Sigma_-$ and corresponding to locally rotationally symmetric cosmologies.

The closure of the phase space also contains a circle of fixed points, called the *Kasner circle* $\mathcal{K}$.[1] It is precisely the unit circle in the (Σ_+, Σ_-)-plane and the corresponding solutions are vacuum solutions of Bianchi type I, corresponding to the Kasner solutions. For the statement of Theorem 2.1 below, let us note that the Kasner circle contains three special points, $T_i, i \in \{1,2,3\}$, also known as the Taub points and corresponding to the flat Kasner solutions. After removing these from $\mathcal{K}$ we are left with three components called $\mathcal{K}_j, j = 1,2,3$. The arcs $\mathcal{K}_j$ can be characterized alternatively as $\mathcal{K}_j := \mathcal{K} \cap \{\beta_j < 0\}$, where the β_j are

$$\begin{aligned} \beta_1 &:= q - 4\Sigma_+, \\ \beta_2 &:= q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-, \\ \beta_3 &:= q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-, \end{aligned}$$

cf. Definition 6.1 of [49]. Note that $q|_{\mathcal{K}} \equiv 2$, by the Hamiltonian constraint (2.3). In particular, we have exponential decay of N_j near the corresponding arc $\mathcal{K}_j$ for $j = 1,2,3$, since the β_j amount exactly to $\ln(N_j)'$. The quantities β_j are strictly positive on the arcs $\mathcal{K}_i$ if $i \neq j$ and also satisfy $\beta_i(T_j) = 6\delta_{ij}$, $i,j = 1,2,3$. See Figure 2.1 for a depiction of the arc $\mathcal{K}_1$ relative to the Taub points and the rest of the Kasner circle.

We are now ready to formulate the main result which concerns the convergence of the backward orbits, and resolves the conjecture in Section 6.3.3 of [58] in the affirmative.

Theorem 2.1 (Paper A, Theorem 1.6)**.** *Let $x \in B_1^+(\mathrm{VI}_0)$ with $\Omega(x) > 0$ and $\gamma \in (2/3, 2)$. Then, either*

(a) x is the equilibrium $P_1^+(\mathrm{VI}_0)$,

(b) x lies in the unstable manifold of one of the equilibria F, $P_2^+(\mathrm{II})$ or $P_3^-(\mathrm{II})$, or

(c) $\varphi^\tau(x)$ converges to a point in $\mathcal{K}_1$ as $\tau \to -\infty$.

[1] This symbol is also used for the expansion-normalized Weingarten map in the introduction, as well as in the summary of Paper C, but in Paper A only refers to the Kasner circle.

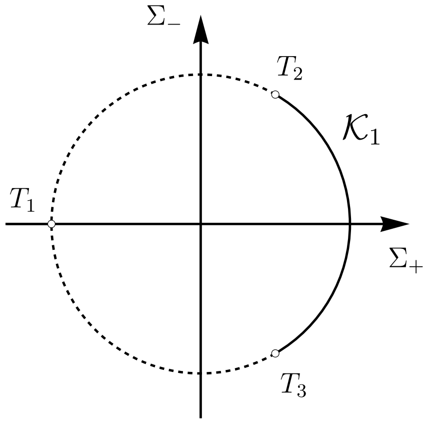

Figure 2.1: (Paper A, Figure 1). The arc $\mathcal{K}_1$ relative to the Taub points $T_i, i \in \{1, 2, 3\}$ as well as the rest of the Kanser circle (dashed).

The unstable manifolds of case *(b)* are of co-dimension at least two, so it follows that case *(c)* is generic. To be precise, the set of orbits satisfying *(c)* is an open and dense subset of full measure in the expansion-normalized phase space and is thus Lebesgue and Baire generic.

2.1.2 Intuition and sketch of the proof of Theorem 2.1

The intuition behind the proof of Theorem 2.1 is that $\mathcal{K}_1$ acts as a source. Consider the full set of evolution equations (1.35), i.e. without the specialization that $N_{11} = 0$. The non-special fixed points on the Kasner circle always have one negative eigenvalue β_j with eigenspace in the direction $\partial_{N_{jj}}$, $j \in \{1, 2, 3\}$. What we prove in effect is that the extended phase space $\mathrm{cl}\, B_1^+(\mathrm{VI}_0)$, i.e. the subset $N_{11} = 0$, is the centre-unstable manifold to $\mathcal{K}_1$ of certain Bianchi type VIII strata. The fact that $\mathcal{K}_1$ acts as a source is reflected in the global dynamics of (2.2), because if Σ_+ is ever positive, then it decreases monotonically. From there convergence of the backward orbits follows if Σ_+ ever becomes positive. Indeed, by rewriting the evolution equation for Σ_+ using the Hamiltonian constraint, we see that

$$\Sigma_+' \leq -(2 - q^*)\Omega\Sigma_+.$$

In practice the proof proceeds through a series of vacuous statements, in the sense that we the assumptions never occur yet we need the implication. One may deduce that there must be some α-limit point in $\mathcal{K}_1$ for points outside of the unstable manifolds of case *(b)* or the fixed point $P_1^+(\mathrm{VI}_0)$ of case *(a)*. Then one may conclude convergence to that point. By using monotone functions, in particular the function

$$Z_1 := (\Sigma_-^2 + N_+^2)/(N_-^2 - N_+^2),$$

one can show that the α-limit set of any orbit consists of type I and II points. Then by invariance of the α-limit set, the α-limit set contains a point on the Kasner circle $\mathcal{K}$, which we moreover show is not one of the special points $T_i, i \in \{1, 2, 3\}$.

Next, we use the Kasner map K, which takes the ω-limit point of a vacuum type II orbit and produces the corresponding α-limit point. The Kasner map also plays an important role in describing the α-limit set of Bianchi type VI_0, and we have the following lemma.

Lemma 2.2 (Paper A, Lemma 2.9)**.** *Assume $x \in B_1^+(\mathrm{VI}_0)$ for $\gamma \in (2/3, 2)$ has an α-limit point $y \in \mathcal{K}_2 \cup \mathcal{K}_3$. Then the image $K(y)$ of y under the Kasner map is contained in $\alpha(x)$ as well.*

If this α-limit point is in $\mathcal{K}_2$ or $\mathcal{K}_3$, then we may apply the Kasner map a finite number of times to obtain an α-limit point in $\mathcal{K}_1$. It follows that we must have an α-limit point in $\mathcal{K}_1$ as well. Using the monotonicity of Σ_+ we obtain in the end convergence to this latter α-limit point.

2.1.3 The Klein-Gordon equation on generic backgrounds

Next, we consider the results regarding the Klein–Gordon equation on type VI_0 orthogonal perfect fluid developments as backgrounds[2] by combining results from [43] with Theorem 2.1 above. We recall the notions of a Bianchi spacetime, of a Bianchi orthogonal perfect fluid data as well as the corresponding development from Section 1.5.

There are two time-coordinates appearing in the statement of the theorem, for which we require a preliminary to define them. Let (M, h) be a Bianchi spacetime, cf. Definition 1.9. Then t_- is said to be a *monotone volume singularity* if there exists a time $t_0 \in I$ such that the mean curvature $\theta(t)$ of

[2]In other words, the Klein–Gordon equation is not coupled to Einstein's equations.

the hypersurface $G \times \{t\}$ is strictly positive on the interval (t_-, t_0), and if the function τ, defined as

$$\tau(t) := \tfrac{1}{3} \log \sqrt{\det a(t)}, \tag{2.6}$$

satisfies $\tau(t) \to -\infty$ as $t \to t_-$. The function τ is known as the *logarithmic volume density* and satisfies $\tau'(t) = \theta/3$. In what follows we employ τ as a new time-coordinate. It is also precisely the expansion-normalized time that appears in the formulation of the system of equations (2.2).

In Theorem 2.3 we also encounter the time coordinate σ, defined implicitly by

$$\frac{d\sigma}{dt} = \frac{1}{3} \frac{1}{\sqrt{\det a(t)}}, \tag{2.7}$$

together with the condition that $\sigma(t) = 0$ for the same time t for which $\tau(t) = 0$. We note that if the Lie group is of type VI_0 then $\sigma \to -\infty$ corresponds to $\tau \to -\infty$, cf. Lemma 52 of [43].

Recall that for a Lorentz manifold (M, h), the Klein-Gordon equation reads

$$\Box_h u = m^2 u \tag{2.8}$$

where $m \in \mathbb{R}$ is a constant. In the statement of Theorem 2.3 below $\langle \sigma \rangle := \sqrt{1 + \sigma^2}$ denotes the Japanese bracket. Moreover, with generic data we mean data that corresponds to case *(c)* in Theorem 2.1.

Theorem 2.3 (Paper A, Theorem 7.1)**.** *Let (G, g, k, ρ) be Bianchi orthogonal perfect fluid data for $\gamma \in (\frac{2}{3}, 2)$, with G of type VI_0 and with $\rho > 0$. Assume that the data are generic and denote the corresponding development by (M, h).*

Then there exist constants $\lambda, \nu > 0$ such that for any smooth solution $u \in C^\infty(M)$ of the Klein-Gordon equation (2.8) on (M, h), there are smooth functions $v, w, \phi, \psi \in C^\infty(G)$, such that for each compact subset $K \subseteq G$ and for each $l \in \mathbb{N}_0$, we have

$$\|u_\tau(\cdot, \tau) - v\|_{C^l(K)} + \|u(\cdot, \tau) - \tau \cdot v - w\|_{C^l(K)} \leq C_{K,l} \langle \tau \rangle e^{\lambda \tau} \tag{2.9}$$

for all $\tau \leq 0$, as well as

$$\|u_\sigma(\cdot, \sigma) - \phi\|_{C^l(K)} + \|u(\cdot, \sigma) - \sigma \cdot \phi - \psi\|_{C^l(K)} \leq C_{K,l} \langle \sigma \rangle e^{\nu \sigma} \tag{2.10}$$

for all $\sigma \leq 0$.

In the above, $C_{K,l} > 0$ denotes a constant depending only on K and l. Relevant results can be formulated also for case *(b)* of Theorem 2.1, see Example 35 of [43].

2.1.4 Discussion

Just like the class B Lie groups, a Lie group of Bianchi type VI_0 has a two-dimensional Abelian subgroup. This gives rise to an Abelian subgroup G_2 of the isometry group in a Bianchi type VI_0 cosmology. As explained in Subsection 1.5.5, in the case of an orthogonal perfect fluid, the g_2 necessarily acts orthogonally-transitively, and this fact is related to the quiescence which occurs for Bianchi type VI_0 orthogonal perfect fluid cosmologies. Moreover, the shear-invariant orbits correspond to cosmologies with a hypersurface-orthogonal Killing vector field just like the polarized orbits that are encountered in Paper B. In particular, groups of type VI_0 and type $\mathrm{VI}_{-1/9}$ share common geometrical features, and the same holds for type VI_0 and type $\mathrm{VI}_{-1/9}$ cosmologies. However, these do not appear as prominently in type VI_0 cosmologies in the special case of an orthogonal perfect fluid. Therefore it would be of interest to compare type VI_0 cosmologies with non-isotropic matter, for example a tilted fluid or a magnetic field, with certain type $\mathrm{VI}_{-1/9}$ cosmologies. The are various results on the tilted type VI_0 cosmologies, cf. [29, 27, 14], and also on the magnetic type VI_0 cosmologies, cf. [35, 61]. If the Abelian G_2 does not act orthogonally-transitively then we expect similar dynamics to occur as in the case of Bianchi type $\mathrm{VI}_{-1/9}$. Bianchi type $\mathrm{VI}_{-1/9}$ cosmologies are the subject of Paper B, though with orthogonal stiff fluid matter suppressing the oscillatory behaviour. It is of interest to understand these links and perhaps even equivalences between dynamics. This could yield insights in favourable choices of the propagation of the orthonormal frame, or as a step forward to understand the attractor for the Bianchi type $\mathrm{VI}_{-1/9}$ vacuum cosmologies, cf. [30].

2.2 Summary of Paper B

In Paper B we consider orthogonal stiff fluid cosmologies of the exceptional Bianchi type $\mathrm{VI}_{-1/9}$. Type $\mathrm{VI}_{-1/9}$ is known as *exceptional* because, both in vacuum or when Einstein's equations are coupled to an orthogonal perfect fluid, the momentum constraints have a degeneracy for cosmologies of this type. As a result, the Abelian subgroup G_2 of the isometry group does not necessarily act orthogonally-transitively. This is manifested in the description of the expansion-normalized variables in one more degree of freedom compared to cosmologies of the other Bianchi types VI_η and VII_η, $\eta \in \mathbb{R}$, and thus having the same degrees of freedom as vacuum or orthogonal perfect fluid cosmologies of types VIII or IX.

The main result of Paper B may be stated as follows: *Generically, the initial singularity of a Bianchi type* $\mathrm{VI}_{-1/9}$ *cosmology with orthogonal stiff fluid matter is anisotropic, silent, quiescent and contracting in all directions. In particular, the limits of the eigenvalues of the expansion-normalized Weingarten map exist, are generically distinct, and are all strictly positive.* The quiescence that is shown to occur is generically the result of the presence of stiff fluid matter, whose expansion-normalized energy density increases monotonically toward the past, and related to the regime of quiescence characterized by the eigenvalues of $\mathcal{K}$ being strictly positive.

There are also secondary mechanisms leading to quiescence that are important for a non-generic set of spacetimes. On the one hand, there is the possibility of the G_2 acting orthogonally-transitively, which is also the case for the cosmologies under consideration in Paper A. On the other hand, one of the Killing vector fields of the spacetime may be hypersurface-orthogonal, which gives rise to the set of polarized orbits, analogous to the shear-invariant set encountered in Paper A. The results concerning these two types of orbits may be summarized as follows: *There are co-dimension one sets of orbits for which the smallest of the limits of the eigenvalues of the expansion-normalized Weingarten map is strictly negative, for which either the* G_2 *acts orthogonally-transitively in the corresponding spacetime, or which asymptotically satisfy a polarization condition.* This is in contrast with cosmologies of type VIII or IX, where for all orbits the limits of the eigenvalues of the expansion-normalized Weingarten map are all three strictly positive.

Another important result is the system of equations itself, which is based on choosing two sets of angular coordinates. In particular, we obtain a system of evolution equations with five variables which only reflect the inherent degrees of freedom of the system, along with two constraints, which can be written as two functions in terms of the five variables. The formulation has the added benefit that the functions $M, P, \sin(\theta)$ and Ω have homogeneous evolution equations. In fact, using a product of powers of these functions we obtain that, in Bianchi type $\mathrm{VI}_{-1/9}$ orthogonal perfect fluid cosmologies with $\gamma \geq 4/3$, the expansion-normalized energy density Ω vanishes toward the expanding direction.

2.2.1 The evolution equations and the phase space

From the usual system of expansion-normalized evolution equations specialized to a stiff fluid, i.e. $\gamma = 2$, we may obtain the following set of evolution equations:

$$M' = \big[q + 2\Sigma_+ + 2\sqrt{3}\cos(\theta)\Sigma_-\big]M, \tag{2.11a}$$

$$P' = \big[-(2-q) - (3+\sin^2(\theta))\Sigma_+ + \sqrt{3}\cos(\theta)\Sigma_-\big]P, \tag{2.11b}$$

$$\theta' = -2\sqrt{3}\Sigma_- \sin(\theta), \tag{2.11c}$$

$$\Sigma_+' = -(2-q)\Sigma_+ - 2M^2 + 3P^2, \tag{2.11d}$$

$$\Sigma_-' = -(2-q)\Sigma_- - \sqrt{3}\cos(\theta)\big(2M^2 + P^2 - \tfrac{2}{3}\sin^2(\theta)\Sigma_+^2\big), \tag{2.11e}$$

where q is here shorthand for

$$q = 2 - 2\big(1 + \tfrac{1}{3}\sin^2(\theta)\big)M^2. \tag{2.11f}$$

We also have the constraints defining Ω and $\Sigma_\times$, namely

$$\Sigma_\times = \tfrac{1}{\sqrt{3}}\Sigma_+ \sin(\theta), \tag{2.11g}$$

$$\Omega = 1 - \big(1 + \tfrac{1}{3}\sin^2(\theta)\big)\big(M^2 + \Sigma_+^2\big) - P^2 - \Sigma_-^2, \tag{2.11h}$$

and the corresponding auxiliary equations

$$\Sigma_\times' = [-(2-q) - 2\sqrt{3}\Sigma_- \cos(\theta)]\Sigma_\times - \tfrac{1}{\sqrt{3}}\sin(\theta)\big(2M^2 - 3P^2\big), \tag{2.11i}$$

$$\Omega' = -2(2-q)\Omega. \tag{2.11j}$$

The derivation of this system of equations may be found in Appendix B of Paper B. Here we note that M is the expansion-normalized difference of the eigenvalues of the matrix n. We recall for type $\mathrm{VI}_{-1/9}$ the eigenvalues of n have opposite signs by the Bianchi classification. The variable P represents the extra degree of freedom due to the G_2 not acting orthogonally-transitively, and $\cos(\theta)$ is the ratio of the sum and the difference of the eigenvalues of n. Note also that $\tan(\theta)$ is the ratio of the squares of the off-diagonal shear.

Due to symmetries, we may assume $M \geq 0$, $P \geq 0$ and $\sin(\theta) \geq 0$, and naturally we shall consider only non-negative energy densities so that $\Omega \geq 0$. The sets $\{M = 0\}$ and $\{\sin(\theta) = 0\}$ correspond respectively to type I and type II orbits. By the above considerations we define the *phase space* to be the set

$$\mathrm{B}_+ := \Big\{(M, P, \theta, \Sigma_+, \Sigma_-) \in \mathbb{R}^5 \mid M > 0, P \geq 0, \theta \in (0,\pi), \Omega \geq 0\Big\}. \tag{2.12}$$

We also distinguish the vacuum orbits $\mathrm{B}_+^0 = \mathrm{B}_+ \cap \{\Omega = 0\}$ and the non-vacuum orbits $\mathrm{B}_+^\Omega = \mathrm{B}_+^\Omega \cap \{\Omega > 0\}$. Moreover, we denote

$$\mathrm{OT}_+^\Omega := \mathrm{B}_+^\Omega \cap \{P = 0\}, \tag{2.13}$$

for the set of non-vacuum orthogonally-transitive orbits, as well as

$$\mathrm{S}_+^\Omega := \mathrm{B}_+^\Omega \cap \{\cos(\theta) = 0 = \Sigma_-\} \tag{2.14}$$

for the set of non-vacuum polarized orbits.

2.2.2 Fixed points and the convergence of backward orbits

In the boundary of the phase space, in particular the non-vacuum boundary, there exist fixed points which play an important role in describing the character of the initial singularity. In particular, we have the following subsets fixed points in of $(\mathrm{cl}\,\mathrm{B}_+^\Omega) \setminus (\mathrm{cl}\,\mathrm{B}_+^0)$, namely

$$\mathcal{D}^+ := \left\{(0,0,0,\Sigma_+,\Sigma_-) \mid \Sigma_+^2 + \Sigma_-^2 < 1\right\}, \tag{2.15a}$$

$$\mathcal{D}^- := \left\{(0,0,\pi,\Sigma_+,\Sigma_-) \mid \Sigma_+^2 + \Sigma_-^2 < 1\right\}, \tag{2.15b}$$

$$\mathcal{F} := \{(0,0,\theta,0,0) \mid \theta \in [0,\pi]\}, \tag{2.15c}$$

$$\mathcal{P} := \left\{(0,0,\pi/2,\Sigma_+,0) \mid \Sigma_+^2 < 3/4\right\}. \tag{2.15d}$$

The projection of the discs $\mathcal{D}^\pm$ on the $\Sigma_+\Sigma_-$-plane coincides with the more familiar Jacobs disc defined by $\mathcal{D} := \{(\Sigma_+,\Sigma_-) : \Sigma_+^2 + \Sigma_-^2 < 1\}$. Note that $\mathcal{D}^+, \mathcal{D}^-$ and $\mathcal{P}$ are connected through $\mathcal{F}$, in particular $\mathcal{D}^\pm \cap \mathcal{F} \neq \emptyset$ as well as $\mathcal{P} \cap \mathcal{F} \neq \emptyset$. For the conclusions that follow, it is convenient to define certain subsets of $\mathcal{D}^\pm$ depending on which eigenvalue of the expansion-normalized shear is largest. Of importance for the orbits OT_+^Ω are the subsets of $\mathcal{D}^\pm$ given by

$$\mathcal{M}_{23}^- = \left\{(0,0,\pi,\Sigma_+,\Sigma_-) \mid \Sigma_+^2 + \Sigma_-^2 < 1, \Sigma_+ - \sqrt{3}\Sigma_- > -1 \text{ and } \Sigma_- > 0\right\},$$

$$\mathcal{M}_{32}^+ = \left\{(0,0,0,\Sigma_+,\Sigma_-) \mid \Sigma_+^2 + \Sigma_-^2 < 1, \Sigma_+ + \sqrt{3}\Sigma_- > -1 \text{ and } \Sigma_- < 0\right\},$$

both of which will play a role in the subsequent analysis. The projections of $\mathcal{M}_{23}^\pm$ and $\mathcal{M}_{32}^\pm$ onto the Jacobs disc $\mathcal{D}$, i.e. in the $\Sigma_+\Sigma_-$-plane, are depicted in Figure 2.2.

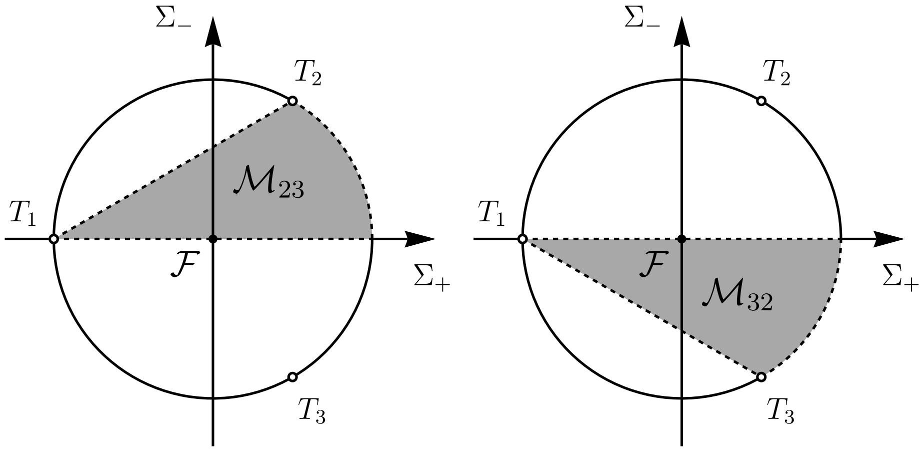

Figure 2.2: (Paper B, Figure 1). The projections of the sets $\mathcal{M}_{23}^{\pm}$ (left) and $\mathcal{M}_{32}^{\pm}$ (right) onto the Jacobs disc $\mathcal{D}$ in the $\Sigma_+\Sigma_-$-plane. Note that the boundaries of the regions are not part of the respective sets. The projection of the set $\mathcal{F}$, in the center of the circle, is also indicated.

Also the following two subsets of $\mathcal{D}^-$ and $\mathcal{D}^+$ play an important role in the statement of the results:

$$\Delta_{123}^{-} = \left\{(0,0,\pi,\Sigma_+,\Sigma_-) \mid \Sigma_+ < -\tfrac{1}{\sqrt{3}}\Sigma_- < 0 \text{ and } \Sigma_+ - \sqrt{3}\Sigma_- > -1\right\},$$
$$\Delta_{132}^{+} = \left\{(0,0,0,\Sigma_+,\Sigma_-) \mid \Sigma_+ < \tfrac{1}{\sqrt{3}}\Sigma_- < 0 \text{ and } \Sigma_+ + \sqrt{3}\Sigma_- > -1\right\}.$$

The projections of the sets $\Delta_{123}^{\pm}$ and $\Delta_{132}^{\pm}$ onto the Jacobs disc $\mathcal{D}$, i.e. in the $\Sigma_+\Sigma_-$-plane, are shown in Figure 2.3.

For an orthogonal stiff fluid, the expansion-normalized matter density Ω is monotonically increasing toward the past, and from this fact it may be shown that any past orbit converges to a point in one of the above sets. In the proposition that follows, we note that $X := \sqrt{3}\Sigma_- \cos(\theta)$.

Proposition 2.4 (Paper B, Proposition 3.5)**.** *Let* $x \in \mathrm{B}_+^{\Omega}$ *for* $\gamma = 2$*. Then* $\lim_{\tau\to-\infty}\varphi^{\tau}(x)$ *exists and* $\lim_{\tau\to-\infty} X(\tau) \le 0$*. Moreover, we have that either*

(a) $\lim_{\tau\to-\infty}\varphi^{\tau}(x) \in (\mathcal{D}^+ \cup \mathcal{D}^-) \setminus \mathcal{F}$*, or*

(b) $\lim_{\tau\to-\infty}\varphi^{\tau}(x) \in \mathcal{P} \setminus \mathcal{F}$*, or*

(c) $\lim_{\tau\to-\infty}\varphi^{\tau}(x) \in \mathcal{F}$*.*

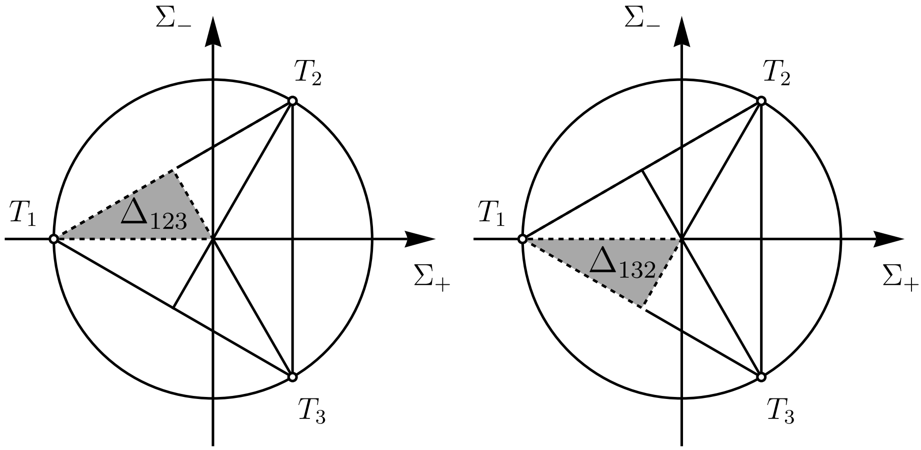

Figure 2.3: (Paper B, Figure 2). The projections of the sets $\Delta^{\pm}_{123}$ (left) and $\Delta^{\pm}_{132}$ (right) onto the Jacobs disc $\mathcal{D}$ in the $\Sigma_+\Sigma_-$-plane. Note that the boundaries of the regions are not part of the respective sets. The triangle, given by the three line segments corresponding to one of the eigenvalues of the expansion-normalized Weingarten map vanishing, is also depicted, as well as the Taub points lying at their intersections.

This proposition thus establishes a trichotomy of asymptotic behaviour. In particular, we may define three sets of orbits, based on their asymptotic behaviour, that together encapsulate all of the phase space. These are the set AD^{Ω}_{+} of *asymptotically-diagonalizing* orbits, the set HP^{Ω}_{+} of *half-polarized* orbits, and the set IS^{Ω}_{+} of (past) *isotropizing* orbits, defined as

$$
\begin{aligned}
\mathrm{AD}^{\Omega}_{+} :=& \{x \in \mathrm{B}^{\Omega}_{+} \mid \lim_{\tau \to -\infty} \varphi^{\tau}(x) \in (\mathcal{D}^{+} \cup \mathcal{D}^{-}) \setminus \mathcal{F}\},\\
\mathrm{HP}^{\Omega}_{+} :=& \{x \in \mathrm{B}^{\Omega}_{+} \mid \lim_{\tau \to -\infty} \varphi^{\tau}(x) \in \mathcal{P} \setminus \mathcal{F}\},\\
\mathrm{IS}^{\Omega}_{+} :=& \{x \in \mathrm{B}^{\Omega}_{+} \mid \lim_{\tau \to -\infty} \varphi^{\tau}(x) \in \mathcal{F}\},
\end{aligned}
$$

respectively We note that $\mathrm{S}^{\Omega}_{+} \subset \mathrm{HP}^{\Omega}_{+}$ and it is a corollary of Lemma 4.2 of Paper B that $\mathrm{IS}^{\Omega}_{+} \subset \mathrm{OT}^{\Omega}_{+}$.

2.2.3 The initial singularity for a generic set of orbits

We may now state the main result.

Theorem 2.5 (Paper B, Theorem 6.1)**.** *For $\gamma = 2$, any backward orbit in the invariant set $\mathrm{AD}_+^\Omega \setminus \mathrm{OT}_+^\Omega$, which is a subset of B_+^Ω of full measure that moreover contains a countable intersection of dense and open subsets of B_+^Ω, converges to a point in $\Delta_{123}^- \cup \Delta_{132}^+$.*

In particular, for the orbits in $\mathrm{AD}_+^\Omega \setminus \mathrm{OT}_+^\Omega$ the past limits of the eigenvalues of the expansion-normalized Weingarten map exist, are distinct and are all strictly positive.

Let us briefly discuss the logical structure leading to this result. Once Proposition 2.4 above is established, the proof of Theorem 2.5 follows in a straightforward manner. In Section 4 of Paper B we obtain Proposition 2.6 below, which is concerned with the behaviour of exceptional asymptotically diagonalizing orbits. In practice the proof is built up as a sequence of lemmata each of which place restrictions on the asymptotic data. In this process special care is needed to exclude convergence to the boundary cases.

Proposition 2.6 (Paper B, Theorem 4.7)**.** *Let $x \in \mathrm{AD}_+^\Omega \setminus \mathrm{OT}_+^\Omega$ for $\gamma = 2$. Then $\varphi^\tau(x)$ converges to a point in $\Delta_{123}^- \cup \Delta_{132}^+$ as $\tau \to -\infty$. In particular, the limits of the eigenvalues of the expansion-normalized Weingarten map exist, are distinct and strictly positive.*

After noting that OT_+^Ω is a smooth submanifold of co-dimension 1, the only part that is left is showing that the half-polarized orbits are non-generic. This is done in Section 5 of Paper B, from which we note in particular the following proposition.

Proposition 2.7 (Paper B, Proposition 5.3)**.** *The set HP_+^Ω of half-polarized non-vacuum orbits, for $\gamma = 2$, is contained in a countable union of C^r submanifolds of B_+^Ω of co-dimension 1, where r may be chosen to be any positive integer.*

Moreover, there exists a C^r submanifold of B_+^Ω of co-dimension 1, where r is as above, consisting of half-polarized orbits for which the smallest limit of the eigenvalues of the expansion-normalized Weingarten map is strictly negative.

The second statement in Proposition 2.7 is included explicitly in order to indicate that it is not only orthogonally-transitively of the G_2 that may lead to non-positive limits of the eigenvalues of the expansion-normalized Weingarten map $\mathcal{K}$. There are also half-polarized solutions with this property which do not have an analogue in the orthogonal stiff fluid Bianchi type VIII and IX cosmologies.

2.2.4 Discussion

This article completes the analysis of orthogonal stiff fluid cosmologies of all Bianchi types. However, besides [28, 30], there are scarce results concerning the vacuum and non-stiff orthogonal perfect fluid case for Bianchi type $\mathrm{VI}_{-1/9}$. The nature of the initial singularity for both of those cases is expected to be markedly different, because oscillatory as opposed to quiescent behaviour is expected to hold generically for both cases. cf. in particular Section 5 of [30]. We expect the presented system of evolution equations to be useful for progress in understanding the initial singularity for the vacuum and non-stiff perfect fluid cases as well. In particular, using the monotonic functions from Section 7 of Paper B, one can make some conclusions both on the future and past of orthogonal perfect fluid cosmologies with $\gamma \geq 4/3$, i.e. with relatively high pressure. However, if it can be shown that $\Sigma_+ \cos(\theta)^2 \geq 0$ in some invariant set, then the range for which these conclusions hold can be pushed to $\gamma > 10/9$, at least within this invariant set. This is of interest because at the value $\gamma = 10/9$ a bifurcation occurs in the global dynamics. In particular, for $\gamma = 10/9$ a new family of fixed points emerges, the so-called Wainwright arc of equilibrium points, and as we pass through the $\gamma = 10/9$ there is a change in the stability of a certain isolated non-vacuum fixed point which is expected to be the global future attractor for $\gamma \in (2/3, 10/9)$. We refer to Section 3 of [30] for more details on this bifurcation.

On a different note, it would be of great interest to be able to characterize the half-polarized solutions in terms of initial data. So far, these types of quiescent solutions, i.e. those that are not polarized but have similar asymptotic behaviour, have been characterized only at the singularity, on the one hand using Fuchsian methods, for example in [33], and on the other hand using centre-unstable manifolds (which can be seen as a poor man's Fuchsian method), such as in [40] and [39], as well as in Paper B itself.

Lastly, we wish to comment that vacuum and orthogonal perfect fluid cosmologies of Bianchi type $\mathrm{VI}_{-1/9}$ are of interest for studies in the inhomogeneous setting, such as encountered in Paper C, because of the complications that arise when choosing how to propagate the frame. In that sense the current analysis is somewhat unfortunate; we make a choice of frame that can not be translated directly to the inhomogeneous setting, as two of the frame vectors are chosen to be tangent to the Abelian G_2, and the other orthogonal to it. This relies on the fact that $\hat{n}^{ij}\hat{a}_j = 0$ as a consequence of the Jacobi identity, but this is in a sense a peculiarity of the spatially homogeneous setting.

2.3 Summary of Paper C

Paper C is concerned with developments of CMC initial data to the Einstein-nonlinear scalar field equations. The initial data is not assumed to be isotropic or homogeneous; instead the assumptions place the initial data in a regime in which quiescent singularities are expected to generally occur. One of the assumptions is an algebraic condition on the eigenvalues $\bar{p}_1, ..., \bar{p}_n$ of the expansion-normalized Weingarten map $\bar{\mathcal{K}}$ of the initial data, namely for $I \neq J$:

$$\bar{p}_I + \bar{p}_J - \bar{p}_K < 1. \tag{2.16}$$

In the four-dimensional setting, this condition is equivalent to all the eigenvalues being positive as is also the case for the limits of the eigenvalues in the generic case of Paper B. The other important assumption is the demand of certain bounds on the expansion-normalized quantities $\bar{\mathcal{H}}, \bar{\mathcal{K}}, \bar{\Phi}_1$ and $\bar{\Phi}_0$. There are also technical assumptions concerning non-degeneracy of the eigenvalues of $\bar{\mathcal{K}}$, the existence of a global frame, and the scaling of the potential compared to the mean curvature, which we delve into below. Notwithstanding those, we can informally state the result. Based on the bounds mentioned above, we obtain a threshold for the initial mean curvature $\bar{\theta}$, which, if surpassed, guarantees that the development of the initial data has a quiescent big bang singularity. For a precise statement we refer to Theorem 2.10 below, but what is meant in particular is past global existence of the development until the blowup of the Kretschmann scalar as well as convergence of the eigenvalues of the expansion-normalized Weingarten map $\mathcal{K}$ of the CMC foliation. The conclusions also include detailed asymptotics for the eigenvalues of $\mathcal{K}$ and expansion-normalized quantities related to the scalar field.

The main result, stated as Theorem 2.10 below, is particularly relevant for initial data close to that induced by spatially locally homogeneous solutions (as constructed by Ringström in as of yet unpublished work). Combining the results of Paper C with what is known about the future stability of spatially locally homogeneous solutions to the Einstein-nonlinear scalar field equations, in particular from [51, 48], this leads to a large class of spatially locally homogeneous solutions which are globally nonlinearly stable. By stability we do not mean the stability of the solution per se, or even stability within the symmetry class, but rather we mean that the developments remain close to one another (in some sense we do not specify here, but the details are available in Paper C), and share many important characteristics, in particular concerning curvature blowup and geodesic completeness.

2.3.1 A criterion for the formation of big bang singularities

As mentioned, the natural starting point is initial data to the Einstein-nonlinear scalar field equations, say $\mathfrak{I} := (\Sigma, \bar{h}, \bar{k}, \bar{\phi}_1, \bar{\phi}_0)$, and let us denote the potential to the scalar field by V. Recall that we encountered the expansion-normalized quantities $\mathcal{H}, \mathcal{K}, \Phi_1$ and Φ_0 in Definitions 1.4, 1.3 and 1.5. If these, or any other object, are constructed from the initial data $\mathfrak{I}$, then we denote it with a bar overhead, e.g. $\bar{\mathcal{K}}$. Before we can state the main result, there are several parameters that need to be defined.

When it comes to the potential, we need to make some restrictions so as to be able to control how $V(\phi)/\theta^2$ behaves toward the initial singularity. The class of potentials under consideration is the following:

Definition 2.8 (Paper C, Definition 1)**.** Fix $\sigma_V \in (0, 1)$. If $V \in C^\infty(\mathbb{R})$ is non-negative and has the property that for each $0 \leq k \in \mathbb{Z}$, there is a constant $c_k > 0$ such that

$$\textstyle\sum_{l\leq k}|V^{(l)}(x)| \leq c_k e^{2(1-\sigma_V)|x|} \tag{2.17}$$

for all $x \in \mathbb{R}$, then V is said to be a *σ_V-admissible potential.*

Next is the aforementioned algebraic condition on the eigenvalues of $\bar{\mathcal{K}}$. In four-dimensional spacetime it is equivalent to the eigenvalues being positive. In what follows we use a parameter, denoted by σ_p, that measures how well the algebraic condition (2.16) is satisfied.

Definition 2.9 (Paper C, Definition 9)**.** Let $\mathfrak{I} = (\Sigma, \bar{h}, \bar{k}, \bar{\phi}_0, \bar{\phi}_1)$ be CMC initial data, with $\bar{\theta} = \mathrm{tr}_{\bar{h}}(\bar{k}) > 0$. Let $\sigma_p \in (0, 1)$. Then $\mathfrak{I}$ and the eigenvalues $\bar{p}_1, ..., \bar{p}_n$ of $\bar{\mathcal{K}}$ are said to be *σ_p-admissible* if

$$\bar{p}_I + \bar{p}_J - \bar{p}_K < 1 - \sigma_p, \tag{2.18}$$

for all I, J, K, such that $I \neq J$.

There are some technical details that appear in the statement of the main theorem, which we briefly wish to address here. First is the assumption of non-degeneracy, i.e. that the eigenvalues $\bar{p}_1, ..., \bar{p}_n$ satisfy an inequality of the form

$$\min_{x\in\Sigma} \min_{I\neq J} |\bar{p}_I - \bar{p}_J| > \zeta_0^{-1} \tag{2.19}$$

for some $\zeta_0 > 0$, so that in particular the eigenvalues are distinct at every $x \in \Sigma$. On the other hand, we shall require that Σ is parallelizable, i.e. that there exists a global frame, say $\{E_i\}_{i=1}^n$. Given the assumption of non-degeneracy this is not a large restriction, as a finite cover of Σ must then be parallellizable, cf. Remark 11 of Paper C. Moreover, the manifold Σ on which the initial data lives is assumed to be closed. In the four-dimensional setting the assumption is not a large restriction either as all orientable, closed 3-manifolds are in fact parallellizable, cf. e.g. [10].

The main result of Paper C is then the following:

Theorem 2.10 (Paper C, Theorem 12)**.** *Fix admissibility thresholds σ_V, $\sigma_p \in (0,1)$ and let*

$$\sigma := \min\left(\tfrac{\sigma_V}{3}, \tfrac{\sigma_p}{5}\right). \tag{2.20}$$

Fix $3 \leq n \in \mathbb{N}$ and regularity degrees k_0, $k_1 \in \mathbb{N}$, such that

$$k_0 \geq \left\lceil \tfrac{n+1}{2} \right\rceil, \tag{2.21a}$$

$$k_1 \geq \tfrac{(2n+3)(1+2\sigma)}{\sigma}\left(k_0 + 3 + \left\lceil \tfrac{n+1}{2} \right\rceil\right). \tag{2.21b}$$

Let $(\Sigma, h_{\mathrm{ref}})$ be a closed Riemannian manifold of dimension n with smooth global orthonormal frame $(E_i)_{i=1}^n$, and let $V \in C^\infty(\mathbb{R})$ be a σ_V-admissible potential. For any $\zeta_0 > 0$, there is then a $\zeta_1 > 0$ such that:

If $\mathfrak{I}$ are σ_p-admissible CMC initial data on Σ for the Einstein-nonlinear scalar field equations with potential V, such that the associated expansion-normalized initial data $(\Sigma, \bar{\mathcal{H}}, \bar{\mathcal{K}}, \bar{\Phi}_0, \bar{\Phi}_1)$ satisfy

$$\|\bar{\mathcal{H}}^{-1}\|_{C^0(\Sigma)} + \|\bar{\mathcal{H}}\|_{H^{k_1+2}(\Sigma)} + \|\bar{\mathcal{K}}\|_{H^{k_1+2}(\Sigma)} + \textstyle\sum_{i=0}^1 \|\bar{\Phi}_i\|_{H^{k_1+2}(\Sigma)} < \zeta_0; \tag{2.22}$$

$|\bar{p}_I - \bar{p}_J| > \zeta_0^{-1}$ for $I \neq J$; and the mean curvature satisfies $\bar{\theta} > \zeta_1$, then the maximal globally hyperbolic development of $\mathfrak{I}$, say (M, g, ϕ), with associated embedding $\iota : \Sigma \hookrightarrow M$, has a past crushing big bang singularity in the following sense:

CMC foliation: *There is a diffeomorphism* Ψ *from* $(0, t_0] \times \Sigma$ *to* $J^-(\iota(\Sigma))$, *i.e. the causal past of the Cauchy hypersurface* $\iota(\Sigma)$, *such that* $\Psi(\{t_0\} \times \Sigma) = \iota(\Sigma)$ *and the hypersurfaces* $\Psi(\Sigma_t) \subset M$, *where* $\Sigma_t := \{t\} \times \Sigma$, *are spacelike Cauchy hypersurfaces with constant mean curvature* $\theta = \frac{1}{t}$, *for each* $t \in (0, t_0]$.
Asymptotic data: *There are unique everywhere distinct functions* $\mathring{p}_1, \dots, \mathring{p}_n \in C^{k_0+1}(\Sigma)$ *and functions* $\mathring{\Phi}_0$, $\mathring{\Phi}_1 \in C^{k_0+1}(\Sigma)$, *satisfying*

$$\textstyle\sum_I \mathring{p}_I = \sum_I \mathring{p}_I^2 + \mathring{\Phi}_1^2 = 1, \tag{2.23}$$

i.e., the generalized Kasner conditions, and, for all I, J, K *with* $I \neq J$,

$$\mathring{p}_I + \mathring{p}_J - \mathring{p}_K < 1. \tag{2.24}$$

There is also a constant $C > 0$ *such that, for all* $t \in (0, t_0]$,

$$\|p_I(t,\cdot) - \mathring{p}_I\|_{C^{k_0+1}(\Sigma)} \leq Ct^{\sigma}, \tag{2.25a}$$

$$\|\Phi_0(t,\cdot) - \mathring{\Phi}_0\|_{C^{k_0+1}(\Sigma)} + \|\Phi_1(t,\cdot) - \mathring{\Phi}_1\|_{C^{k_0+1}(\Sigma)} \leq Ct^{\sigma}, \tag{2.25b}$$

where $p_1, \dots, p_n$ *are the eigenvalues of* $\mathcal{K}$, *and* $\mathcal{K}$, Φ_1 *and* Φ_0 *are the expansion-normalized quantities induced on* Σ_t *by appealing to Definitions 1.3 and 1.5.*
Curvature blow-up: *There is a constant* $C > 0$ *such that* $\mathfrak{R}_g := \mathrm{Ric}_{g,\mu\nu}\,\mathrm{Ric}_g^{\mu\nu}$ *and* $\mathfrak{K}_g := \mathrm{Riem}_{g,\mu\nu\xi\rho}\,\mathrm{Riem}_g^{\mu\nu\xi\rho}$ *satisfy*

$$\left\| t^4\mathfrak{K}_g(t,\cdot) - 4\left[\textstyle\sum_I \mathring{p}_I^2(1-\mathring{p}_I^2) + \sum_{I<J}\mathring{p}_I^2\mathring{p}_J^2\right]\right\|_{C^{k_0+1}(\Sigma)} \leq Ct^{2\sigma}, \tag{2.26a}$$

$$\|t^4\mathfrak{R}_g(t,\cdot) - \mathring{\Phi}_1^4\|_{C^{k_0+1}(\Sigma)} \leq Ct^{2\sigma} \tag{2.26b}$$

for all $t \in (0, t_0]$, *so that* (M, g) *is* C^2 *past inextendible. Moreover, every past directed causal geodesic in* M *is incomplete and* $\mathfrak{R}_g$ *and* $\mathfrak{K}_g$ *blow up along every past inextendible causal curve.*

In Subsections 1.4 and 1.5, examples that fit the assumptions of the theorem are identified. One important example is initial data induced by developments that induce data on the singularity, following the framework of [46, 45]. This includes a large class of Bianchi class A developments as a special case, cf. Corollary 31 of Paper C. Combining this corollary of Theorem 2.10 with results concerning the future stability of spatially locally homogeneous solutions of class A for certain types of potentials leads to the nonlinear stability of a large class of spatially locally homogeneous solutions, cf. Theorems 38 and 44 of Paper C. Moreover, there are important examples of initial data, falling under what we call the degenerate case, for which Theorem 2.10 itself does not apply directly, but for which the underlying arguments do still apply with some modifications, so that similar conclusions can be obtained. This concerns in particular some of the recent results in the literature on quiescent big bang formation, in particular the results in [23] (except for the statements concerning polarized $U(1)$-symmetric developments), as well as results in [54] and in [20].

2.3.2 A regime of quiescence

Following the lines of Subsection 1.7 of Paper C, we wish to discuss some of the key ideas that go into the proof of Theorem 2.10. Of particular importance are the scaffold, the asymptotic Hamiltonian constraint and the dual role of the mean curvature. In Paper C, we are trying to precisely identify a regime in which quiescence generally occurs, based on known heuristics, and these tools and concepts help us formalize this regime.

Recall that we have the assumption of the algebraic condition (2.18) and the initial bound on the expansion-normalized quantities (2.22). If moreover the initial mean curvature is sufficiently large, we then may expect the convergence of the expansion-normalized quantities $\mathcal{H}$, $\mathcal{K}$, Φ_1, and Φ_0. For the Kasner scalar field solutions of Example 1.6, the quantities $\mathcal{H}$, $\mathcal{K}$, Φ_1, and Φ_0 in fact do not evolve at all, which is a strong clue to what the behaviour for a general quiescent solution should look like, at least in the regime of large mean curvature, where the expansion-normalized dynamics have fettered out. It is thus of interest to measure how well the actual development fits this picture of invariant expansion-normalized quantities. In order to achieve this, we introduce the *scaffold*, which is, just like the Kasner solutions, a spacetime with a CMC foliation with zero shift, with lapse identically equal to one, and with a scalar function representing the scalar field, such that the induced expansion-normalized quantities $\mathcal{H}$, $\mathcal{K}$, Φ_1, and Φ_0 are equal to the initial ones for all times. Note that the scaffold typically does not satisfy any particular field equations, with of course the notable exception of the Kasner solutions. We may regard the scaffold as the zeroth order approximation of the solution if the solution is in this regime.

Another concept that plays a role in the analysis and in the identification of this regime of quiescence is the *approximate satisfaction of the asymptotic Hamiltonian constraint*. This is informed by the notion of initial data on the singularity as in [46, 45], as the initial data on the singularity is required to satisfy precisely this constraint. By the asymptotic Hamiltonian constraint, we simply mean the equation

$$\mathcal{K}_{IJ}\mathcal{K}^{IJ} + \Phi_1^2 = 1. \tag{2.27}$$

One can understand this equation as the limit of the Hamiltonian constraint when the mean curvature goes to infinity under the algebraic condition on the eigenvalues of $\mathcal{K}$, as given in Definition 2.9, the bounds on the expansion-normalized quantities (2.22) and the conditions on the potential as given in Definition 2.8. In fact, the assumptions of Theorem 2.10 naturally force this equation to be approximately satisfied. Indeed, for initial mean curvature $\bar{\theta} > 0$, the Hamiltonian constraint as satisfied by the initial data may be written as

$$\bar{\mathcal{K}}_{IJ}\bar{\mathcal{K}}^{IJ} + \bar{\Phi}_1^2 + 2\bar{\theta}^{-2}V \circ \bar{\phi}_0 = 1 + \bar{\theta}^{-2}\left(\mathrm{Scal}_{\bar{h}} - 2|\mathrm{d}\bar{\phi}_0|_{\bar{h}}^2\right). \tag{2.28}$$

However, the last term on the right-hand side may be shown to decay as $\bar{\theta}^{-\varepsilon}$ for some $\varepsilon > 0$, due to the algebraic condition (2.18), the bounds on the expansion-normalized quantities (2.22); similar conclusions can be shown to hold for the term involving the potential. Thus, for large enough initial mean curvature, the asymptotic Hamiltonian constraint is approximately satisfied. On the other hand, the mean curvature also serves to define the time coordinate: $t = \theta^{-1}$. The equation for the lapse $\bar{N}$ at the initial time $t_0 := \bar{\theta}^{-1}$ is then given by

$$(t_0^2\Delta - 1)(\bar{N} - 1) + (1 - \bar{\mathcal{K}}_{IJ}\bar{\mathcal{K}}^{IJ} - \bar{\Phi}_1^2 + t_0^2\tfrac{2}{n-1}V \circ \bar{\phi})\bar{N} = 0, \tag{2.29}$$

which in fact is equivalent to the Raychaudhuri equation for the normal vector field to the CMC foliation. Hence the approximate satisfaction of the asymptotic Hamiltonian constraint (2.27) and the bounds on the potential ensure the approximate equality $\bar{N} \approx 1$ if $t_0 = \bar{\theta}^{-1}$ is small enough. Here we note that the $t_0^2 V \circ \bar{\phi} = \bar{\theta}^{-2}V \circ \bar{\phi}$ becomes negligible due to the assumptions on the potential as in Definition 2.8. This argument is important to obtain good estimates for the lapse. These in turn are necessary to be able to start the bootstrap argument which is discussed below.

By the reasoning above we see that $\bar{\theta}^{-1}$ (as a time-coordinate) serves as a measure of the distance to the initial singularity, as the inverse mean curvature approximates proper time, but it also serves as a smallness parameter, ensuring that the initial data is in the quiescent regime. This latter feature is important in the bootstrap argument, as we delve into below/ What happens in practice in the energy estimates is that negative powers of the mean curvature appear on the right-hand sides. Therefore, by choosing the initial mean curvature large enough, we can ensure that these errors are small enough to close the bootstrap argument. In particular, the lower bound on $\bar{\theta}$ works as a threshold for the asymptotic regime.

2.3.3 The bootstrap argument

In practice, the core of the proof of Theorem 2.10 is based on a bootstrap argument, where the development of the initial data is compared to the scaffold after fixing the gauge and the propagation of the frame. The idea of the bootstrap argument is that, on the one hand, one proves that the development remains close (in an expansion-normalized sense) to the scaffold for some time. On the other hand, one shows that the assumptions that are required to prove the first statement remain true; e.g. the algebraic condition remains satisfied, with certain quantitative estimates. Then as a result one can show that the development stays close all the way up the initial singularity, by choosing the smallness parameter small enough to ensure the quantitative estimate is good enough to keep applying the same argument. We wish to give some details here concerning how the argument works in practice, and to recount some of the analytical difficulties encountered.

The gauge in which the analysis takes place is that of a CMC foliation with zero shift with the time-coordinate given by the inverse of the mean curvature, i.e. $t = \theta^{-1}$, as well as a Fermi-Walker propagated frame. This is the same setup as in the work [23] by Fournodavlos, Rodnianski and Speck, in which the stability (in the same sense as here) of the past of the development of the Kasner scalar field solutions is proven. (This is the reason that the equations, associated variables and so on are called the FRS equations, FRS variables and so on.) Many other aspects of the analysis, for example the energy estimates and Theorem 115 of Paper C regarding the asymptotics, are based on the analysis in [23], though with modifications in various areas. Just as in [23], the bootstrap argument relies, on the one hand, on detailed lower-order estimates where a low number of derivatives is controlled in C^0, and, on the other hand, on rough higher-order estimates where a high number of derivatives is controlled in L^2. The level of control is reflected in particular in the dependence in time of the quantities that are being estimated. While the energy estimates for the lower-order estimates are more straightforward as they are of an ODE-nature, the higher order estimates require appealing to the momentum constraint to be able to close the argument (which also leads to a loss of a derivative in our analysis compared to that of [23]).

Now, in the bootstrap argument of Paper C, the smallness parameter is not the distance to any particular solution. Instead it is given by the inverse of the initial mean curvature which is also the initial time, i.e. $t_0 = \bar{\theta}^{-1}$. The bounds that are derived allow us to conclude that, given the algebraic condition (2.18), the expansion-normalized bounds (2.18) et cetera, the size of the initial mean curvature is sufficient to control the past behaviour of the solution enough for us to make the conclusions that we need concerning curvature blowup, asymptotic information et cetera. Here we should note that in the degenerate case described in Section 1.6 one actually does compare to a specific background, yet the smallness parameter is still given in terms of $\bar{\theta}^{-1}$.

We wish to highlight two important and related differences in the analysis in Paper C compared to that in [23]. In [23] the smallness parameter is the distance of the initial data to that induced by a Kasner scalar field solution, which we recall is spatially homogeneous. On the one hand, we do not have this smallness assumption. Instead, the smallness parameter in the bootstrap argument is the inverse of the initial mean curvature $\bar{\theta}^{-1}$, ensuring that the initial data is in the regime of quiescence.

On the other hand, in Paper C, one compares with the scaffold, which is constructed from the initial data, and which is typically neither a solution, nor is it spatially homogeneous. This potential inhomogeneity of the scaffold leads to analytical difficulties, mostly relating to spatial derivatives of the eigenvalues $\bar{p}_I$ of $\mathcal{K}$. For example for the estimates for the structure coefficients, one needs to commute spatial derivatives with an operator of the form $-t^{-1}(\bar{p}_I + \bar{p}_J - \bar{p}_K) - \partial_t$; this commutator vanishes in the analysis of [23]. However, in the setting of Paper C, the spatial derivatives of the eigenvalues $\bar{p}_I$ could be large if the initial data is inhomogeneous. This causes complications, which necessitate a different definition of the lower-order energy, with weights depending on the number of derivatives and lower-order bounds on the $\bar{p}_I$. Moreover, in the higher-order energy estimates, we require estimates that are tailored to extract terms in which the eigenvalues $\bar{p}_I$ are not differentiated, and estimate the remainder separately based on interpolation. Yet another consequence of the spatial inhomogeneity of the scaffold is that extra terms appear in the evolution equations for the structure coefficients of the scaffold. However, by discriminating between the frame components and the structure coefficients in the bootstrap assumptions, using different powers of t, the required estimates for the structure coefficients actually simplify.

2.3.4 Structure of the proof

With the above ideas in mind, let us now walk through the logical structure of the proof of Theorem 2.10. The proof can be found in Section 7 of Paper C. The first step is to translate the bounds on the geometric initial data, in particular the bound (2.22) on the expansion-normalized quantities, into estimates for the initial data to the FRS equations. These estimates are formalized in Definition 56 of Paper C in the notion of *diagonal FRS initial data* satisfying the *FRS expansion normalized bounds.* The derivation of these is the content of Proposition 57 of Paper C. Moreover, if the initial mean curvature $\bar{\theta}$ of the initial data is large enough, bounds may also be deduced for the initial lapse, which is shown in Proposition 63 of Paper C. These bounds for the initial lapse are essential for initializing the bootstrap argument, for they are the only obstruction of closeness between the development and the scaffold initially. However, the approximate satisfaction of the asymptotic Hamiltonian constraint enforces that the lapse will be close to 1 for sufficiently large initial mean curvature $\bar{\theta}$.

The next logical step in the proof of Theorem 2.10 is Theorem 73 of Paper C, concerning past global existence as well as an energy estimate controlling the dynamical variables in terms of the initial mean curvature. The proof of Theorem 73 is based on a bootstrap argument based on energy estimates, as discussed above. For the reasons explained in the subsection above, for large enough initial mean curvature $\bar{\theta}$ the bounds that go into the main result can be bootstrapped, as the expansion-normalized quantities do not move around much at all, so in particular they stay close to the corresponding quantities of the scaffold.

Using the control of the dynamical variables in terms of the initial mean curvature, we can obtain detailed asymptotics for the eigenvalues of $\mathcal{K}$, in particular their convergence (in the lower order norms) as $t \downarrow 0$. This argument is rather similar in nature to the one found in [23], except for the role of the inverse of the initial mean curvature as a smallness parameter, the regularity that is recovered based on the seperation of the eigenvalues, and a proof of the past incompleteness of causal geodesics. Consequently, the blowup of the Kretschmann scalar and the Ricci tensor contracted with itself may be obtained at the rates specified in Theorem 2.10 above.

Finally, by Proposition 53, the results we obtain in the gauge-fixed setting actually correspond to properties of the development of the initial data, so that in particular we obtain a solution to the Einstein-nonlinear scalar field equations that inherits all the properties of the solution to the FRS equations.

2.3.5 Discussion

The choice of a CMC foliation, with its infinite speed of propagation due to the elliptic equation for the lapse, has certain favourable properties for the overall analysis. However it would be of interest if a similar framework could be developed for a local gauge. For example, in [11, 12] results are obtained in a local gauge, though only for nearly isotropic initial data. (Note that the second article is in the Einstein-Euler-scalar field setting, so it involves both a scalar field and a fluid.) One would naturally expect the analysis in Paper C to localize toward the past given the silent causal structure; however, many details regarding the setup of the lapse and potentially a non-zero shift are to be developed, notwithstanding the local existence theory.

On a similar note, the Fermi-Walker propagated frame is not well-suited to recover asymptotic information concerning the expansion-normalized Weingarten map. On the other hand, it is rather unclear at this stage whether wellposedness can be proven for the choice of frame adapted to the Weingarten map, i.e. being an eigenframe of $\mathcal{K}$ at all times. One way forward, in 3+1-dimensions at least, could be to adapt the frame diagonalizing the matrix n constructed from the structure coefficients, as was done in the analysis of Paper B concerning Bianchi type $\mathrm{VI}_{-1/9}$ cosmologies. However, it is unclear whether this approach would be effective in recovering the asymptotic behaviour of the eigenspaces of $\mathcal{K}$, if those converge at all.

In various places in the analysis, the following one-form (the same as in Paper B) appears:

$$a := \tfrac{1}{2}\gamma_{IJJ}\omega^I = -\tfrac{1}{2}\,\mathrm{div}_h(e_I)\omega^I.$$

Unsatisfactory estimates of the size of a lead to complications, when it comes to the application of the momentum constraints. This in turn leads to large lower bounds for the size of the constant A that appears in the quantities that control the difference of the solution to the scaffold in higher-order norms. On the other hand, heuristically one expects (with respect to the frame diagonalizing the second fundamental form) that a_I scales as $\bar{\theta}^{-p_I}$, whereas γ_{IJK} scales as $\bar{\theta}^{-p_I-p_J+p_K}$, so it is conceivable that better estimates can be bootstrapped for the components of a compared to the structure coefficients γ_{IJK} with $I \neq K \neq J$. This could lead to substantially better control of the higher derivatives of the solution, for which with the current analysis we do not obtain asymptotic information of interest.

The assumption of non-degeneracy in Theorem 2.10, i.e. the separation of the eigenvalues of $\mathcal{K}$, is somewhat unfortunate. Although it is important for the conclusions concerning regularity of the asymptotic data of $\mathcal{K}$, it puts unfortunate restriction, including topological ones, on the initial data. Perhaps with a local gauge one could localize to regions in Σ where a separation of eigenvalues applies and thus circumvent these restrictions. Moreover, looking beyond the quiescent setting, it is to be expected that eigenvalues frequently coalesce in the oscillatory setting, so it is of interest to be able to handle analytically the coalescence of eigenvalues as well.

Lastly, in [23] there are also results concerning $U(1)$-symmetric polarized solutions, which are similar to the polarized solutions that appear in Paper B, in [33] et cetera. The polarization condition should also be possible in other topologies, and similar results as the ones obtained here should apply in the polarized case.

Contributions by the respondent

Papers A and B are authored solely by the respondent. The respondent is solely responsible for the writing and the mathematical contents. For both papers the topics were suggested by the main advisor, Hans Ringström, who also gave many comments and suggestions, as mentioned in the acknowledgements.

Regarding Paper C, the respondent's main contribution was in most of the writing and mathematical contents of Sections 4, 5 and 6, as well as Appendices A and B, which mainly concern the various estimates required for the bootstrap argument, in particular the energy estimates. For the remainder, i.e. the abstract, introduction and Sections 2,3 and 7, which concern the setup of the bootstrap argument, the a-priori estimates, and the proof of the main theorem, the respondent also contributed to parts of the writing, and participated in regular discussions on the mathematical contents.

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