ROOK MATROIDS AND LOG-CONCAVITY OF P-EULERIAN POLYNOMIALS

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Abstract. We define and study rook matroids, the bases of which correspond to non-nesting rook placements on a skew Ferrers board. We show that rook matroids are closed under taking duals and direct sums, but not minors. Rook matroids are also transversal, positroids, and bear a subtle relationship to lattice path matroids that centers around not having the quaternary matroid Q_6 as a minor. The enumerative and distributional properties of non-nesting rook placements stand in contrast to that of usual rook placements: the non-nesting rook polynomial is not real-rooted in general, and is instead ultra-log-concave. We leverage this property together with a correspondence between rook placements and linear extensions of a poset to show that if P is a naturally labeled width two poset, then the P-Eulerian polynomial W_P is ultra-log-concave. This takes an important step towards resolving a log-concavity conjecture of Brenti (1989) and completes the story of the Neggers–Stanley conjecture for naturally labeled width two posets.

Contents

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1. Introduction

The distributional properties — unimodality, log-concavity and real-rootedness — of combinatorially defined sequences have been the focus of extensive research in the last decade [Brä15, AHK18, BH20, ADH23, Huh12]. A landmark result in this field dates back to Heilmann and Lieb's theorem on the real-rootedness of the matching polynomial of a graph [HL72]. Later, Nijenhuis [Nij76] proved that the rook polynomial of a board is real-rooted, which corresponds to a special case of the previous result when the graph is bipartite. This was one of the early results concerning distributional properties of sequences in rook theory, which is concerned with enumerative aspects of rook configurations on a board. Since then, rook theory has been studied in various other contexts including Garsia and Remmel's q-rook theory [GR86], combinatorial reciprocity [GH00, Cho96] and the cohomology of a special class of Schubert varieties [Din97, DMR07].

In this paper, we draw a connection between rook theory and matroid theory, by considering non-nesting rook placements on a skew-shaped board; the combinatorics of these restricted rook placements is underpinned by a matroidal structure that we dub the rook matroid. We probe distributional properties of the corresponding *non-nesting rook polynomial* and show that while it is not real-rooted in general — for subtle reasons that relate to the Neggers–Stanley conjecture — it is always ultra-log-concave, a fact that we deduce from the Lorentzianity of the basis-generating polynomial of a matroid [BH20]. The matroidal properties are similarly rich: rook matroids are transversal and bear a subtle relationship to another well-studied class of matroids defined on skew shapes: lattice path matroids [BdMN03]. We show in particular, that a single skew shape λ/μ supports two (generally inequivalent) matroidal structures and give a precise criterion for when they are isomorphic to one another. Curiously enough, this involves the quaternary matroid on six elements Q_6.

Similar to lattice path matroids, rook matroids are seen to be positroids. Discovered and studied by Blum in the guise of base-sortable matroids [Blu01], positroids were reintroduced by Postnikov in [Pos06] as matroids that parameterize a certain stratification of the totally non-negative Grassmannian. These strata are the so-called open positroid varieties and their rich geometric properties closely mirror those of the well-studied Richardson varieties; a comprehensive algebro-geometric study of these objects was undertaken in [KLS13]. The multifarious combinatorial parametrizations of positroids—in terms of Grassmann necklaces, bounded affine permutations, Le diagrams, planar bipartite graphs and more—have since been thoroughly studied [Oh11, KLS13, HLO23]. We prove that rook matroids are positroids via Oh's theorem [Oh11], the Grassmann necklace approach.

We then consider a poset-theoretic perspective of non-nesting rook placements by relating bases of a rook matroid to linear extensions of a corresponding width two poset. This allows us to show that for naturally labeled posets P of width two, the P-Eulerian polynomial W_P is ultra-log-concave. The context of this result is the Neggers–Stanley conjecture [Neg78, Sta86], which asserts that W_P is real-rooted. This was disproved by Brändén in 2004 [Brä04b] and by Stembridge in 2007 [Ste07], the latter of whom provided a naturally labeled counter-example of width two. By deducing the strongest possible distributional property for W_P, our results thus complete the picture of the Poset conjecture in the width two case. We do so by refining a well-known bijection between linear extensions of a poset and lattice paths to show that a certain multivariate analog of W_P — first defined by Brändén and Leander in [BL16]— is Lorentzian in the sense of [BH20].

The following is a summary of our main contributions together with the organization of the paper:

- In Section 2.1, we introduce our main combinatorial object – non-nesting rook placements on skew Ferrers shapes, its generating polynomial and compute a few examples. In Section 3.1, we define the rook matroid and study some of its structural properties: namely being transversal (Theorem 3.3), closed under taking duals and direct sums, but not closed under taking minors (Example 3.8).
- In Section 3.2, we characterize precisely when rook matroids are isomorphic to lattice path matroids, which is when the underlying skew shape does not contain 332/1 as a subshape (Theorem 3.24). This criterion can be rephrased in terms of the rook matroid not having the matroid Q_6 as a minor. We also show that rook matroids generalize lattice path matroids in the following sense: lattice path matroids can be obtained as contractions of large enough rook matroids (Proposition 3.27). In Section 3.3, we show that the rook matroid is a positroid (Theorem 3.36). We then consider the Tutte polynomial of a rook matroid in Section 3.4 and show that it is the same as that of the corresponding lattice path matroid (Theorem 3.38).
- In Section 4 we prove that the non-nesting rook polynomial is ultra-log-concave via its matroidal interpretation. We then consider the half-plane properties of lattice path matroids and give the first example of a lattice path matroid that *does not* have the half-plane property (Theorem 4.3). We also give new and self-contained proofs of the half-plane property of snake matroids and panhandle matroids (Proposition 4.8).
- In Section 5, we derive a bijective correspondence between skew shapes and width two posets that maps non-nesting rook placements to linear extensions, yielding a P-Eulerian interpretation of the basis-generating polynomial of the rook matroid (Theorem 5.4). Consequently, the univariate polynomial W_P is ultra-log-concave which settles the log-concavity part of the Neggers–Stanley conjecture in the case of naturally labeled width two posets (Corollary 5.10); the log-concavity question has also been posed by Brenti [Bre89, Conjecture 1.1] and Brändén [Brä15, Question 2]. We also use this correspondence in the opposite direction to deduce enumerative information about the the non-nesting rook polynomial such as (a) an example of when it is not real-rooted (Corollary 5.13) and (b) a precise criterion as to when the non-nesting rook polynomial is gamma-positive (Corollary 5.16).

2. Preliminaries

In this section, we define the basic combinatorial objects that will be the focus of our interest: non-nesting rook placements on boards and their generating polynomials. We will also recall the necessary background on matroids and notions from the geometry of polynomials that will be used in later sections.

2.1. **Boards, skew shapes and rook placements.** We use $[n]$ to mean the set $\{1, 2, \ldots, n\}$ and $[m, n]$ to mean the discrete interval $\{m, m+1, \ldots, n\}$. The set of size k subsets of $[n]$ is denoted by $\binom{[n]}{k}$.

A *board* B with r rows and c columns is a subset of the rectangular grid $[r] \times [c]$. We label the rows with elements from $[r]$ and the columns with elements from $[r+1, r+c]$ respectively, as in Example 2.1. The row labeling is done from the top to bottom and the column labeling is done from left to right. We refer to cells (i, j) of B with respect to this labeling until otherwise specified. Given a board B with r rows and c columns, the *bipartite graph corresponding to B* is the graph $\Gamma_B = ([1, r] \sqcup [r + 1, c]), E$ where $i \in [1, r]$ is connected to $j \in [r = 1, r + c]$ if $(i, j) \in B$. For most of what follows, we restrict ourselves to special classes of boards coming from integer partitions.

An *integer partition* $\lambda = (\lambda_1, \dots, \lambda_n)$ is any non-increasing tuple of positive integers. Following the English convention, the *Ferrers diagram* of λ is a left-justified, weakly decreasing array of boxes. Two special partitions correspond to Ferrers diagrams in the shape of a staircase and rectangle, respectively. For the former, δ_n denotes the partition $(n, n - 1, \dots, 1)$ and for the latter, $(a)^b$ denotes the partition consisting of b parts each of size a.

Given two integer partitions $\lambda \supseteq \mu$ of length at most r, we let the *skew Ferrers board* associated with λ/μ be the board with cells

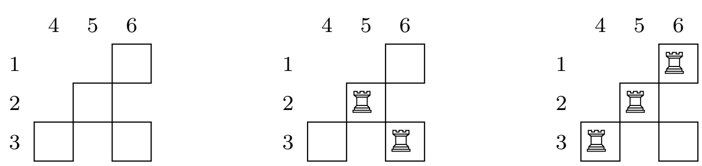
$$\{(i, r + j) : 1 \leq i \leq r \text{ and } \mu_i < r + j \leq \lambda_i\}.$$

When μ is the empty partition, we treat μ as the all-zeroes vector. In what follows, we also refer to a skew Ferrers board as a *skew shape*. A *subshape* of a skew shape λ/μ is the skew shape obtained by deleting rows or columns of λ/μ. The *size* of a skew shape λ/μ is the number of boxes in its diagram and is denoted $|\lambda/\mu|$. An *outer corner* of λ/μ is a cell $(i, j) \notin \lambda/\mu$ such that $(i, j - 1), (i - 1, j) \in \lambda/\mu$. An *inner corner* of λ/μ is a cell $(i, j) \notin \lambda/\mu$ such that $(i + 1, j), (i, j + 1) \in \lambda/\mu$.

By a *non-attacking* placement of rooks on a board, we mean a configuration of rooks where no two rooks can take each other in the sense of chess. More formally, a non-attacking rook placement ρ on a board B is a subset of B no two cells of which share a row or column index. Two rooks in a non-attacking rook placement form a *nesting* if one rook lies South-East of another. Hereafter, a *non-nesting rook placement* (or just rook placement, when it is clear that the setting is a non-nesting one) on a board B is a non-attacking rook placement such that no pair of rooks forms a nesting. It is straightforward to see that a non-attacking rook placement on B corresponds to a matching of the graph Γ_B.

We will frequently move between the settings of non-nesting rook placements and Dyck paths in a skew shape. A *Dyck path* in a skew shape λ/μ is a lattice path starting from the Southwest corner of λ/μ and terminating at the Northeast corner of λ/μ that consists only of North and East steps. Examples can be found in Section 3.2.

Example 2.1. From left to right below is a board, a nesting rook placement and a non-nesting rook placement.



We denote the set of non-nesting rook placements on a skew shape λ/μ by $\mathrm{NN}_{\lambda/\mu}$ and define the generating polynomial of its k-subsets as follows.

Definition 2.2. Given a skew shape λ/μ with c columns, the *non-nesting rook polynomial* of λ/μ is defined by

$$M_{\lambda/\mu}(t) = \sum_{k=0}^{c} r_k(\lambda/\mu) t^k,$$

where $r_k(\lambda/\mu)$ is the number of non-nesting rook placements on λ/μ of size k. Note that the degree of $M_{\lambda/\mu}$ equals the length of the longest Northwest to Southeast diagonal that fits inside λ/μ, which is not necessarily equal to c.

Example 2.3. Let $\lambda_{a,b} = (a^b)$. Then we have

$$M_{\lambda_{a,b}}(t) = \sum_{k=0}^{\min(a,b)} \binom{a}{k}\binom{b}{k} t^k.$$

This can be seen directly: choose a set $\{j_1 < \ldots < j_k\}$ of columns in $\binom{a}{k}$ ways and choosing a set of rows $\{i_1 > \ldots > i_k\}$ independently in $\binom{b}{k}$ ways. Form the rook placement with cells $\{(i_{k+1-\ell}, j_\ell) : \ell = 1, \ldots k\}$; by virtue of the ordering, this must be non-nesting.

By specialising $a = b = n$, we obtain the type B Narayana polynomial:

$$W(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

Example 2.4 (Narayana polynomials)**.** The non-nesting rook polynomial for the staircase Ferrers board $\lambda = \delta_n = (n, n-1, n-2, \ldots, 2, 1)$ is

$$M_{\delta_n}(t) = t^{-1} N_{n+1}(t),$$

where $N_n(t) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} t^k$ denotes the n^{th} Narayana polynomial. This can be seen by identifying rook placements on δ_n with arc diagrams of a set partition and then using the standard bijection to show that non-nesting set partitions of $[n]$ correspond to Dyck paths contained in δ_n. (Recall that the Narayana polynomial N_n can be seen as the peak-generating polynomial of Dyck paths of size n.) In this way, non-nesting rook placements on skew shapes generalize the study of non-nesting set partitions by considering general (skew) Ferrers boards in place of δ_n. A leitmotif in this paper is a bijective correspondence between rook placements and lattice paths; a proof of this for general skew shapes can be found later in Proposition 3.11.

Distributional properties and multivariate generalizations of $M_{\lambda/\mu}$, together with its realizability as a P-Eulerian polynomial of a poset, will be studied in later sections.

2.2. **Matroids.** We use the following axiomatization of matroids in terms of its bases; for any undefined terms hereafter we refer the reader to [Oxl11].

Definition 2.5. A *matroid* M is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ is a collection of subsets of E, called *bases*, satisfying the following two properties:

(1) $\mathcal{B} \neq \emptyset$.

(2) For each distinct pair B_1, B_2 in $\mathcal{B}$ and element $a \in B_1 \setminus B_2$, there exists $b \in B_2 \setminus B_1$ such that $(B \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

The second property above is referred to as the *basis-exchange axiom* for matroids. Several other axiomatizations of matroids exist; an exhaustive list can be found in [Oxl11, Chapter 1]. One such axiomatization can be made in terms of independent sets. A set $I \subset E$ is an *independent set* of a matroid $M = (E, \mathcal{B})$ if $I \subseteq B$, for some $B \in \mathcal{B}$.

An important example of matroids is the *uniform matroid* $U_{k,n}$, defined as the matroid with ground set $[n]$ and bases given by all k-subsets of $[n]$.

We briefly recount the various operations one can apply to a matroid to produce a new matroid.

(1) Given a matroid $M = (E, \mathcal{B})$ and an element $e \in E$, the *deletion* and *contraction* of M by e are the matroids with ground set $E \setminus e$ that are respectively denoted by $M \setminus e$ and M/e and the bases of which are respectively given by:

$$\mathcal{B}(M \setminus e) = \{B \in \mathcal{B} : e \notin B\},$$
$$\mathcal{B}(M/e) = \{B \setminus e : B \in \mathcal{B}, e \in B\}.$$

A matroid obtained from M by a sequence of deletions and contractions is called a *minor* of M. A class $\mathcal{C}$ of matroids is *minor-closed* if given a matroid $M \in \mathcal{C}$, every minor of M is also in $\mathcal{C}$.

(2) The *dual* of a matroid $M = (E, \mathcal{B})$ is the matroid M^* on the ground set E and bases $\mathcal{B}(M^*) = \{E \setminus B : B \in \mathcal{B}\}$.

(3) An element $e \in E$ is a *loop* of M if e appears in no basis of M.

(4) An element $e \in E$ is a *coloop* of M if $e \in E$ appears in every basis of M

(5) The *direct sum* of matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ is the matroid $M_1 \oplus M_2$ with ground set $E_1 \sqcup E_2$ and bases $\mathcal{B}_1 \sqcup \mathcal{B}_2$.

(6) The *single-element series extension* of $M = (E, \mathcal{B})$, denoted $S(M, U_{1,2})$, is the matroid $N = (E \cup \{f\}, \mathcal{B}')$, where

$$\mathcal{B}' = \{B \cup \{g\} : B \in \mathcal{B},\ g \in \{e, f\} \setminus B\},$$

and $U_{1,2}$ is the rank 1 uniform matroid on ground set $\{e, f\}$ where $e \in E$ and $f \notin E$.

(7) Given a matroid $M = (E, \mathcal{B})$ of rank k and an element $e \notin E$, the *free extension* of M by e is the matroid $M + e$ on ground set $E \cup e$ and bases

$$\mathcal{B}(M + e) = \mathcal{B} \cup \{I \cup \{e\} : |I| = k - 1,\ I \text{ is independent in } M\}.$$

Transversals of set systems give rise an important class of matroids. Given a set system $\mathcal{N} = (N_1, \ldots, N_r)$ of not necessarily distinct subsets N_i of some ground set E, a *transversal* X of $\mathcal{N}$ is a subset of E consisting of r distinct elements, one from each set in $\mathcal{N}$: that is, $X = \{a_i : i = 1, \ldots, r\}$ such that $a_i \in N_i$ for $i = 1, \ldots, r$. A *partial transversal* of $\mathcal{N}$ is a transversal for some subsystem $\mathcal{M} = (N_i)_{i \in J}$ indexed by a subset $J \subset [r]$. The following theorem due to Edmonds and Fulkerson laid the foundation for the theory of transversal matroids:

Theorem 2.6. [EF65] *The partial transversals of $\mathcal{N}$ are the independent sets of a matroid on E.*

Matroids that arise in this way are called *transversal matroids* and $\mathcal{N} = (N_1, \ldots, N_r)$ is called a *presentation* of the matroid. The bases of a transversal matroid are given by the maximal partial transversals of $\mathcal{N}$. When $\mathcal{N}$ admits a transversal, the maximal partial transversals are precisely the transversals of $\mathcal{N}$.

2.3. **Real-rootedness, log-concavity and multivariate generalizations.** In this subsection, we recall some basic notions in the geometry of polynomials.

Given a sequence of non-negative integers $(a_k)_{k=0}^n$, it is convenient to work instead with its generating polynomial $A(t) = \sum_{k=0}^n a_k t^k$. The following notions, listed in descending order of strength, encode distributional properties of the sequence:

(1) The generating polynomial $A(t)$ is *real-rooted* if A has only real roots as univariate polynomial. Note that by the non-negativity of the a_k, the roots of A will necessarily be non-positive.

(2) The sequence $(a_k)_{k=0}^n$ is *ultra-log-concave* if

$$\left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} \quad \text{for all } 1 \leq k \leq n-1.$$

(3) The sequence $(a_k)_{k=0}^n$ is *log-concave* if

$$a_k^2 \geq a_{k-1} \cdot a_{k+1} \quad \text{for all } 1 \leq k \leq n-1.$$

(4) The sequence $(a_k)_{k=0}^n$ is *unimodal* if there exists some $0 \leq m \leq n$ such that:

$$a_0 \leq a_1 \leq \ldots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \ldots \geq a_n.$$

The sequence $(a_k)_{k=0}^n$ has *no internal zeros* if

$$a_i a_j > 0 \implies a_k > 0 \quad \text{for all } 0 \leq i < j < k \leq n.$$

As mentioned above, the following implications hold: (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Here (3) implies (4) only if $(a_k)_{k=0}^n$ has no internal zeros. For a comprehensive and modern survey of these notions, see [Brä15].

The sequence $(a_k)_{k=0}^n$ is *palindromic* if $a_k = a_{n-k}$ for $0 \leq k \leq n$. This is equivalent to the generating polynomial $A(t)$ satisfying $t^n \cdot A\left(\frac{1}{t}\right) = A(t)$. We will use the term palindromic for both the sequence and its generating polynomial. Any palindromic polynomial $A(t)$ can be expanded in the basis $\{t^i(1+t)^{n-2i}\}_{i=0}^{\lfloor \frac{n}{2} \rfloor}$:

$$A(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k}.$$

If $\gamma_i \geq 0$ for all $i = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, then A is said to be *γ-positive*. By pairing up reciprocal roots and noting that products of gamma-positive polynomials are gamma-positive, one can show that palindromic, real-rooted polynomials with non-negative coefficients are γ-positive.

Showing that a polynomial is real-rooted directly is often hard; multivariate techniques have streamlined and simplified these ideas.

Definition 2.7. A polynomial $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if

$$\operatorname{Im}(x_1) > 0, \ldots, \operatorname{Im}(x_n) > 0 \implies P(x_1, \ldots, x_n) \neq 0.$$

If all the coefficients of P are real we say that P is *real stable*. This will always be the case for us, and hereafter every instance of the word stable implicitly means real stable.

Real stability is a generalization of real-rootedness in the following precise sense: a univariate polyomial is stable if and only if it is real-rooted. Examples of stable polynomials include multivariate spanning tree polynomials, multivariate Eulerian polynomials and so-called Plücker polynomials: determinants of linear combinations of rank one positive semi-definite matrices [Brä15]. A less restrictive notion is that of Lorentzian polynomials.

Lorentzian polynomials are a class of multivariate polynomials containing stable, homogeneous polynomials and generalizing the notion of ultra-log-concavity. To define Lorentzian polynomials, we need two additional notions: that of M-convex sets and the other of quadratic forms of Lorentzian signature.

Definition 2.8. A finite set $M \subset \mathbb{N}^d$ is *M-convex* if for all $\alpha, \beta \in M$ and $i \in [d]$ such that $\alpha_i > \beta_i$ there exists $j \in [d]$ such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in M$.

Definition 2.9. A symmetric $n \times n$ matrix Q with non-negative real entries has *Lorentzian signature* if it has exactly one positive eigenvalue.

By [BH20], the following can be taken as the definition of Lorentzian polynomials.

Definition 2.10. A homogeneous polynomial $P(x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ of degree d is *Lorentzian* if the following two conditions hold:

(1) The support of P is M-convex.

(2) For every $\alpha = (\alpha_1, \dots \alpha_n) \in [d]^n$ with total sum $d-2$, the quadratic form $\partial^\alpha P$ has Lorentzian signature. That is, the quadratic form $\partial^\alpha P$ has exactly one positive eignenvalue.

Important examples of Lorentzian polynomials include volume polynomials of convex bodies and projective varieties, and basis-generating polynomials of matroids [BH20].

3. The non-nesting rook matroid

In this section, we show that there is a natural matroid structure underpinning the set of non-nesting rooks on a skew Ferrers board. The associated class of matroids is a dual-closed subclass of both transversal matroids and positroids. However, it is not minor-closed and therein lies the subtlety in the relationship between rook matroids and lattice path matroids. The schematic in Figure 1 summarises the various containment and characterization properties that rook matroids satisfy with respect to other known classes of matroids; we will prove these in Sections 3.2 and 3.3.

We begin by studying the structural properties of the rook matroid that arise from its transversal description.

3.1. Structural results.

Let λ/μ be a skew shape. Throughout we draw and refer to Young diagrams λ and μ in terms of the English convention, that is as a left-justified array of boxes. With this convention in place, we index the rows of λ/μ by $\{1, 2, \dots, r\}$ and the columns by $\{r+1, \dots, r+c\}$, and we set

$$E_{\lambda/\mu} := \{1, 2, \dots, r\} \cup \{r+1, \dots, r+c\}.$$

Given a non-nesting rook placement ρ on λ/μ, we associate to it the set $R(\rho) \cup C(\rho)$ where $R(\rho)$ is the set of row indices occupied by ρ and $C(\rho)$ is the set of column indices that are not occupied

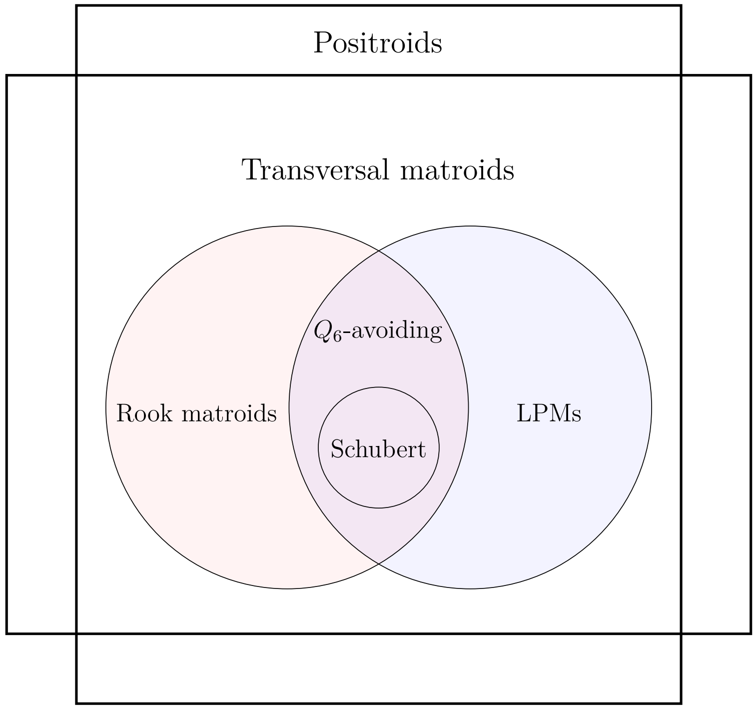


FIGURE 1. Various matroid subclasses.

by ρ. First note that for each non-nesting rook placement ρ, we have that $|R(\rho) \cup C(\rho)| = c$, the number of columns of the shape. We gather these sets into a single collection :

$$\mathcal{R}_{\lambda/\mu} \coloneqq \{R(\rho) \cup C(\rho) : \rho \in \mathrm{NN}_{\lambda/\mu}\},$$

where as before $\mathrm{NN}_{\lambda/\mu}$ is the set of non-nesting rook placements on λ/μ.

Secondly, we note that the correspondence $\rho \mapsto R(\rho) \cup C(\rho)$ is bijective.

Lemma 3.1. *Let λ/μ be a skew shape. The map from $NN_{\lambda/\mu}$ to $\mathcal{R}_{\lambda/\mu}$ defined by $\rho \mapsto R(\rho) \cup C(\rho)$ is a bijection.*

Proof. Note that this map can be written as

$$\{(i_1, j_1), \ldots, (i_k, j_k))\} \mapsto \{i_1, \ldots, i_k\} \cup ([r+1, r+c] \setminus \{j_1, \ldots, j_k\}) \tag{1}$$

which is clearly injective. The inverse map takes the set of occupied rows and unoccupied columns $\{i_1, \ldots, i_k\} \cup \{j_{k+1}, \ldots j_c\}$ to $\rho = \{(i_1, \ell_1), \ldots, (i_k, \ell_k))\}$, where $\{\ell_1 > \ell_2 \ldots > \ell_k\}$ is the set $[r+1, r+c] \setminus \{j_{k+1}, \ldots, j_c\}$. The order on the column indices ℓ_s implies that that ρ is a non-nesting rook placement. □

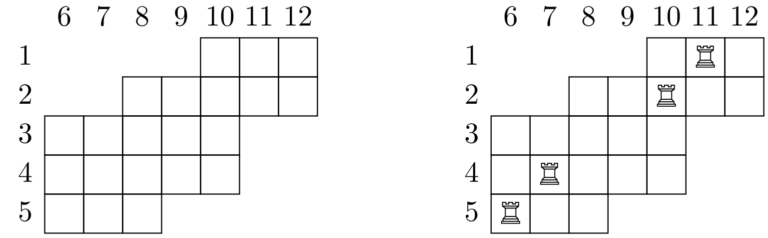
In the light of Lemma 3.1, we can speak of ρ and $R(\rho) \cup C(\rho)$ interchangeably.

Given a skew shape, we may define a set system as follows. For $j \in [c]$, let A_j be the set of row indices occupied by column $r + j$ together with $r + j$, that is

$$A_j = \{i \in [r] : (i, r+j) \in \lambda/\mu\} \cup \{r + j\}.$$

Then $\mathcal{A}_{\lambda/\mu} \coloneqq (A_1, \ldots, A_c)$ is a set system on $E_{\lambda/\mu}$ which defines a transversal matroid, which we also denote by $\mathcal{A}_{\lambda/\mu}$.

Example 3.2. Consider the skew shape $\lambda/\mu = 77553/42$.



This skew shape has corresponding set system

$$A_1 = \{3,4,5,6\} \qquad A_2 = \{3,4,5,7\} \qquad A_3 = \{2,3,4,5,8\}$$
$$A_4 = \{2,3,4,9\} \qquad A_5 = \{1,2,3,4,10\} \qquad A_6 = \{1,2,11\}$$
$$A_7 = \{1,2,12\}.$$

The non-nesting rook placement shown on the right above can be represented by

$$R(\rho) \cup C(\rho) = \{1,2,4,5,8,9,12\} \quad \text{where} \quad 5 \in A_1, \quad 4 \in A_2, \quad 8 \in A_3,$$
$$9 \in A_4, \quad 2 \in A_5, \quad 1 \in A_6,$$
$$12 \in A_7.$$

Theorem 3.3. *The set $\mathcal{R}_{\lambda/\mu}$ is the set of maximal partial transversals of the set system $\mathcal{A}_{\lambda/\mu}$. In particular, $\mathcal{R}_{\lambda/\mu}$ is the set of bases of a transversal matroid on the ground set $E_{\lambda/\mu}$.*

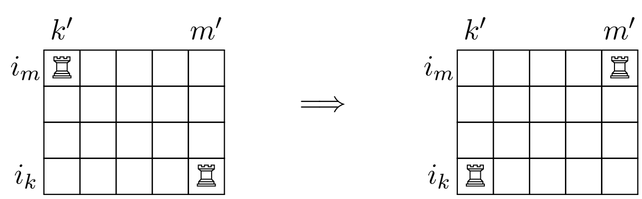
Proof. Let $\mathcal{T}_{\lambda/\mu}$ be the set of transversals of the set system $\mathcal{A}_{\lambda/\mu}$. We claim that the map $\rho \mapsto R(\rho) \cup C(\rho)$ is the necessary bijection from $\mathrm{NN}_{\lambda/\mu}$ to $\mathcal{T}_{\lambda/\mu}$. Given a non-nesting rook placement ρ, the set $R(\rho) \cup C(\rho)$ is indeed a transversal of $\mathcal{A}_{\lambda/\mu}$ since we choose $i \in A_j$ if there is a rook in column $r+j$ and row i; if column $r+j$ is empty, we choose $r+j \in A_j$. Further, by Lemma 3.1 we know that this map is a bijection, so it suffices to show that every transversal of $\mathcal{T}_{\lambda/\mu}$ is of the form $R(\rho) \cup C(\rho)$ for some $\rho \in \mathrm{NN}_{\lambda/\mu}$.

Suppose we are given a transversal T of $\mathcal{A}_{\lambda/\mu}$. The fact that the row indices in T correspond to distinct sets of $\mathcal{A}_{\lambda/\mu}$ implies that T defines a rook placement ρ on λ/μ —possibly containing nestings—with no rooks in the column indices of T. We need to show that ρ can be transformed into a non-nesting rook placement. This will follow from the claim below.

Claim: Let the occupied row indices be ordered as $r \geq i_1 > i_2 > \cdots > i_\ell \geq 1$. Then

$$i_1 \in A_1, \; i_2 \in A_2, \; \dots \; i_\ell \in A_\ell. \tag{2}$$

Proof of claim: Suppose that we have $i_k \in A_m$ and $i_m \in A_k$ for some $k < m$. This corresponds to placing rooks on the squares marked in the diagram below, where we have used k' and m' for $r+k$ and $r+m$ respectively. By virtue of the cells (i_m, k') and (i_k, m') being occupied and the board being a skew Ferrers board, we know that the cells (i_m, m') and (i_k, k') must also lie on λ/μ. Thus by swapping the row indices of the two rooks we can be sure that the rook configuration is still contained within λ/μ with the same unoccupied columns and with one less nesting. That is, $i_k \in A_k$ and $i_m \in A_m$ and this proves the claim.



By performing a sequence of such swaps for each pair of nested rooks, we can ensure that (2) holds, thus canonically realizing the transversal T as a non-nesting rook placement ρ on λ/μ. □

Identifying the resulting matroid by its set of bases, we call $\mathcal{R}_{\lambda/\mu}$ the *rook matroid* on λ/μ; for any matroid M, we say that M is a rook matroid if it is isomorphic to a rook matroid of the form $\mathcal{R}_{\lambda/\mu}$. The uniform matroid $U_{k,n}$ corresponds to the rook matroid on the $(n-k)\times k$ rectangle. Later on, we will see how rook matroids interact with other well-studied classes of matroids, including lattice path matroids and positroids.

Remark 3.4. It is natural to ask if the skew shape assumption is necessary in the above theorem. The answer is yes; consider the board B in Figure 2 below. If $\mathrm{NN}(B)$ denotes the set

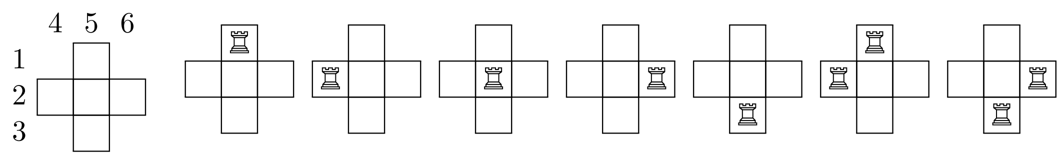


FIGURE 2. Board which does not admit a rook matroid structure, and the seven additional non-nesting rook placements. This does not form a matroid.

of non-nesting rook placements on B, then the collection of would-be bases of the rook matroid on B is (with same order as in Figure 2):

$$\{R(\rho)\cup C(\rho) : \rho \in \mathrm{NN}(B)\} = \{456, 146, 256, 246, 245, 346, 126, 234\}.$$

However, this does not satisfy the basis-exchange axiom: $6 \in \{1,4,6\}\setminus\{2,3,4\}$ but neither of $\{1,4,2\}$ or $\{1,4,3\}$ are contained in the collection above. Indeed, neither of 142 nor 143 can represent a valid non-nesting rook placement on B. This fact—together with the argument in the proof above that allows us to interpret any transversal of $\mathcal{A}_{\lambda/\mu}$ as a non-nesting rook placement by carrying out swaps— emphasizes the importance of the board being a skew shape in order to support the structure of a rook matroid.

More can be said about the transversal structure of the rook matroid. A *fundamental transversal matroid* is a matroid M that has a basis B (called a fundamental basis) such that each cyclic flat F of M is spanned by some subset of B. An equivalent characterization identifies fundamental matroids as those transversal matroids that have a presentation $(A_1,\dots,A_c)$ such that for each $j\in[c]$ there exists an element $i\in A_j$ such that $i\notin A_k$ for all $k\in[c]\setminus\{j\}$. This allows us to immediately note that rook matroids are fundamental transversal, as the distinguished element in each set of the presentation of $\mathcal{R}_{\lambda/\mu}$ is simply the associated column label. The fundamental basis of the rook matroid then corresponds to the empty rook placement. See [BKdM11] for more about fundamental transversal matroids.

Lemma 3.5. *The class of rook matroids is closed under direct sums.*

Proof. A direct sum of rook matroids corresponds to the rook matroid on the *direct sum* of the corresponding skew shapes. That is, if λ_1/μ_1 and λ_2/μ_2 are two skew shapes, then the direct sum $\lambda/\mu := (\lambda_1/\mu_1)\oplus(\lambda_1/\mu_1)$ is obtained by positioning the North-East corner of λ_1/μ_1 to the immediate South-West of the South-West corner of λ_2/μ_2; see Figure 3. Then by the obvious isomorphism, $\mathcal{R}_\sigma \cong \mathcal{R}_{\lambda_1/\mu_1}\oplus\mathcal{R}_{\lambda_2/\mu_2}$. □

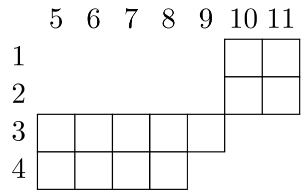


FIGURE 3. The direct sum $\lambda_1/\mu_1 \oplus \lambda_2/\mu_2$ of the rook matroids on $\lambda_1/\mu_1 = 54$ and $\lambda_2/\mu_2 = 22$ is the rook matroid on the direct sum of their shapes, $\lambda/\mu = 7754/55$.

We now examine rook matroids through the structural properties of being dual-closed and being minor-closed. The former is rather straightforward while the latter turns out to be quite subtle.

The *conjugate* of a partition λ is the partition λ' obtained by taking the transpose of the Ferrers diagram of λ. Duals of rook matroids are simply rook matroids on conjugated shapes.

Proposition 3.6. *Let λ/μ be a skew shape and let λ'/μ' denote its conjugate. Then the following isomorphism holds: $\mathcal{R}^*_{\lambda/\mu} \cong \mathcal{R}_{\lambda'/\mu'}$.*

Proof. Suppose λ/μ has r rows and c columns. Define a map $f : [r+c] \to [r+c]$ by

$$f(i) = \begin{cases} i + c, & \text{for } i \in [r], \\ i - r, & \text{for } i \in [r+1, r+c]. \end{cases}$$

Then f is clearly a bijection that sends rows and columns of λ/μ to columns and rows of λ'/μ' respectively. This corresponds to conjugating the skew shape and thus f is the necessary isomorphism. See Figure 4 for an example. $\square$

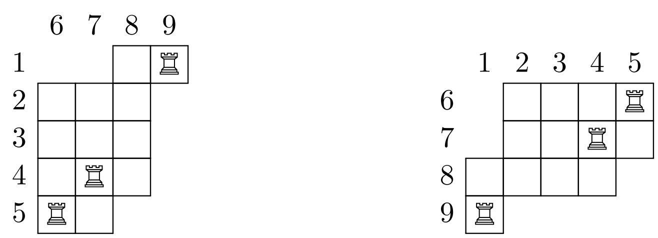


FIGURE 4. Duals of rook matroids on skew shapes are rook matroids on conjugates of shapes.

The following lemma shows that minors of rook matroids — in two cases — are straightforward to describe combinatorially. We make use of this lemma several times throughout the paper.

Lemma 3.7. *Let λ/μ be a skew shape and $\mathcal{R}_{\lambda/\mu}$ be the corresponding rook matroid. Suppose x is a row index and y is a column index of λ/μ. Then*

- *The deletion $\mathcal{R}_{\lambda/\mu} \setminus x$ is isomorphic to the rook matroid obtained from the shape λ/μ with row x removed;*
- *The contraction $\mathcal{R}_{\lambda/\mu}/y$ is isomorphic to the rook matroid obtained from the shape λ/μ with column y removed;*

- *The bases in the contraction* $\mathcal{R}_{\lambda/\mu}/x$ *correspond to non-nesting rook placements on* λ/μ *where row* x *is occupied by a rook;*
- *The bases in the deletion* $\mathcal{R}_{\lambda/\mu} \setminus y$ *correspond to non-nesting rook placements on* λ/μ *where column* y *is occupied by a rook.*

Proof. This follows directly from definitions. □

We turn our attention to the property of being minor-closed, which fails for rook matroids. This can be seen by taking any extension of the lattice path matroid $\mathcal{P}_{332/1}$ that happened to also be a rook matroid; see Example 3.8. The deeper reason for this failure is best understood in terms of the observation that minors of fundamental transversal matroids are rarely fundamental transversal.

Example 3.8. Consider $\lambda/\mu = 3322/22$, the skew shape shown in Figure 5 and $M = \mathcal{R}_{\lambda/\mu}$. The matroid $M \setminus 6$ can be seen to be isomorphic to $\mathcal{P}_{332/1}$, which is not a rook matroid. This is because the only possible candidate is $\mathcal{R}_{332/1} \cong Q_6$ (proven below), which is a known excluded minor of lattice path matroids. Thus $M \setminus 6$ is not a rook matroid and rook matroids are not minor-closed[1].

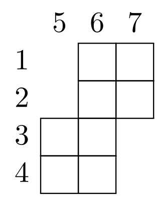


FIGURE 5. The matroid $\mathcal{R}_{3322/22} \setminus 6 \cong \mathcal{P}_{332/1}$ is not a rook matroid.

Below, we show that matroid $\mathcal{R}_{332/1}$ is isomorphic to Q_6, the quaternary matroid of rank 3 on 6 elements. To see that Q_6 is not a lattice path matroid, it suffices to note than Q_6 is an excluded minor for the class of lattice path matroids; see [Bon10, Theorem 3.1] where Q_6 appears in the list as A_3. For further properties of Q_6, see [Oxl11, pg. 641].

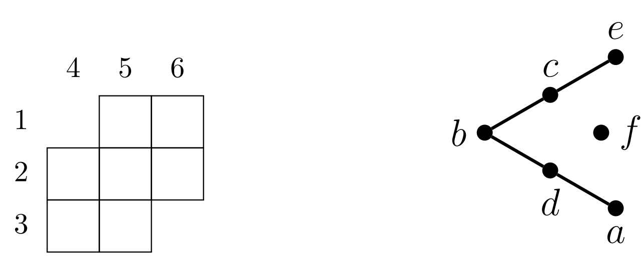


FIGURE 6. The rook matroid $\mathcal{R}_{332/1}$ and the geometric representation of the Q_6 matroid; the two matroids are isomorphic.

Lemma 3.9. *The rook matroid* $\mathcal{R}_{332/1}$ *is isomorphic to the quaternary matroid* Q_6.

[1]We thank Joe Bonin for suggesting an example that did not rely on exhaustive computer search, and for pointing out that the class of rook matroids on 332/1-avoiding skew shapes is *not* minor-closed, as was claimed in an earlier version of this manuscript.

Proof. By checking the three element circuits of each matroid, it is not hard to see that the map from $\{1, 2, \ldots, 6\}$ to $\{a, b, c, d, e, f\}$ defined as $1 \mapsto c, 2 \mapsto f, 3 \mapsto d, 4 \mapsto a, 5 \mapsto b, 6 \mapsto e$ is a matroid isomorphism between $\mathcal{R}_{332/1}$ and Q_6. □

We will return to the link between 332/1-avoiding shapes and lattice path matroids in Section 3.2 by considering the representation of a lattice path as a permutation.

3.2. **Relation to lattice path matroids.** In what follows we collect enumerative and matroidal results linking rook matroids to lattice path matroids, a well-studied family of transversal matroids that also arise from skew shapes. The goal of this subsection is to identify precisely when a rook matroid and a lattice path matroid on the same skew shape are isomorphic.

By *lattice path* we always mean a path from $(0, 0)$ to some integral endpoint consisting only of North and East steps. We will identify a lattice path with its set of *East* steps; note that this convention differs from the traditional treatment of lattice path matroids in [BdMN03, BdM06], where North steps identified a lattice path. Since lattice path matroids are a dual-closed class of matroids, this point of difference has no structural consequences. We use the convention of East steps to make clear a certain bijection between rook placements and lattice paths that we define later.

Let $m, r \geq 0$ and U, L be two lattice paths from $(0, 0)$ to (m, r) such that U never goes below L. Let λ/μ be the skew shape defined by the region enclosed by U and L. Lattice paths can be uniquely identified by the set of their east steps in the following way. For a given lattice path L, traverse L from the bottom left corner of λ/μ to the top right and record the indices of the East steps taken. See Figure 7 below for an example.

Denote by $M[U, L]$ the transversal matroid on $[m + r]$ with presentation given by $\mathcal{E} = (E_1, \ldots, E_m)$ where E_j are East step sets defined by

$$E_j = \{i : i \text{ is the index of the } j^{\text{th}} \text{ East step of some lattice path } P \text{ in } \lambda/\mu\}.$$

Definition 3.10. A *lattice path matroid* (LPM) is a matroid that is isomorphic to the matroid $M[U, L]$ for some lattice paths U, L.

To emphasize the perspective of the matroid arising from a skew shape we will instead use the notation $\mathcal{P}_{\lambda/\mu} := M[U, L]$, where U, L are respectively the upper and lower boundary paths of the diagram defined by the skew shape λ/μ. With this choice we will be studying only loopless and coloopless lattice path matroids since U and L never share a step.

Note that when μ is the empty partition, the lattice path matroid $\mathcal{P}_\lambda$ is called generalized Catalan matroid [BdM06] or *Schubert matroid*.

Our first result connecting rook matroids and lattice path matroids is a straightforward bijection, that naturally leads to the question of when they are isomorphic.

Proposition 3.11. *For any skew shape λ/μ, we have that the sets of bases of $\mathcal{P}_{\lambda/\mu}$ and $\mathcal{R}_{\lambda/\mu}$ have equal cardinality.*

Proof. Given a lattice path inside a skew shape, a cell of the skew shapeis a *valley* if it is immediately above an East-North segment of the path. It is straightforward to see that if we know the location of the valley squares, we can recover the path. Note also that valleys of a path must be located in a non-nesting configuration. Thus given a path, we simply place a rook in each of its valleys in order to produce a non-nesting rook placement. See Figure 8 for an example. Conversely, given a non-nesting rook placement, we fix the valleys of the path precisely

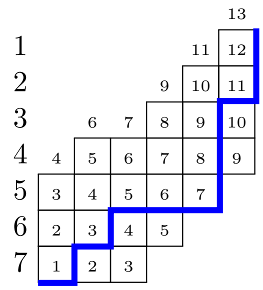


FIGURE 7. The diagram of the skew shape λ/μ for $\lambda = 6666543, \mu = 5431$. Here $m = 6, r = 7$. The numbers in the j^{th} column of the shape represent the elements of the east step set E_j for $j = 1, \ldots, 6$. The set of East steps of the lattice path in blue is $\{1, 3, 5, 6, 7, 11\}$.

at each cell containing a rook. The non-nesting nature of the rook placement ensures that this a valid lattice path. □

Remark 3.12. In particular, when λ is the staircase $(n, n-1, \ldots, 1)$, we obtain the bijection described in Example 2.4.

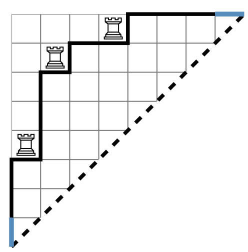


FIGURE 8. Dyck paths and non-nesting rook placements are in bijection by fixing the valleys of the lattice path at the rook positions.

For the exact count of the number of lattice paths contained in λ/μ, there is an elegant determinantal formula that can be deduced from the Lindström–Gessel–Viennot lemma.

Corollary 3.13 (See [Kra15, Theorem 10.7.1] and [SP02, Eq. (6)])**.** *Let λ and μ be partitions with at most ℓ parts (some of which may be zero) such that λ/μ defines a skew shape. Then the number of lattice paths contained in λ/μ equals*

$$\mathcal{P}_{\lambda/\mu} = \det\left[\binom{\lambda_i - \mu_j + 1}{i - j + 1}\right]_{1 \le i,j \le \ell.}$$

Hence this is also the cardinality of $\mathcal{R}_{\lambda/\mu}$.

Observe that the bijection in Proposition 3.11 is not a matroid isomorphism in general, since the outer corners–to–rooks bijection described above was not shown have been induced by a bijection between the groundsets of the matroids. The goal for the remainder of this subsection is to characterize when $\mathcal{R}_{\lambda/\mu}$ and $\mathcal{P}_{\lambda/\mu}$ are isomorphic. As we noted in Section 3.1, the obstruction to this isomorphism comes from the skew shape 332/1 where the lattice path matroid structure and rook matroid structure are not isomorphic. The more surprising fact is that this is the only obstruction for a rook matroid to be a lattice path matroid.

The next sequence of results develops the tools needed to define this matroid isomorphism, using the combinatorics of skew shapes. The first notion involves dividing up the skew shape into rectangles.

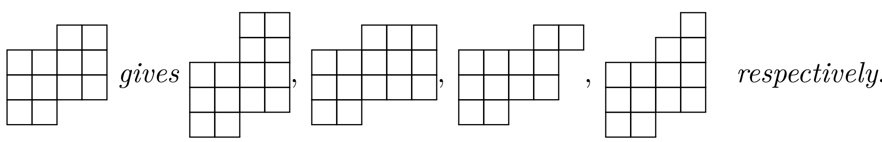
Definition 3.14. The *rectangular decomposition* of a skew shape λ/μ is the collection of rectangles partitioning the skew shape, each of which is formed by grouping together columns that share the same set of row indices.

An example of the rectangular decomposition of a skew shape is shown in Figure 11. We use the rectangular decomposition to give a characterization of 332/1-*avoiding* skew shapes, that is the class of skew shapes *not* containing 332/1 as a subshape. The following proposition gives two ways of describing this class.

Proposition 3.15. *Let λ/μ be a connected skew shape. Then the following are equivalent:*

(i) The shape λ/μ is 332/1*-avoiding.*

(ii) The shape λ/μ can be built from the skew shape consisting of a single cell by using a sequence of following four operations: (I) duplicating the first row, (II) duplicating the last column, (III) adding a box to the first row, (IV) creating a new first row with a single box. These four operations are illustrated on the shape below:



(iii) The rectangular decomposition of λ/μ satisfies the property that for any three consecutive rectangles R_1, R_2, R_3, we do not simultaneously have *that*

(a) the top of R_1 is strictly above the bottom of R_3,

(b) the top of R_1 is strictly below the top of R_2,

(c) the bottom of R_3 is strictly above the bottom of R_2.

Proof. Let λ/μ occupy r rows and c columns. The equivalences are clear if either $r \leq 2$ or $c \leq 2$, so assume for the remainder of this proof that $r, c \geq 3$.

$(i) \implies (ii)$ Let λ/μ be a skew shape not satisfying (iii). We must show that the diagram must then contain 332/1 as a subshape. We may assume that λ/μ is a minimal shape having this property. (Here by minimal with respect to its size $|\lambda/\mu|$) t

By minimality and the assumption above, this implies that λ/μ must satisfy the following properties:

(1) The first two rows of λ/μ have different length.

(2) The last two columns of λ/μ have different length.

(3) The first and second row have their last cells in the same column.

(4) The first row contains more than one box.

Property (1) and (3) imply that λ/μ has an inner corner at the first row; let the corresponding column index of that corner be k. Let the common column index in which the first two rows have their last cell be j. The last two columns of the skew shape will then be $j-1$ and j. By property (2), columns $j-1$ and j have different length which implies that λ/μ has an outer corner in column j; let the corresponding row index of that corner be $s \geq 3$, by the assumption on the number of rows and columns. Then λ/μ contains 332/1 as a subshape since it has an inner corner at $(1,k)$, an outer corner at (s,j) and columns k to $j-1$ — which by the assumption $r, c \geq 3$ are at least two in number — all have the same length.

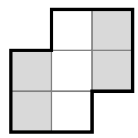


FIGURE 9. Rectangular decomposition of 332/1 shown with shaded rectangles. The skew shape 332/1 is characterized by its rectangles satisfying properties (a),(b),(c) in point (iii) of Proposition 3.15.

(ii) $\implies$ *(iii)* Proceed by induction on $|\lambda/\mu|$; the base case holds vacuously. Assume the result holds for all skew shapes with size less than $|\lambda/\mu|$. Let the skew shape λ_1/μ_1 be the shape before λ/μ to which one of the operations (I)-(IV) was applied and let $\mathcal{D}$ and $\mathcal{D}'$ respectively be the collection of rectangles in the rectangular decompositions of λ/μ and λ_1/μ_1. Consider the operation that transformed λ_1/μ_1 to λ/μ. If it was (I), then duplicating the top row maintains the relative heights of any three rectangles in $\mathcal{D}'$ and hence (iii) holds for $\mathcal{D}$. If it was (II), then only the width of the last rectangle in $\mathcal{D}$ changes while its heights stays the same, so (iii) holds for $\mathcal{D}$. If the last operation was (III) and the number of rectangles in $\mathcal{D}'$ increases, then the location of the bottom of the last rectangle in $\mathcal{D}$ strictly increases, which ensures that property (a) and (b) in point (iii) cannot hold simultaneously with R_3 as this last rectangle. Finally, if the last operation is (IV), then in the non-trivial case, $\mathcal{D}$ has one more rectangle than $\mathcal{D}'$ but this rectangle has the same bottom as the last rectangle in $\mathcal{D}'$ so (c) does not hold.

(iii) $\implies$ *(i)* Suppose λ/μ contains 332/1. We must show that there are three consecutive rectangles in the rectangular decomposition of λ/μ such that *(a)*, *(b)*, *(c)* hold for these.

Since λ/μ contains 332/1 as a subshape, we can then find an inner corner (i_1, j_1) and an outer corner (i_2, j_2) of λ/μ such that the subshape of λ/μ consisting of rows i_1 to i_2 and columns j_1+1 to j_2-1 is a rectangle; call this R_2. Then if R_1 is the rectangle consisting of column j_1 and R_3 is the rectangle consisting of column j_2, it is clear that rectangles R_1, R_2, R_3 satisfy properties (a), (b), (c). See Figure 9 for an example of this on 332/1 itself. This concludes the proof. □

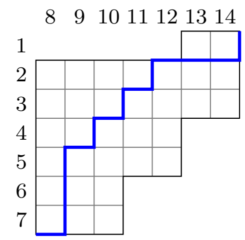
We now introduce a tool that will give us a way to correspond between rook placements and lattice paths. The idea is to represent a lattice path by a permutation the i^{th} entry of which keeps track of which row or column index of the skew shape the i^{th} step of the lattice path occupies, as counted from the bottom.

Definition 3.16 (Path permutation)**.** Let λ/μ be a skew shape with r rows and c columns labeled in the usual way. The *path permutation* corresponding to L is $\psi_L = \psi_1 \ldots \psi_{r+c} \in \mathfrak{S}_{r+c}$

as below:

$$\psi_i = \begin{cases} \text{column index of the column occupied by step } i & \text{if } i \text{ is an East step,} \\ \text{row index of the row occupied by step } i & \text{if } i \text{ is a North step.} \end{cases}$$

Example 3.17. Consider the blue lattice path L drawn in the figure below:



Then the corresponding path permutation is given (in 2-line notation) as

$$\psi_L = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 8 & 7 & 6 & 5 & 9 & 4 & 10 & 3 & 11 & 2 & 12 & 13 & 14 & 1 \end{bmatrix}.$$

Below, we define a very special lattice path inside λ/μ that only exists when λ/μ is 332/1-avoiding. The permutation corresponding to this path turns out to be precisely the bijection that will induce the matroid isomorphism between $\mathcal{P}_{\lambda/\mu}$ and $\mathcal{R}_{\lambda/\mu}$ shown ahead. In Section 5, the skew shape is given a different labeling; the path permutation with respect to this labeling plays a crucial ingredient in the proof of the main result in that section.

Definition 3.18 (Spine path)**.** Suppose that λ/μ is a 332/1-avoiding skew shape contained in an $r \times c$ rectangle. Let $R_1, \dots, R_k$ denote the rectangles in the rectangular decomposition of λ/μ. (Note that these rectangles satisfy item (iii) in Proposition 3.15.) We construct a lattice path L of length $r + c$ as follows.

Start from the lower left hand corner of R_1.

(i) If there is only one rectangle ($k = 1$) then let L be the North–East boundary of R_1.

(ii) If the bottom of R_2 is not at the same level as the bottom of R_1, take North steps until the bottom of R_2 is reached, and then East steps until the lower left corner of R_2 is reached. Recursively construct the rest of L from here based on the shape $R_2, R_3, \dots, R_k$.

(iii) If $k \geq 3$, and if the bottom of R_1 and R_2 are the same, and the top of R_1 is at the same level as the bottom of R_3, then take North steps until the top of R_1 is reached, and then take East steps until the bottom left corner of R_3 is reached. Recursively construct the rest of L based on the shape $R_3, \dots, R_k$.

(iv) In any other case, follow the North–East border of R_1. This ends somewhere on the left side of R_2, say at (p, q). Now construct the rest of L using the part of the shape which is North–East of (p, q).

Observe that we apply this case if (a) $k = 2$, (b) the bottoms of R_1, R_2 and R_3 are the same, or (c) when the bottom of R_3 is strictly above the top of R_2.

(Observe, the case when the bottom of R_3 is strictly below the top of R_1 does not appear, since this would violate the 332/1-avoiding property, see Proposition 3.15, so we have covered all cases.)

See Figure 10 for an illustration of the four cases.

The resulting lattice path is called the *spine path* of λ/μ. By definition, the spine path is always contained inside λ/μ.

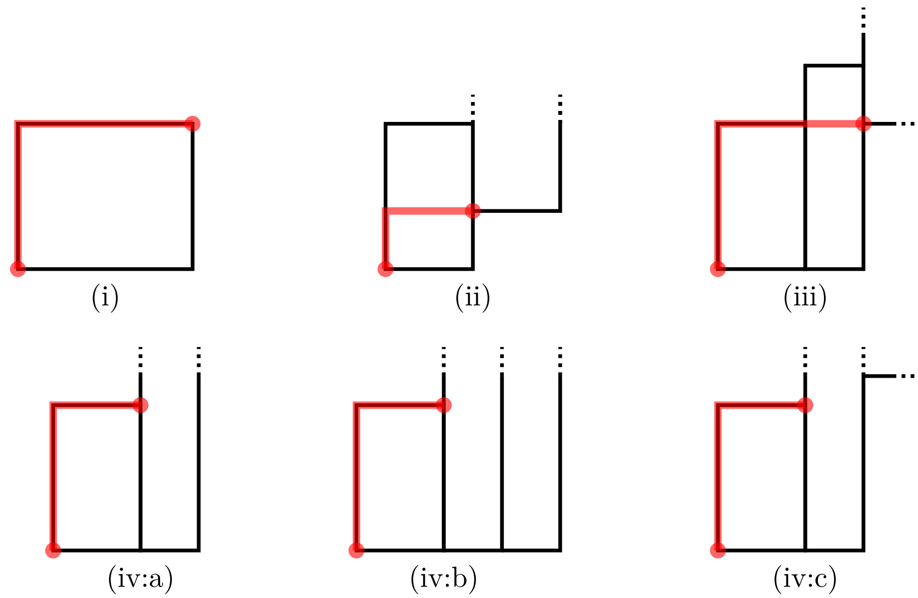


Figure 10. The four cases for the construction of the spine path (shown in thick red) detailed in Definition 3.18.

Example 3.19. The spine path is particularly easy to construct for Ferrers shapes, see the first shape in Figure 11.

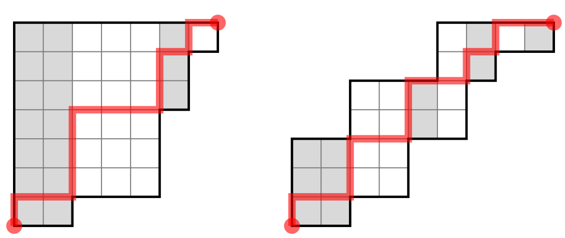


Figure 11. Two examples of the spine path in two 332/1-avoiding skew shapes; the rectangular decomposition of the shapes is indicated by the shading.

Below, when we refer to a North-East segment of L, we mean the word `NE` as a subword of L when written as a sequence of `N`s and `E`s.

Lemma 3.20. *Suppose L is the spine path of a 332/1-avoiding skew shape λ/μ, and consider a corner of the spine path formed by a North-East segment of L occuring in row p and column q of λ/μ. Then the subshape $\{(i,j) \in \lambda/\mu : i \geq p, j \geq q\}$ of λ/μ is a rectangle.*

Similarly, if an East–North corner of L occurs in column q and row p, then the subshape $\{(i,j) \in \lambda/\mu : i \leq p, j \leq q\}$ of λ/μ is a rectangle.

Remark 3.21. See Figure 11 for an example of this property. The idea behind this lemma is that no run of East steps of the spine path intersects the interior of more than one rectangle in the skew shape.

Proof. We only prove the statement concerning the North-East segments of L since the other case can be proved analogously. Let $R_1, \dots, R_k$ be the rectangles in the rectangular decomposition of λ/μ. We note that each rectangle in the rectangle decomposition of a skew shape can contain at most one subshape induced by a North-East segment; this is because any run of East steps of the spine path intersects the interior of at most one rectangle in the skew shape. Let $D_1, \dots, D_k$ be these (possibly empty) Ferrers shapes. If there is indeed a `NE` segment of L in R_i, we will call D_i non-degenerate.

We proceed by induction on k. When $k = 1$, λ/μ is itself a rectangle and L is, by construction, the North-East boundary of λ/μ. Hence the subshape formed by the only North-East segment of L is also a rectangle.

If $k = 2$ or 3, the result follows by a casewise inspection of the relative heights of the rectangles R_1, R_2, R_3 and the construction of the spine path in Definition 3.18.

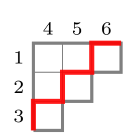
Suppose now $k > 3$ and the result holds for all skew shapes with at most $k-1$ rectangles in the rectangle decomposition. Let L be the spine path of λ/μ. By construction of L, the restriction of L to the rectangles $R_1, \dots, R_{k-3}$ is the spine path L' of the skew shape with rectangular decomposition $R_1, \dots, R_{k-3}$. Applying the induction hypothesis to this skew shape and L', we see that each of $D_1, \dots, D_{k-3}$ is a rectangle.

It remains to consider D_i for $i = k-2, k-1$ and k. If D_{k-2} is non-degenerate, one of two cases can occur: (a) L crosses the interior of R_{k-2}. This corresponds to case (ii) in the spine path construction and hence D_{k-2} is clearly a rectangle; or else we have (b) L does not cross the interior of R_{k-2}, in which instance either case (iii) or (iv) from the spine path construction holds. In each of these cases, the rectangle(s) succeeding R_{k-2} have the same bottom as R_{k-2} so once again D_{k-2} is a rectangle. If D_{k-1} is non-degenerate, only cases (ii) and (iv a) of the spine path construction can occur; in both cases it is immediate that D_{k-1} is a rectangle. Finally by construction of the spine path, L travels along the North-East boundary of R_k and hence D_k equals R_k itself and is hence a rectangle. □

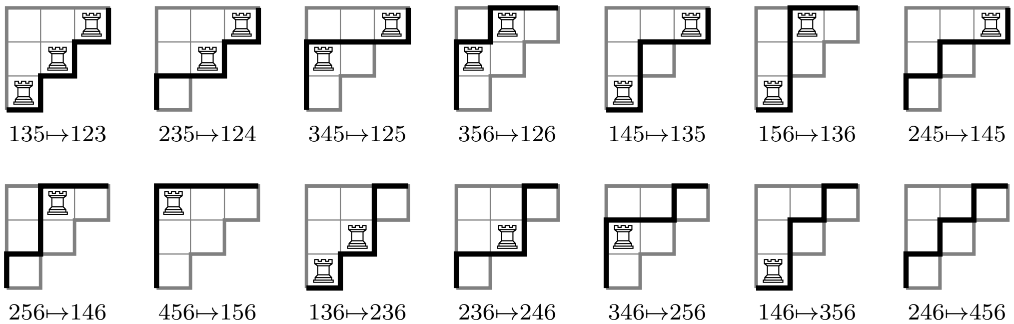
For brevity, let $\psi_{\lambda/\mu}$ be the path permutation determined by the spine path of λ/μ whenever λ/μ is a 332/1-avoiding skew shape. This is called the *spine bijection* for short. We will show that if λ/μ is a 332/1-avoiding skew shape then its spine bijection induces a bijection from the bases of $\mathcal{P}_{\lambda/\mu}$ to the bases of $\mathcal{R}_{\lambda/\mu}$. By an abuse of notation, we call the induced map between set systems $\psi_{\lambda/\mu}$. Note that the spine bijection sends the basis in $\mathcal{P}_{\lambda/\mu}$ corresponding to the spine path to the non-nesting rook placement in $\mathcal{R}_{\lambda/\mu}$ without any rooks. Before proceeding with the proof, we first give an example.

Example 3.22. Consider the case $\lambda = 321$. The spine bijection ψ_λ (in 2-line notation) and the corresponding spine path are shown below.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 5 & 1 & 6 \end{bmatrix}$$



We can see by inspection that ψ_λ induces a bijection from lattice paths in $\mathcal{P}_{321}$ to non-nesting rook placements in $\mathcal{R}_{321}$. We illustrate the induced bijection $\psi_{321} : \mathcal{P}_{321} \to \mathcal{R}_{321}$ below by drawing the lattice paths together with the corresponding non-nesting rook placements. Each shape is labeled with the lattice path basis on the left and its corresponding non-nesting rook placement basis on the right.



A crucial ingredient in generalizing the idea above is the presence of a spine path, which we know can only be defined in 332/1-avoiding shapes. The following theorem shows that a rook matroid and lattice path matroid on the same skew Ferrers shape are isomorphic if and only if 332/1 does not occur as a subshape of λ/μ.

Before we prove this, we introduce a definition that allows one to move between lattice paths contained inside a skew shape.

Definition 3.23. Two lattice paths L and L' are related by a *flip* if L can be obtained from L' by either (a) flipping an East-North segment of L' to a North-East segment, or (b) flipping a North-East segment to an East-North segment.

We are now ready to state the main theorem of this subsection.

Theorem 3.24. *The lattice path matroid $\mathcal{P}_{\lambda/\mu}$ is isomorphic to the non-nesting rook matroid $\mathcal{R}_{\lambda/\mu}$ if and only if λ/μ is a* 332/1*-avoiding skew shape. In that case, an explicit bijection from $\mathcal{P}_{\lambda/\mu}$ to $\mathcal{R}_{\lambda/\mu}$ is given by $\psi_{\lambda/\mu}$.*

Proof. For the only if direction: suppose λ/μ contains 332/1 as a subshape, then $\mathcal{R}_{\lambda/\mu}$ would contain Q_6 as a minor by Lemma 3.9 and Lemma 3.7. But Q_6 is an excluded minor (see [Bon10, Thm. 3.1]) for the class of lattice path matroids so $\mathcal{R}_{\lambda/\mu}$ cannot be isomorphic to $\mathcal{P}_{\lambda/\mu}$.

For the if direction: Let λ/μ be a 332/1-avoiding skew shape on r rows and c columns. Since λ/μ avoids 332/1, we can consider its spine path L_0 and the associated spine bijection $\psi_{\lambda/\mu}$, which for the purpose of brevity we occasionally refer to as ψ. Recall that we are identifying a lattice path by its set of East steps.

We introduce a working example on the 332/1-avoiding skew shape 7775533/5 to illustrate the steps that follow. See Figure 12, from where we obtain the two-line notation of the spine bijection $\psi_{775533/33}$:

$$\psi_{775533/33} = \left[\begin{array}{cc|ccc|cc|cc|ccc|cc} 1 & 2 & \textcircled{3} & \textcircled{4} & \textcircled{5} & 6 & 7 & \textcircled{8} & \textcircled{9} & 10 & 11 & 12 & \textcircled{13} & \textcircled{14} \\ 7 & 6 & 8 & 9 & 10 & 5 & 4 & 11 & 12 & 3 & 2 & 1 & 13 & 14 \end{array}\right].$$

In the two-line notation of the spine path permutation, we partition the columns into *blocks*, formed by runs of steps in the same direction (North or East). We then circle the East steps of the spine path.

Since $\psi_{\lambda/\mu}$ is a bijection from $[r+c]$ to itself, the result will follow if we can show that $\psi_{\lambda/\mu}$ maps lattice paths inside λ/μ to non-nesting rook placements on λ/μ. To that end, we make the following claim.

Claim: If L is a lattice path contained inside λ/μ, then $\psi_{\lambda/\mu}(L)$ defines a non-nesting rook placement on λ/μ.

We prove the claim by induction on k, the number of cells between L and the spine path. For the $k=0$ case, L is the spine path itself, the East steps of which, by construction, correspond to the column indices of λ/μ which implies that $\psi_{\lambda/\mu}(L_0)$ is the empty rook placement.

Now suppose $k > 1$ and consider any lattice path L^+ contained inside λ/μ with k cells between it and L_0. Then L^+ can be obtained from some lattice path L in λ/μ with $k-1$ cells between L and L_0, via a flip. In the two-line notation of $\psi_{\lambda/\mu}$, circle the East steps of L. By induction hypothesis, the values below the circled entries, $\psi_{\lambda/\mu}(L)$, defines a valid non-nesting rook placement ρ_L on λ/μ. In the case of our example, this becomes:

$$\psi_{775533/33} = \left[\begin{array}{cc|ccc|cc|cc|ccc|cc} 1 & \textcircled{2} & 3 & \textcircled{4} & 5 & 6 & \textcircled{7} & 8 & \textcircled{9} & 10 & \textcircled{11} & \textcircled{12} & \textcircled{13} & 14 \\ 7 & 6 & 8 & 9 & 10 & 5 & 4 & 11 & 12 & 3 & 2 & 1 & 13 & 14 \end{array}\right].$$

Figure 12 illustrates this on our running example. At the level of the circled entries of the first line of $\psi_{\lambda/\mu}$, the flipping of an `EN` segment to a `NE` segment or vice versa corresponds to *moving a circle* (the index of an East step) either one position left or right. In the running example, there are four such North-East segments and three East-North segments which can be flipped.

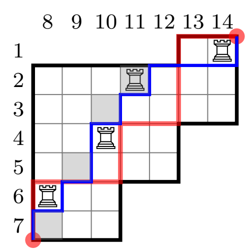


FIGURE 12. The skew shape $\lambda/\mu = 7775533/5$ together with its spine path L_0 in red, a representative lattice path $L = \{2,4,7,9,11,12,13\}$ in blue and the corresponding non-nesting rook placement $\psi_{\lambda/\mu}(L) = \{1,2,4,6,9,12,13\}$. The shaded cells from bottom to top correspond to `NE` or `EN` segments the flips of which are of type B, A, D, and C respectively.

We will show that $\psi_{\lambda/\mu}(L^+)$ is also a non-nesting rook placement contained within λ/μ. The set $\psi_{\lambda/\mu}(L^+)$ certainly defines a non-nesting rook placement within the rectangle occupied by λ/μ. It remains to show that this rook placement has its rooks on λ/μ as well. By Lemma 3.20, we know that any such flip occurs in a rectangular region determined by the spine path. Note that such a rectangular region is given by row and column indices in two adjacent blocks in the two-line notation. For example, the fifth and sixth block in our example, corresponds to the rectangle in rows $1,2,3$ and columns $13,14$.

Let i and $i+1$ be consecutive steps of the lattice path L such that exactly one of them is an East step. (Recall that a circled index is an East step). Consider the row/column indices $\psi(i)$ and $\psi(i+1)$ and recall that we have partitioned the columns of the two-line array of the spine bijection $\psi_{\lambda/\mu}$ into blocks. There are four cases to consider:

A. *$\psi(i)$ and $\psi(i+1)$ are both column indices from the same block.*

B. *$\psi(i)$ and $\psi(i+1)$ are both row indices from the same block.*

C. *$\psi(i)$ and $\psi(i+1)$ are in different blocks, and the East step (recall, exactly one of i and $i+1$) is mapped to a column index.*

D. *$\psi(i)$ and $\psi(i+1)$ are in different blocks, and the East step of L is mapped to a row index.*

These four cases appear in the black lattice path L shown in Figure 12. We have a Case A flip for step $i = 4$, Case B for step $i = 1$, Case C for step $i = 9$ and Case D for step $i = 7$.

We now describe how a flip on a segment of L affects the rook placement ρ_L in each case. It is clear in each case that $\psi_{\lambda/\mu}(L^+)$ is obtained from $\psi_{\lambda/\mu}(L)$ by swapping $\psi(i)$ or $\psi(i+1)$ for the other element. We will see that in every case this corresponds to moving the rooks of ρ_L such that they still stay on λ/μ, since all operations are performed on some *rectangle* above or below the spine path as guaranteed by Lemma 3.20. We consider each of the cases above.

A. In this case we have two adjacent columns $\psi(i)$ and $\psi(i+1)$ where one is empty the other is occupied by a rook of ρ_L. We can then move the rook horizontally to the adjacent column and obtain a new valid rook placement on λ/μ.

B. In this case we have two adjacent rows $\psi(i)$ and $\psi(i+1)$ where one is empty the other is occupied by a rook. The rook is moved vertically to the adjacent row to obtain a new new valid rook placement on λ/μ.

C. There are two sub-cases to consider.

   In the first case, we have an empty column $p = \psi(i)$ and an empty row $q = \psi(i+1)$ of ρ_L and need to make[2] both occupied without affecting the occupancy of the other rows or columns, or the non-nesting nature of the other rooks of ρ_L.

   There are two configurations to consider. Either the rectangle North-West of the cell (q, p) is empty, or it contains some rooks (marked with $\bullet$). In the first case, we simply place a rook in (q, p). This is shown in the figure on the left below.

   In the second configuration, we form a "staircase" of cells (marked $\circ$) based on the location of the existing rooks in the rectangle North-West to (q, p), as seen in the figure above on the right. We then move the rooks of ρ_L to the the locations marked $\circ$, once again obtaining a new valid non-nesting rook placement on λ/μ, without affecting the occupancy of any other row or column of ρ_L.

   In the second case[3], we have an empty row $p = \psi(i)$ and an empty column $q = \psi(i+1)$ of ρ_L. The same operation as above is performed but the figures are rotated 180°.

[2]In the two-line array, this corresponds to moving a circle one step to the right.
[3]In the two-line array, this corresponds to moving a circle one step to the left.

D. As in C, there are two cases. In the first case, column $q = \psi(i)$ and row $p = \psi(i+1)$ are occupied (the second case is treated analogously). Again, two configurations may arise: one where ρ_L has a rook in (p,q) and one where it does not. This situation is precisely the inverse of Case C:

This shows that $\psi_{\lambda/\mu}(L^+)$ also defines a non-nesting rook placement on λ/μ, completing the proof of the claim. With this, the proof of the theorem follows since $\psi_{\lambda/\mu} : \mathcal{P}_{\lambda/\mu} \to \mathcal{R}_{\lambda/\mu}$ is a well-defined map between the collections of bases of two matroids that is induced by a bijection between their groundsets. □

Example 3.25. We describe the results of performing the four types of flips in Figure 12. Recall that the lattice path L is described by the indices of its East steps, $L = \{2,4,7,9,11,12,13\}$.

- The flip of type A at step 4 gives the new path $L^+ = \{2,5,7,9,11,12,13\}$.
- The flip of type B at step 1 gives the new path $L^+ = \{1,4,7,9,11,12,13\}$.
- The flip of type C at step 9 gives the new path $L^+ = \{2,4,7,10,11,12,13\}$.
- The flip of type D at step 7 gives the new path $L^+ = \{2,4,8,9,11,12,13\}$.

Note that in Case B, we need to flip a North-East segment of L in Figure 12 to an East-North segment of L^+ in the second picture from the left in Figure 13, in order to increase the number of cells between L and the spine path.

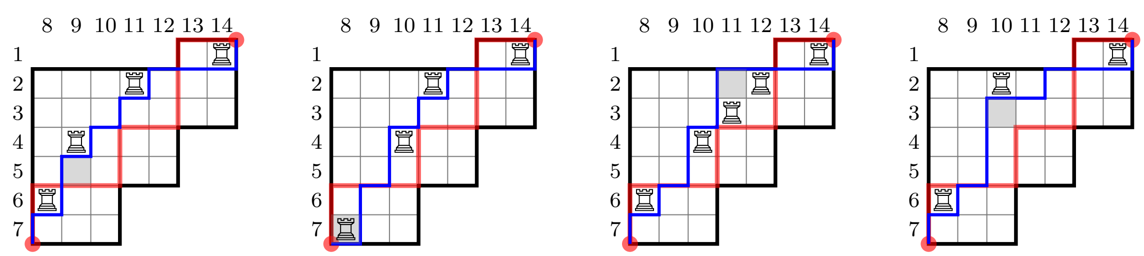


FIGURE 13. From left to right: the four new rook placements induced by the lattice path L^+ (shown in blue) after the four types of flips (A,B,C,D). The shaded cell in each skew shape shows where the flip occurred.

Remark 3.26. It would be of interest to find a matroidal proof of Theorem 3.24. One reason that this might be possible is that the isomorphism $\mathcal{R}_\lambda \cong \mathcal{P}_\lambda$ can be proven from a matroid theoretic perspective as follows: Ferrers shapes can be built from a single box, using top row duplication and single box addition; the former operation corresponds to free extension of a rook matroid by a single element and the latter corresponds to series extension of a rook matroid by a single element. A similar strategy would be to find the correct type of matroid extension that corresponds to operations (I)-(IV) in Proposition 3.15 (ii) for all 332/1-avoiding shapes.

Theorem 3.24 gives us a single criterion for when a rook matroid is isomorphic to a lattice path matroid, in terms of avoiding a fixed subshape.

In a different direction, we can obtain lattice path matroids as contractions of large enough rook matroids, where the meaning of large enough is made precise in the following proposition.

Proposition 3.27. *Suppose λ/μ is a skew shape with r rows. The lattice path matroid $\mathcal{P}_{\lambda/\mu}$ is isomorphic to the contraction $\mathcal{R}_{\lambda^*/\mu^*}/[r]$, i.e.*

$$\mathcal{P}_{\lambda/\mu} \cong \mathcal{R}_{\lambda^*/\mu^*}/[r],$$

where

$$\lambda^* := (\lambda_1 + r, \lambda_2 + r - 1, \ldots, \lambda_{r-1} + 2, \lambda_r + 1),$$
$$\mu^* := (\mu_1 + r - 1, \mu_2 + r - 2, \ldots, \mu_\ell + r - \ell).$$

Proof. Let λ/μ have c columns. Note that $\mathcal{P}_{\lambda/\mu}$ and $\mathcal{R}_{\lambda^*/\mu^*}/[r]$ are matroids on the ground sets $[r+c]$ and $[r+1, 2r+c]$, respectively. Let $f : [r+c] \to [r+1, 2r+c]$ be defined by $f(i) = i + r$; then f is clearly a bijection. It suffices to show that f sends bases of $\mathcal{P}_{\lambda/\mu}$ to bases of $\mathcal{R}_{\lambda^*/\mu^*}/[r]$. In other words, f must send lattice paths contained in λ/μ to rook placements contained in λ^*/μ^* such that every row is occupied. The map f modifies the lattice path L in λ/μ to obtain a lattice path L' in λ^*/μ^* such that every North step of L becomes an East-North step of L'. An example is shown in Figure 14. A rook is then placed in every outer corner of the path, and since we have one outer corner for each row, we are done. ☐

(A) Skew shape $\lambda/\mu = 332/1$ with the path $L = 156$ marked in blue.

(B) Skew shape $\lambda^*/\mu^* = 653/3$ with the path L' marked in blue.

FIGURE 14. The map $i \mapsto i + r$ in Proposition 3.27 adds an `E` step before every `N` step in L to yield a lattice path L' with a rook in each of its outer corners.

Rook matroids are shown to be positroids in Section 3.3, and positroids are known to be closed under taking minors [ARW16, Proposition 3.5] and under a cyclic shift of the groundset [ARW16, Lemma 3.3]. Since the map f in the proof of Proposition 3.27 is a cyclic shift, we can thus deduce that every lattice path matroid is a positroid, a fact first proven in [Oh11].

The following corollary is immediate from Proposition 3.27.

Corollary 3.28. *Let λ be a partition of length ℓ, and*

$$\lambda^{\triangleright} := (\lambda_1 + \ell, \lambda_2 + \ell - 1, \ldots, \lambda_{\ell-1} + 2, \lambda_\ell + 1).$$

We then have the following matroid isomorphism:

$$\mathcal{R}_\lambda \cong \mathcal{R}_{\lambda^{\triangleright}}/\{1, 2, \ldots, \ell\}.$$

This also gives us a matroidal way to interpret *maximal* non-nesting rook placements on a Ferrers diagram and note that the matroid structure is the same as that of a Schubert matroid.

3.3. Relation to positroids.

Definition 3.29. Let A be a $d \times n$ matrix with real entries such that all its maximal minors are non-negative. The representable matroid $M(A)$ associated to A is called a *positroid*.

The property of being a positroid is strongly dependent on the total ordering on the ground set of the matroid. The assertions we make in this subsection depend on the ordering previously introduced on the ground set of the rook matroid; for convenience, we give this ordering a name.

Definition 3.30. We say that a skew shape λ/μ with r rows and c columns has the *matroid labeling* if its rows are labeled 1 to r from top to bottom and the columns are labeled $r+1$ to $r+c$ from left to right.

Throughout this subsection, every rook matroid will have the matroid labeling. This turns out to be the correct total ordering on the ground set in order for the rook matroid to be a positroid; the reason for this is hinted at when determining the Grassmann necklace of this matroid.

As alluded to before, there are several classes of combinatorial objects with which positroids are in bijection[Pos06]. The Grassmann necklaces make up one such class; we recall their definition below.

Definition 3.31. Let $d \leq n$ be positive integers. A Grassmann necklace of type (d, n) is a sequence $(I_1, \dots, I_n)$ of d-subsets $I_k \in \binom{[n]}{d}$ such that for any $i \in [n]$,

(1) If $i \in I_i$, then $I_{i+1} = I_i \setminus \{i\} \cup \{j\}$ for some $j \in [n]$.

(2) If $i \notin I_i$, then $I_{i+1} = I_i$, where $I_{n+1} \coloneqq I_1$.

The *i-order* on $[n]$ is the total order defined by

$$i <_i i+1 <_i \dots <_i n <_i 1 <_i \dots <_i i-2 <_i i-1.$$

Fixing $i \in [n]$, the *Gale order* with respect to $<_i$ is the partial order on $\binom{[n]}{d}$ defined by:

$$S \leq_i T \iff s_j \leq_i t_j \quad \text{for all } j \in [d],$$

where $S = \{s_1 <_i \dots <_i s_d\}$ and $T = \{t_1 <_i \dots <_i t_d\}$ are ordered in the i-order.

Grassmann necklaces arise naturally from matroids in the following way. Let $M = ([n], \mathcal{B})$ be a matroid of rank d and let I_k be the lexicographically minimal basis in $\mathcal{B}$ with respect to $<_k$. Then $\mathcal{I}(M) = (I_1, \dots, I_n)$ is the *Grassmann necklace of a matroid*, from the following proposition.

Proposition 3.32. [Pos06, Lemma 16.3] *For any matroid $M = ([n], \mathcal{B})$ of rank d, $\mathcal{I}(M)$ is a Grassmann necklace of type (d, n).*

Conversely, matroids—more specifically positroids—can be cooked up from Grassmann necklaces by the theorem below. This also yields necessary and sufficient criteria for a matroid to be a positroid in terms of Grassmann necklaces. This was first conjectured by [Pos06] who proved necessity; the sufficiency was proven by Oh [Oh11].

Theorem 3.33 ([Oh11, Pos06])**.** *Let $\mathcal{I} = (I_1, \dots, I_n)$ be a Grassmann necklace of type (r, n) and let*

$$B(\mathcal{I}) = \left\{ B \in \binom{[n]}{r} : B \geq_j I_j \text{ for all } j \in [n] \right\}.$$

Then $M(\mathcal{I}) = ([n], B(\mathcal{I}))$ is a matroid of rank r on $[n]$ with collection of bases $B(\mathcal{I})$. Furthermore, if $M = ([n], \mathcal{B})$ is a matroid with Grassmann necklace $\mathcal{I} = \mathcal{I}(M)$, then $\mathcal{B} \subseteq B(\mathcal{I})$, with equality holding if and only M is a positroid.

With this background in place, we make the following definition to distinguish certain non-nesting rook placements.

Definition 3.34. Let λ/μ be a skew shape with r rows and c columns. For $i \in [r+c]$, the *i-extremal non-nesting rook placement* on λ/μ is the rook placement obtained by the following procedure.

If $i \in [1, r]$:

(1) Place a rook in the last cell of row i.

(2) For $j \in [i+1, r]$, place a rook in row j as far to the right as possible, while ensuring the non-nesting condition holds.

If $i \in [r+1, r+c]$:

(1) For $j \in [i, r+c]$, leave column j unoccupied.

(2) For $j \in [1, r]$, place a rook in row j as far to the right as possible while maintaining the unoccupied status of the columns in steps (1).

The resulting rook placement is non-nesting since the non-nesting condition is maintained at every step. An example is illustrated in Figure 15. The name is justified by the following lemma which characterises Grassmann necklaces of rook matroids in terms of extremal rook placements.

FIGURE 15. The 3-extremal and 8-extremal non-nesting rook placement on 54421/31 on the left and right respectively.

Proposition 3.35. *Let $\mathcal{R}_{\lambda/\mu}$ be the rook matroid on skew shape λ/μ. Let $\mathcal{I} = (I_1, \dots, I_{r+c})$ be the associated Grassmann necklace. Then for each $i \in [r+c]$, I_i is the i-extremal non-nesting rook placement on λ/μ.*

Proof. Let $\mathcal{B}$ be the collection of bases of $\mathcal{R}_{\lambda/\mu}$. Fix $i \in [r+c]$. By definition, I_i is the lexicographically minimal element in $\mathcal{B}$ with respect to the i-order $<_i$; let the corresponding non-nesting rook placement be τ_i. Let the i-extremal non-nesting rook placement on λ/μ be ρ_i. Order the elements of I_i and $R(\rho_i) \cup C(\rho_i)$ in the i-order as follows:

$$I_i = R(\tau_i) \cup C(\tau_i) = \{j_1 <_i j_2 <_i \dots <_i j_c\},$$
$$R(\rho_i) \cup C(\rho_i) = \{k_1 <_i k_2 <_i \dots <_i k_c\}.$$

Claim: The equality $I_i = R(\rho_i) \cup C(\rho_i)$ holds.

We will prove that $j_s = k_s$ by induction on s. The $s = 1$ case is immediate since since $k_1 = i$ by construction and $i \leq_i j_1 \leq_i k_1$. Fix $2 \leq s \leq c$ and suppose $j_1 = k_1, \dots, j_{s-1} = k_{s-1}$. Since

$i = j_1 <_i j_s$, we only consider the case when $j_s \in [i, r+c]$ in the i-order which coincides with the usual order on this interval; we will henceforth use $<$ in place of $<_i$.

Suppose i is a row index. Consider the element $k_s \in R(\rho_i) \cup C(\rho_i)$. Two cases arise:

Case 1: If $k_s \leq r$, then ρ_i has a rook as far to the right as possible in row k_s and by the procedure to generate extremal rook placements, there must necessarily be a rook in the row above; i.e. $k_s = k_{s-1} + 1$. Putting this together with the induction hypothesis, we get that

$$j_{s-1} < j_s \leq k_s = k_{s-1} + 1 = j_{s-1} + 1,$$

so we must have equality between j_s and k_s.

Case 2: If $k_s \geq r+1$, then column k_s is empty. If $k_{s-1} \geq r+1$, then there is no rook in column k_{s-1} either. By the same reasoning as in Case 1, we must necessarily have $j_s = k_s$.

If $k_{s-1} \leq r$ then columns $r+1$ to $k_s - 1$ are occupied by a rook of ρ_i. This means that j_s cannot equal any column index $r+1, \dots k_s - 1$, since τ_i and ρ_i have rooks in rows corresponding to the columns above. This also means that the rook of ρ_i in column $r+1$ occurs in row k_{s-1}. If j_s were to be a row index between $k_{s-1}+1$ and r, then there would be two rooks of τ_i in column $r+1$, which is infeasible. Hence j_s is not equal to any index in the interval (in the usual order) $[k_{s-1}+1, k_s - 1]$. By minimality, we must then have $j_s = k_s$, and this completes the proof for the case when i is a row index.

When i is a column index, the proof proceeds along similar lines. Assume $2 \leq s \leq c$ and $k_1 = j_1, \dots, k_{s-1} = j_{s-1}$ holds. Again, two cases arise:

Case 1: If $k_s \leq r+c$, then column k_s is empty and by construction of ρ_i, column k_{s-1} must also be empty and hence $k_{s-1} = k_s - 1$ which by the same reasoning used above implies that $j_s = k_s$.

Case 2: If k_s is a row index, then ρ_i has a rook in row k_s. If k_{s-1} is also a row index, which by the procedure that defines extremal rook placements, again implies that $k_{s-1} = k_s - 1$, from which $j_s = k_s$. If, on the other hand, k_{s-1} is a column index, then by virtue of k_s being a row index and the definition of ρ_i, it must be that $k_{s-1} = r+c$ and the rook in row k_s is in column $i-1$. If j_s were any row index between 1 and $k_s - 1$ then τ_i, and hence ρ_i, would have a rook in some column between i and $r+c$, which contradicts the fact that these columns are unoccupied by construction. By minimality, we must then have $j_s = k_s$, thereby completing the proof. □

We will make use of the above characterization to prove that rook matroids are positroids.

Theorem 3.36. *Let λ/μ be a skew shape and $\mathcal{R}_{\lambda/\mu}$ be the corresponding rook matroid. Then $\mathcal{R}_{\lambda/\mu}$ is a positroid.*

Proof. We will use the criterion in Theorem 3.33. Let $\mathcal{I} = (I_1, \dots, I_{r+c})$ be the Grassmann necklace of $\mathcal{R}_{\lambda/\mu}$. By Lemma 3.35, each I_i corresponds to the i-extremal non-nesting rook placement on λ/μ. Consider $B \in \binom{[r+c]}{c}$ satisfying $B \geq_j I_j$ for all $j \in [r+c]$, where $\geq_j$ denotes the j^{th} Gale order. We need to show that B is a basis of $\mathcal{R}_{\lambda/\mu}$. Since B is some c-subset of $[r+c]$, there is some non-nesting rook placement ρ on the $r \times c$ rectangle such that $B = R(\rho) \cup C(\rho)$. We will be done if we can show that ρ lies within λ/μ. Suppose the rooks of ρ occupy the cells (k_i, j_i), for $i = 1, \dots, m$ where $k_1 < \ldots < k_m$. For each $i = 1, \dots, m$, consider the corresponding k_i-extremal non-nesting rook placements $I_{k_i} = R(\sigma_i) \cup C(\sigma_i)$ and let (k_i, ℓ_i) be the first rook of σ_i in the k_i-order. Since ℓ_i is the last cell in row k_i, we need to show that $j_i \leq \ell_i$ for $i = 1, \dots, m$. We will prove this one rook at a time. Suppose to the contrary that $j_1 > \ell_1$.

Let the elements of I_{k_1} and B be respectively ordered in the k_1-order as

$$I_{k_1} = \{t_1 <_{k_1} t_2 <_{k_1} \dots <_{k_1} t_c\},$$
$$B = \{k_1 <_{k_1} k_2 <_{k_1} \dots <_{k_1} k_c\}.$$

Since (k_1, j_1) contains a rook of ρ, we must have $j_1 \in I_{k_1}$, since by the definition of the k_1-extremal rook placement, all the rooks of σ_1 occur in the interval $[k_1, r]$. So $j_1 = t_n$, for some $n \in [1, c]$.

Note that ℓ_1 is the last occupied column of I_{k_1}, so there exists t_q such that $[\ell_1 + 1, r + c] = [t_q, t_c] \subset I_{k_1}$. Also, the maximal element of B in the k_1-order must necessarily be a column index, since otherwise we would have a rook in σ_1 with a row index smaller than k_1. Since $B \geq_{k_1} I_{k_1}$, we must have $k_i \geq_{k_1} t_i$ for $i \in [q, c]$. Since the elements on both sides of the inequality are column indices, the previous inequality also holds in the usual order. Then the sequence $(t_i)_{i=q}^{c}$ forms an interval of column indices terminating at the last column $r + c$ so we must necessarily have $k_i = t_i$ for $i \in [q, c]$. Since $j_1 > \ell_1 = t_q$, this implies in particular that $k_n = t_n = j_1$. Then $j_1 \in B$, contradicting the fact that ρ has a rook in (k_1, j_1). Thus $j_1 \leq \ell_1$.

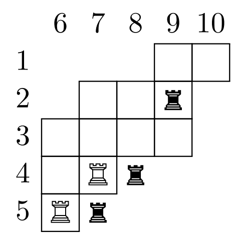


FIGURE 16. The black and white rook placements respectively represent $B = \{4 <_4 5 <_4 6 <_4 10 <_4 2\}$ and $I_4 = \{4 <_4 5 <_4 8 <_4 9 <_4 10\}$. Here $B \geq_4 I_4$ fails to hold since $6 <_4 8$.

Now consider (k_2, j_2); if $j_2 > \ell_2$ then by similar reasoning as above, we must have $j_2 \in I_{k_2}$. Also note that the maximal element of B in the k_2-order is k_1. Since ℓ_2 is the last occupied column index of σ_2, the interval $[\ell_2 + 1, r + c]$ is a subset of I_{k_2}, forming its last $r + c - \ell_2$ elements in the k_2 order. Let the column indices of ρ corresponding to $\ell_2 + 1, \dots, r + c - 1$ in the k_2-order be $k_{s+1}, \dots, k_{s+r+c-\ell_2-1}$. Since $B \geq_{k_2} I_{k_2}$, all of these column indices must take distinct values in the interval $[\ell_2 + 1, r + c]$ but cannot take on the values j_1 or j_2, since ρ has a rook in these columns. Thus $r + c - \ell_2 - 1$ indices have a total of $r + c - \ell_2 - 2$ values to take on, which is impossible, since the elements of B are distinct. We must hence have $j_2 \leq \ell_2$. An example is illustrated in Figure 16. Repeating this argument for the remaining row indices $k_3, \dots, k_m$ of ρ, we see that each corresponding column index j_i satisfies $j_i \leq \ell_i$, which completes the proof. □

3.4. **Tutte polynomials.** In this subsection, we show that the Tutte polynomials of rook matroids and lattice path matroids coincide, thereby tightening the correspondence between these classes of matroids. This fact is not obvious a priori, given the fact that, from Theorem 3.24, lattice path matroids and rook matroids are not always isomorphic.

The *Tutte polynomial* is defined as the corank-nullity generating function of the matroid M:

$$T_M(x, y) := \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

For an overview of results on Tutte polynomials, see [EMM22]. The important fact about Tutte polynomials that we will use is that it satisfies the deletion-contraction recursion:

$$T_M(x,y) = T_{M/e}(x,y) + T_{M\setminus e}(x,y), \tag{3}$$

where $e \in E$ is neither a loop nor a coloop.

In order to prove the stated equality of Tutte polynomials, we need the following simple lemma.

Lemma 3.37. *Let M be a rook matroid on a skew shape with r rows and c columns. Then the two contractions M/r and $M/(r+1)$ are isomorphic, and thus have the same Tutte polynomial.*

Proof. Bases of M/r correspond to rook placements with a rook in the last row of the skew shape and bases of $M/(r+1)$ correspond to rook placements with no rook in the first column of the skew shape. Let $f : [r+c] \setminus r \to [r+c] \setminus (r+1)$ be defined by $f(i) = i$, for $i \neq r+1$ and $f(r+1) = r$. Then f is a bijection that sends bases of M/r to bases of $M/(r+1)$: a rook placement with a rook in the cell $(r, r+1)$ gets mapped to itself while rook placements with empty first column and occupied row r gets mapped to the rook placement obtained by moving the rook in row r to the cell $(r, r+1)$. This movement does not affect the non-nesting nature of the rest of the rook placement. □

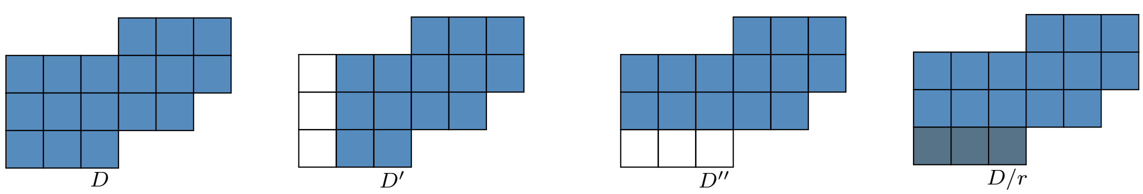
The following is a curious result, since we know that $\mathcal{P}_{\lambda/\mu}$ and $\mathcal{R}_{\lambda/\mu}$ are not isomorphic when λ/μ contains 332/1 as a subshape. For the purpose of readability, we use $T(M;x,y)$ in place of $T_M(x,y)$ ahead.

Theorem 3.38. *Let λ/μ be a skew shape. Then*

$$T(\mathcal{P}_{\lambda/\mu}; x,y) = T(\mathcal{R}_{\lambda/\mu}; x,y).$$

Proof. We proceed by induction over λ/μ, the number of squares in the diagram of the skew shape. The base case is immediate since both the lattice path matroid and the rook matroid on a singe square are equal to the uniform matroid $U_{1,2}$.

Assume that the statement holds for all diagrams with fewer boxes than in λ/μ. Let D denote the diagram λ/μ and let us consider the following additional diagrams:



In D' we have removed the first column of D, in D'' we have removed the last row of D, indexed by r. The last picture indicates that the last row contains a rook.

The Tutte polynomial for the lattice path matroid satisfies

$$T(\mathcal{P}_D; x,y) = T(\mathcal{P}_{D'}; x,y) + T(\mathcal{P}_{D''}; x,y), \tag{4}$$

by deletion-contraction, which corresponds to conditioning on whether or not the first step in the path is an East step or a North step.

Similarly, the Tutte polynomials for the rook matroid satisfies

$$T(\mathcal{R}_D; x,y) = T(\mathcal{R}_D/r; x,y) + T(\mathcal{R}_{D''}; x,y), \tag{5}$$

by considering deletion-contraction of the last row index. By Lemma 3.37, we know that $T(\mathcal{R}_D/r; x, y) = T(\mathcal{R}_D/(r+1); x, y) = T(\mathcal{R}_{D'}; x, y)$. By the induction hypothesis, we have that

$$T(\mathcal{P}_{D''}; x, y) = T(\mathcal{R}_{D''}; x, y) \text{ and } T(\mathcal{P}_{D'}; x, y) = T(\mathcal{R}_{D'}; x, y).$$

Putting these all together, it follows that $T(\mathcal{P}_D; x, y) = T(\mathcal{R}_D; x, y)$. ☐

The strongest form of the Merino-Welsh conjecture, formulated in [MW99], is a log-convexity type statement for evaluations of the Tutte polynomial; it says that if M is a loopless and coloopless matroid then $T_M(2,0) \cdot T_M(0,2) \geq T_M(1,1)^2$. While recently disproved for general matroids [BCCP24], it has been shown to hold for certain classes of matroids including split matroids [FS23] and lattice path matroids [KMSRA18]. In the light of the latter result, Theorem 3.38 shows that this conjecture also holds for rook matroids.

Corollary 3.39. *Rook matroids satisfy the Merino–Welsh conjecture.*

The Tutte polynomial is a valuative invariant of a matroid: it behaves well under subdivisions of matroid polytopes. On the basis of Theorem 3.38 and computer experiments, we make the case that every valuative invariant has the same value on $\mathcal{R}_{\lambda/\mu}$ as it does on $\mathcal{P}_{\lambda/\mu}$.

Conjecture 3.40. *Let λ/μ be a skew shape and let $\mathcal{R}_{\lambda/\mu}$ and $\mathcal{P}_{\lambda/\mu}$ be the corresponding rook matroid and lattice path matroid respectively. Then for every valuative invariant f for matroids, we have*

$$f(\mathcal{R}_{\lambda/\mu}) = f(\mathcal{P}_{\lambda/\mu}).$$

In particular, the same would hold for a universal valuative invariant like Derksen's $\mathcal{G}$-invariant [Der09, DF10]. For a systematic treatment of valuative invariants of matroids, see [FS24].

4. Distributional properties of non-nesting rook numbers

In this section, we deduce the distributional properties of the coefficients of the non-nesting rook polynomial, using the underlying matroid structure. The main result of the following subsection can be seen as a non-nesting anaologue of the bipartite case of the Heilmann-Lieb theorem [HL72, Nij76].

4.1. Ultra-log-concavity of $M_{\lambda/\mu}$.

Let λ/μ be a skew shape with r rows and c columns. The basis-generating polynomial $r_{\lambda/\mu}$ of $\mathcal{R}_{\lambda/\mu}$ serves as the appropriate multivariate generalization of $M_{\lambda/\mu}$, the univariate non-nesting rook polynomial introduced in Section 2.1. To emphasize the role played by row and column variables separately – which will also be useful in Section 5 – we use $\mathbf{x} = (x_1, \ldots, x_r)$ and $\mathbf{y} = (y_{r+1}, \ldots, y_{r+c})$ for the row and column variables of $r_{\lambda/\mu}$ respectively. By Theorem 3.3, bases of rook matroids correspond to occupied row indices taken together with unoccupied column indices of rook placements; we can thus write

$$r_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_{\rho \in \mathrm{NN}_{\lambda/\mu}} \prod_{i \in R(\rho)} x_i \prod_{j \in C(\rho)} y_j, \tag{6}$$

where as before $\mathrm{NN}_{\lambda/\mu}$ denotes the set of non-nesting rook placements on λ/μ and $R(\rho), C(\rho)$ correspond to the set of row indices occupied by ρ and the set of column indices that are not occupied by ρ respectively.

Corollary 4.1. *For every skew shape λ/μ, the coefficient sequence of the non-nesting rook polynomial $M_{\lambda/\mu}(t) = \sum_{k=0}^{d} r_k(\lambda/\mu)t^k$ is ultra-log-concave with no internal zeros. That is,*

$$\left(\frac{r_k}{\binom{d}{k}}\right)^2 \geq \frac{r_{k-1}}{\binom{d}{k-1}} \cdot \frac{r_{k+1}}{\binom{d}{k+1}} \quad \text{for all } 1 \leq k \leq d-1,$$

where d is the degree of $M_{\lambda/\mu}(t)$.

Proof. Let M be the rook matroid $\mathcal{R}_{\lambda/\mu}$, P_M be the corresponding basis-generating polynomial, and $T = [r]$. Consider $f(t,s) = P_M(t \cdot \mathbf{1}_T + s \cdot \mathbf{1}_{T^c})$ where $\mathbf{1}_T$ is the characteristic vector of T. Note that P_M can be expressed as in Equation 6. The result then follows from the fact that P_M is Lorentzian [BH20, Theorem 3.10], specializing variables preserves Lorentzianity, and a homogeneous bivariate polynomial is Lorentzian if and only if its coefficient sequence is ultra-log-concave with no internal zeros. □

Remark 4.2. For general matroids and arbitrary subsets T, the proof above yields a basis-counting inequality, first discovered by Stanley in 1981 for regular matroids [Sta81]. This inequality is one of many arising in the framework of convex geometry and matroid theory that have been investigated from the perspective of characterizing equality conditions; see [CP24, Yan23, vHYZ24] for example.

At this juncture, we note the parallel between the non-nesting rook polynomial and the full rook polynomial: both are ultra-log-concave. We will see that this is essentially the strongest property we can hope for, as it turns out that unlike the full rook polynomial the non-nesting counterpart is *not* real-rooted. We will show this in Section 5.2, where the proper context of this failure of real-rootedness is given. For the moment, we approach the real-rootedness question via its multivariate counterpart, real stability.

4.2. Lattice path matroids are not HPP.

In the previous subsection, we saw that ultra-log-concavity holds for the non-nesting rook numbers. In this subsection, we probe the extent to which real-rootedness holds, using the framework of matroids with the half-plane property.

A robust necessary condition that a homogeneous, multiaffine polynomial is stable is that its support forms the collection of bases of a matroid [COSW04]. The *half-plane property* (HPP) of a matroid M identifies whether the converse is true, namely that the basis-generating polynomial of $M = (E, \mathcal{B})$ defined as

$$P_M(\mathbf{x}) = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i,$$

is stable. We call such a matroid HPP. In [COSW04], an in-depth investigation of the half-plane property for matroids was carried out. One question raised – a “wild speculation” according to [COSW04] – was if transversal matroids have the HPP. Shortly thereafter, Choe and Wagner provided an example of a rank 4 transversal matroid on 12 elements that was not HPP [CW06]. Their example, however, is not a lattice path matroid; this can be straightforwardly checked using a criterion described in [BdM06, Theorem 3.14]. To our knowledge, the question of whether lattice path matroids have the HPP is thus still open. The purpose of this subsection is to give the first example of a lattice path matroid—in fact a Catalan matroid—that does not have the half-plane property.

Theorem 4.3. *There exists a generalized Catalan matroid that is not HPP. Concretely, the basis-generating polynomial of the matroid $\mathcal{R}_\lambda$ for $\lambda = 666333$ is not stable.*

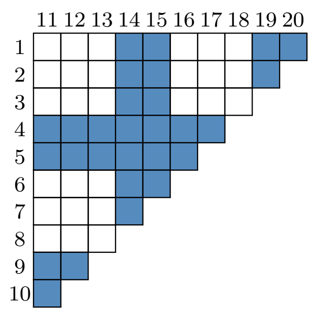
Proof. Let $\lambda = 666333$ and $P(\mathbf{x}, \mathbf{y})$ be the basis-generating polynomial of $\mathcal{R}_\lambda$. It suffices to show that the univariate polynomial obtained from $P(\mathbf{x}, \mathbf{y})$ obtained by setting some variables to t and others to 1 is not real-rooted. Consider the univariate polynomial $f(t) = P(\mathbf{x}, \mathbf{y})$ obtained by the substitution $x_1 = x_2 = x_3 = y_{10} = y_{11} = y_{12} = t$ and $x_4 = x_5 = x_6 = y_7 = y_8 = y_9 = 1$. Using SageMath, we found that

$$f(t) = t^6 + 36t^5 + 225t^4 + 400t^3.$$

This polynomial was found to have non-real roots near $-3.6855 \pm 0.6232i$. □

Corollary 4.4. *The Catalan matroid M_{10} is not HPP. That is, the basis-generating polynomial of M_{10} is not stable.*

Proof. Every generalized Catalan matroid is a minor of a large enough Catalan matroid [BdM06, Theorem 4.2]. In particular, if M_{10} denotes the Catalan matroid of order 10 (realizable as a rook matroid on a staircase shape of size 10), and $X = \{4, 5, 9, 10\}, Y = \{14, 15, 19, 20\}$, then $\mathcal{R}_{666333}$ is isomorphic to $M_{10} \backslash X/Y$ Lemma 3.7.



It follows that M_{10} cannot have the half-plane property. The rook matroid $\mathcal{R}_{666333}$ cannot be obtained as a minor of M_9.

□

Remark 4.5. In [Xu15] it is stated that the class of generalized Catalan matroids is not HPP, however no explicit counterexample is given.

In [Mar16], it was shown that positroids have the Rayleigh property, a weaker property than having the half-plane property, which is known to be equivalent to being strongly Rayleigh [Brä07]. Since lattice path matroids are known to be positroids [Oh11], the results of this section complete the picture about half-plane properties of positroids.

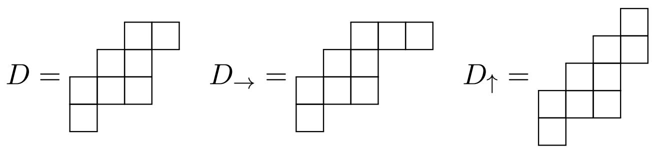
4.3. **Half-plane properties of snake and panhandle matroids.** We briefly turn our attention to two subclasses of lattice path matroids that satisfy the HPP – snake matroids and panhandle matroids – and give a direct proof of the HPP using the Borcea-Brändén symbol theorem.

A skew shape λ/μ is called a *snake*[4] if it is connected and does not contain a 2×2-box of boxes. Lattice path matroids on such shapes are called snake matroids. Snake matroids are known to be graphic [KMSRA18, Theorem 3.2] and graphic matroids are HPP [Brä15, Section 9.1]. Our first goal is to prove the half-plane property of snake matroids without using the fact that snakes are graphic; next, we address the panhandle matroid case.

[4]Also known as 'rim-hook' or 'border strip'.

A priori, one reason to expect that snake matroids might satisfy the HPP is that the non-nesting rook polynomial on a snake is real-rooted. Indeed, the absence of 2×2 subshape implies that no configuration of rooks can be nesting, and hence the non-nesting rook polynomial on a snake is simply the regular rook polynomial and is thus real-rooted [Nij76].

If D is the diagram of any skew shape, we can extend it by one box in one of two ways: by adding the box as a new column in the first row of D or by adding it as a new row in the last column of D. We denote these new diagrams by $D_\rightarrow$ and $D_\uparrow$ respectively. We illustrate these constructions below for $\lambda/\mu = 4331/21$.



Lemma 4.6. *Let λ/μ be a skew shape on r rows and c columns with diagram D and let $D_\rightarrow$ and $D_\uparrow$ be as over. Let $P_D(\mathbf{x};\mathbf{y})$ be the basis-generating polynomial for the rook matroid on λ/μ. Then the following recursions hold:*

$$P_{D_\rightarrow}(\mathbf{x};\mathbf{y}) = y_{c+1}P_D(\mathbf{x};\mathbf{y}) + x_1\left(P_D(\mathbf{x};\mathbf{y})|_{x_1=0}\right), \tag{7}$$

$$P_{D_\uparrow}(\mathbf{x};\mathbf{y}) = P_D(x_2,\ldots,x_{r+1};\mathbf{y}) + x_1\frac{\partial}{\partial y_c}P_D(x_2,\ldots,x_{r+1};\mathbf{y}). \tag{8}$$

Proof. Consider the deletion-contraction recurrence for the basis-generating polynomial of a matroid M:

$$P_M = P_{M\setminus e} + x_e \cdot P_{M/e}.$$

Equations (7) and (8) can be obtained from the above equation by taking $M = \mathcal{R}_{D_\rightarrow}, e = r+c+1$ and $M = \mathcal{R}_{D_\uparrow}, e = 1$ respectively. Alternatively, in each case the equations follow by conditioning on whether or not the new box contains a rook or not. □

Lemma 4.7. *Consider the linear operators T_1 and T_2 acting on multiaffine polynomials in the ring $\mathbb{C}[x_1,\ldots,x_{r+1},y_1,\ldots,y_{c+1}]$ that are defined as*

$$T_1[f] \coloneqq y_{c+1}f + x_1\left(f|_{x_1=0}\right),$$
$$T_2[f] \coloneqq f + x_{r+1}\frac{\partial}{\partial y_c}f.$$

Both T_1 and T_2 are stability preservers.

Proof. We will use the Borcea-Brändén symbol theorem [BB09, Theorem 2.2]. Let

$$S = (x_1+z_1)\cdots(x_{r+1}+z_{r+1})(y_1+w_1)\cdots(y_{c+1}+w_{c+1}).$$

Then T_1 and T_2 preserve stability if and only if the symbols $T_1[S]$ and $T_2[S]$ are stable as polynomials in the $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}$ variables. It then suffices to show that $T_1[S]/S$ and $T_2[S]/S$ are non-vanishing when all variables are in the upper half-plane. We have that

$$\frac{T_1[S]}{S} = y_{c+1} + \left(\frac{1}{x_1}+\frac{1}{z_1}\right)^{-1}.$$

Since addition and inversion respectively preserve and flip the sign of the non-real part of a complex number lying in the upper half-plane, it is clear that the right-hand side lies in the upper-half-plane. Hence, T_1 is stability-preserving. Similarly,

$$\frac{T_2[S]}{S} = \frac{y_c + w_c + x_{r+1}}{y_c + w_c}$$

whose numerator is non-vanishing if all the variables have positive non-real part. So T_2 is also stability-preserving. □

Note that the recursions (7) and (8) are, after a change of variables, represented by the operators T_1 and T_2 in Lemma 4.7. This lemma then gives us a recipe to establish the half-plane property of a lattice path matroid provided that it can be constructed from a smaller HPP matroid using $\rightarrow$ and $\uparrow$. We use this below for snake and panhandle matroids.

Panhandle matroids are a simple class of lattice path matroids corresponding to Ferrers diagrams of the form $(m, (n)^a)$, for $m \geq n \geq 1$ and $a \geq 1$; the additional row of m boxes on top of a $n \times a$ rectangle suggests the name "panhandle". This coinage was made in [HMM+23] where these matroids were studied in the context of their Ehrhart positivity. The proof we give here of the half-plane property of the panhandle matroids is similar in spirit to the original proof [COSW04]. Panhandle matroids can also be seen as series extensions of uniform matroids; taking series extensions preserves the HPP property in matroids by [COSW04, Proposition 4.10].

Proposition 4.8. *The following subclasses of lattice path matroids have the half-plane property:*

(1) Snake matroids.

(2) Panhandle matroids.

Proof. For (1), we proceed by induction on the number of boxes in the snake. In the case of one box we have that $P_\alpha(\mathbf{x}; \mathbf{y}) = x_1 + y_1$, which is stable. Since every snake can be built from one box by repeatedly performing the $\rightarrow$ and $\uparrow$ operations to the diagram, each of which preserves stability of the basis-generating polynomial, by Lemma 4.7, we are done.

For (2), we know that panhandle matroids are series extensions of uniform matroids. Since series extension of rook matroids corresponds to the $\rightarrow$ operation, we are done, since uniform matroids have the half-plane property and $\rightarrow$ preserves stability by Lemma 4.7. □

5. Rook placements as linear extensions

In this section, we make a bijective correspondence between labeled skew shapes and posets of width two. This allows us to provide an interpretation for the non-nesting rook polynomial in terms of a P-Eulerian polynomial of a poset and deduce the following results among others. One, the P-Eulerian polynomial of a width two poset is ultra-log-concave, thereby completing the picture of the Neggers–Stanley conjecture, which was disproved using a naturally labeled width two counterexample [Ste07]. We deduce this from a stronger statement, namely that a suitable multivariate analog of the P-Eulerian polynomial—first considered in [BL16]—is Lorentzian. Returning to the rook polynomial setting, we learn that there exists a skew shape λ/μ for which the non-nesting rook polynomial $M_{\lambda/\mu}$ has non-real roots. We end by considering the gamma-positivity properties of $M_{\lambda/\mu}$.

5.1. **Neggers–Stanley conjecture.** In this subsection we recall the statement and context of the Neggers–Stanley conjecture. For undefined terminology concerning posets, we refer to [Sta11].

Recall that given a poset P, the *width* of P is the size of the largest antichain in P. A labeling of a poset P on n elements is a bijection $\omega : P \to [n]$. We say that ω is a *natural labeling*, or linear extension of P, if $i \prec j$ implies $\omega(i) < \omega(j)$. The *Jordan-Hölder set* of (P, ω) is the set of all permutations of $[n]$ the inverses of which are linear extensions of (P, ω) i.e., it is defined as

$$\mathcal{L}(P, \omega) = \{\sigma \in \mathfrak{S}_n : i \prec j \implies \sigma^{-1}(\omega(i)) < \sigma^{-1}(\omega(j))\}.$$

In other words, $\sigma \in \mathcal{L}(P)$ if for every relation $i \prec j$ in P, we have that $\omega(i)$ precedes $\omega(j)$ in the one-line representation of the permutation σ.

The *(P, ω)-Eulerian polynomial*, also known as the *W-polynomial of P*, is the descent-generating polynomial of the Jordan-Hölder set of (P, ω):

$$W_{P,\omega}(t) = \sum_{\sigma \in \mathcal{L}(P,\omega)} t^{\mathrm{des}(\sigma)}.$$

When ω is natural, we simply write W_P, since the polynomial is independent of the choice of natural labeling. Observe that in the definitions of the multivariate analogs that follow, there is a dependence on the choice of natural labeling that we make explicit.

The distributional properties of $W_{P,\omega}$ was of early interest to combinatorialists working in poset theory. The following conjecture was first formulated by Neggers in 1978 for natural labelings ω; in 1986, Stanley extended it to arbitrary labelings. Subsequent references to this conjecture also called it the Poset conjecture [Bre89]. For further background, the reader can consult [Brä15, Section 6].

Conjecture 5.1 (Neggers–Stanley conjecture [Neg78, Sta86])**.** *Let (P, ω) be a labeled poset. Then $W_{P,\omega}$ is real-rooted.*

The Neggers–Stanley conjecture was of central importance to algebraic combinatorics until its resolution in the negative in the early aughts: first by Brändén [Brä04a] who found a family of counterexamples to Stanley's formulation and then Stembridge [Ste07] who disproved Neggers' counterpart as well. In both cases, the counterexample furnished was of a width two poset; Brändén's construction was non-naturally labeled while Stembridge's (larger) counterexample was naturally labeled. Despite this breakthrough, the question of unimodality or log-concavity of $W_{P,\omega}$ for general (P, ω) remained open. In particular, the following conjecture of Brenti has been open since 1989.

Conjecture 5.2. [Bre89, Conjecture 1.1] *Let (P, ω) be a labeled poset. Then $W_{P,\omega}$ is log-concave with no internal zeroes.*

There have been two positive results towards this end. Reiner and Welker proved that when P is naturally labeled and graded, W_P is unimodal and symmetric [RW05]. Brändén then gave an elegant combinatorial proof of the same fact by demonstrating a stronger property: namely that W_P is γ-positive for the larger class of sign-graded posets [Brä06, Brä08].

In the next subsection, we show a strengthening of Conjecture 5.2 for special posets P. Namely, we show that when P is naturally labeled and of width two, then W_P is ultra-log-concave. We do so by formulating an appropriate multivariate analog of W_P and recognising it as the basis-generating polynomial of a rook matroid.

5.2. **Matroidal lifts of P-Eulerian polynomials.** Given a skew shape λ/μ, our goal is obtain a suitable poset P such that the W-polynomial of P agrees with the non-nesting rook polynomial of λ/μ. We do so in Theorem 5.4. Before we state this, we introduce two multivariate analogs of W_P: one that is defined only for width two posets and one that can be defined independent of the width; after we show the main theorem for this section we show how one can be obtained from the other.

Given a poset P on n elements with a labeling ω, the *multivariate (P,ω)-Eulerian polynomial* $A_{P,\omega} \in \mathbb{N}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ is defined in [BL16] as:

$$A_{P,\omega}(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathcal{L}(P,\omega)} w(\sigma), \tag{9}$$

where $w(\sigma)$ is the following monomial associated to σ:

$$w(\sigma) = \prod_{i\in \mathrm{DB}'(\sigma)} x_i \prod_{j\in\mathrm{AB}'(\sigma)} y_j.$$

Here the sets $\mathrm{DB}'(\sigma)$ and $\mathrm{AB}'(\sigma)$ are defined as follows:

(1) $\mathrm{DB}'(\sigma) = \{\sigma_i \in [n] : \sigma_{i-1} > \sigma_i\}$ is the set of *descent bottoms* of σ and $\sigma_0 \coloneqq \infty$.

(2) $\mathrm{AB}'(\sigma) = \{\sigma_i \in [n] : \sigma_i < \sigma_{i+1}\}$ is the set of *ascent bottoms* and $\sigma_{r+c+1} \coloneqq \infty$.

Now we define the multivariate analog of a (P,ω)-Eulerian polynomial for a width two poset that is similar to $A_{P,\omega}$, but will be more readily recognizable as a basis-generating polynomial of a matroid. Let (P,ω) be a naturally labelled poset of width two. Fix a chain decomposition $C_1 \sqcup C_2$ of P where the chain C_1 has r elements and the chain C_2 has c elements. With respect to this chain decomposition, define[5] $\widetilde{W}_{P,\omega} \in \mathbb{N}[x_e, y_{e'} : e \in C_1, e' \in C_2]$ as

$$\widetilde{W}_{P,\omega}(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathcal{L}(P,\omega)} \prod_{i\in\mathrm{DB}(\sigma)} x_i \prod_{j\in\mathrm{CA}(\sigma)} y_j, \tag{10}$$

where

(1) $\mathrm{DB}(\sigma) = \{\sigma_i \in [r+c] : \sigma_{i-1} > \sigma_i\}$ is the set of descent bottoms of σ.

(2) $\mathrm{CA}(\sigma) = \{\sigma_i \in [r+c] : \sigma_i < \sigma_{i+1},\ \sigma_i \in C_2\}$ is the set of column ascents, i.e. ascent bottoms of σ lying in C_2 and $\sigma_{r+c+1} \coloneqq \infty$.

The latter condition implies that the element σ_{r+c} is an ascent bottom if and only if $\sigma_{r+c} = r+c$. Here CA stands for column ascent, for a reason that will be made clear during the course of the proof of Theorem 5.4. Also note that $\widetilde{W}_{P,\omega}$ is homogeneous of degree $|C_2|$.

Example 5.3. Suppose (P,ω) is the labeled poset in Figure 17. Here, P is irreducible (i.e. it cannot be written as the ordinal sum of two subposets) with chains $C_1 = \{1,3\}$ and $C_2 = \{2,4,5\}$. The Jordan–Hölder set is $\mathcal{L}(P,\omega) = \{24153, 21453, 12453, 21435, 24135, 12435, 21345, 12345\}$. The polynomial $\widetilde{W}_{P,\omega}$ is a polynomial in the variables x_1, x_3, y_2, y_4, y_5 and is equal to

$$\widetilde{W}_{P,\omega} = x_1x_3y_2 + x_1x_3y_4 + x_3y_2y_4 + x_1x_3y_5 + x_1y_2y_5 + x_3y_2y_5 + x_1y_4y_5 + y_2y_4y_5.$$

---

[5]We emphasize that when (P,ω) is naturally labeled and of width two, the multivariate polynomial $\widetilde{W}_{P,\omega}$ depends on the choice of chain decomposition of P, but the univariate polynomial $W_{P,\omega}$ does not. Irreducible width two posets have a unique decomposition into two (maximal) chains [Ste07, Proposition 5.1].

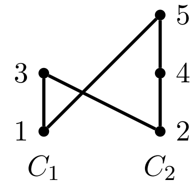


Figure 17. Naturally labeled poset of width two.

Now, recall that if $r_{\lambda/\mu}$ is the basis-generating polynomial of the rook matroid on λ/μ, then we can express it in terms of row variables $x_1, \dots, x_r$ and column variables $y_{r+1}, \dots, y_{r+c}$ as

$$r_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_{\rho \in \mathrm{NN}_{\lambda/\mu}} \prod_{i \in R(\rho)} x_i \prod_{j \in C(\rho)} y_j$$

We can now state the main theorem of the section.

Theorem 5.4. *Let λ/μ be a skew shape and denote by $r_{\lambda/\mu}$ and $\ell_{\lambda/\mu}$ the basis-generating polynomials of the rook matroid and lattice path matroid on λ/μ respectively. There exists a naturally labeled poset (P, ω) of width two such that*

$$r_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) \cong \widetilde{W}_{P,\omega}(\mathbf{x}, \mathbf{y}) \tag{11}$$

$$\ell_{\lambda/\mu}(\mathbf{z}) \cong \sum_{\sigma \in \mathcal{L}(P,\omega)} \prod_{j \in \sigma(C_2)} z_j \tag{12}$$

where $\cong$ denotes equality up to reindexing of the variables.

Remark 5.5. The fact that linear extensions of posets of width k bijectively correspond to lattice paths contained within a compact region of $\mathbb{R}^k$ follows from the standard theory of distributive lattices [Sta11, pg. 296]. Similiar bijections to ours also appear in [CPP22, Lemma 8.1] and an unpublished paper of Stanley [Sta] respectively. The theorem above can be seen as a matroidal and multivariate refinement of the aforementioned results.

Proof. We only address (11), since (12) follows by a similar argument, or alternatively by keeping track of descents and ascents in the bijection provided in [CPP22, Lemma 8.1].

We begin by relabelling the rows and columns of λ/μ in the following manner. Denote the innermost lattice path of the skew shape by $L_{\lambda/\mu}$ and number the rows of λ/μ from bottom to top and the columns from left to right by the North and East steps of $L_{\lambda/\mu}$ respectively. See Figure 18a for an example. Let C_1 and C_2 be the two chains defining P, labeled respectively by the row and column labels of λ/μ that were induced by $L_{\lambda/\mu}$. Let this labeling be ω. Hereafter, in the context of cover relations, we will use i, j in place of $\omega(i), \omega(j)$.

Add the cover relation $i \prec j$ to P for every outer corner (i, j) of λ/μ and similarly the cover relation $j \prec i$ for every inner corner (i, j). This construction is illustrated in Figure 18b. The resulting poset is easily seen to be of width two. We make the following claim:

Claim 1: With the labeling as above, for every cell (i, j) in the skew shape λ/μ, we have that $i < j$.

Proof of claim 1: Since the row labels decrease from top to bottom, it suffices to show this for cell (i, j) that is the topmost box in its column. This holds since every topmost box lies directly under the innermost path of the shape.

To see that P is naturally labeled: consider the cover relation $a \prec b$ where $a \in C_1, b \in C_2$. This implies that the (a, b) is an outer corner of the shape. Since (a, b) is an outer corner, the cell to the left of (a, b), say (a, c), must lie in λ/μ and hence $a < c$, by the claim above. Since the column labelling increases from left to right, we must also have $a < b$. The inner corner case is similarly proved.

(A) Skew shape 54421/31 with innermost path marked with dashes.

(B) Naturally labeled width two poset P corresponding to skew shape 54421/31.

FIGURE 18. Skew shape – poset correspondence. Indices of North and East steps of the dashed path form two disjoint chains, C_1 and C_2. Blue and red cover relations correspond to outer and inner corners respectively.

Now we exhibit a bijection from non-nesting rook placements inside λ/μ to elements of the Jordan-Hölder set $\mathcal{L}(P, \omega)$. This bijection will send non-nesting rook configurations with k occupied rows to inverses of linear extensions with k descents. Formally, define $f : \mathcal{R}_{\lambda/\mu} \to \mathcal{L}(P, \omega)$ by $f = \psi \circ T_{\lambda/\mu}$ where $T_{\lambda/\mu} : \mathcal{R}_{\lambda/\mu} \to \mathcal{P}_{\lambda/\mu}$ is the rooks-to-outer-corners bijection between rook placements and lattice paths described in the proof of Lemma 3.11 and $\psi : \mathcal{P}_{\lambda/\mu} \to \mathcal{L}(P, \omega)$ is the path-to-permutation map defined by $\psi(L) = \sigma = \sigma_1 \ldots \sigma_{r+c}$, as in Definition 3.16. An illustration of this is in Figure 19.

Claim 2: The map f defined above is a bijection.

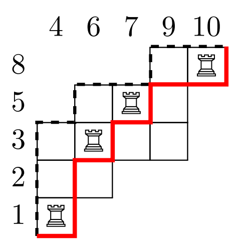


FIGURE 19. A non-nesting rook placement on 54421/31 with its corresponding lattice path in red and path permutation $\sigma = 41263759\,10\,8$. Here $\mathrm{DB}(\sigma) = \{1, 3, 5, 8\}$ and $\mathrm{CA}(\sigma) = \{9\}$. The set $\mathrm{DB}(\sigma) \cup \mathrm{CA}(\sigma)$ represents the rook placement in the picture above.

Proof of Claim 2: By the uniqueness of the permutation representation of the lattice path and the rook-to-path bijection from Proposition 3.11, the map f is well-defined. To show that f is a bijection, it suffices to show that ψ is a bijection.

To that end, we first need to show that $\sigma = \psi(L)$ does indeed lie in $\mathcal{L}(P, \omega)$, whenever L is a lattice path contained inside λ/μ. Suppose $i \prec j$ in P. Two cases arise:

(1) Both of i, j lie in the same chain of P. In this case, they are either both row indices or both column indices; since row indices increase from top to bottom and column indices increase from left to right in the labeling of λ/μ, the path L must traverse step σ_i^{-1} before step σ_j^{-1} and hence i must precede j in σ.

(2) The elements i, j lie in different chains. Assume without loss of generality that i is a row index and j is a column index. Then by construction of P, (i, j) must be an outer corner of λ/μ. Then the $(\sigma_i^{-1})^{\text{th}}$ and $(\sigma_j^{-1})^{\text{th}}$ steps of L are respectively `n` and `e` steps. Let E be the lattice path corresponding to the southern boundary of λ/μ and let $(\psi(E))_k = i$, so that $(\psi(E))_{k+1} = j$. Since (i, j) is an outer corner corresponding to a `ne` step of L, the step at which L takes i can be no later than k while the step at which L takes j can be no earlier than $k+1$. This implies that σ satisfies $\sigma_i^{-1} \leq k < k+1 \leq \sigma_j^{-1}$, as required. The case when $i \prec j$ for i, j equal to column and row indices respectively follows by an analogous argument.

This shows that the image of f lies in $\mathcal{L}(P, \omega)$.

To see that ψ is a surjection, consider $\sigma \in \mathcal{L}(P, \omega)$. Define a lattice path L by letting the i^{th} step of L equal σ_i; such a path certainly satisfies $g(L) = \sigma$ but we need to verify that L is contained inside the shape. Suppose L exits the skew shape λ/μ. Then there exists an outer corner (i, j) of λ/μ such that ji forms an `en` step of L. In particular $\sigma = \sigma_1 \dots ji \dots \sigma_{r+c}$. Now (i, j) is an outer corner which implies that $i \prec j$ and since $\sigma \in \mathcal{L}(P, \omega)$, i must precede j in σ, a contradiction. Thus L is a valid lattice path in $\mathcal{P}_{\lambda/\mu}$ and hence g is a surjection and f is indeed a bijection.

Claim 3: Let ρ be the rook placement that corresponds to σ under the bijection f. Then

$$f(R(\rho)) = \mathrm{DB}(\sigma) \quad \text{and} \quad f(C(\rho)) = \mathrm{CA}(\sigma)$$

That is, f maps the occupied rows of ρ and the unoccupied columns of ρ to descent bottoms of σ and ascent bottoms of σ respectively.

Proof of Claim 3: The map f acts as follows on occupied row and unoccupied column indices.

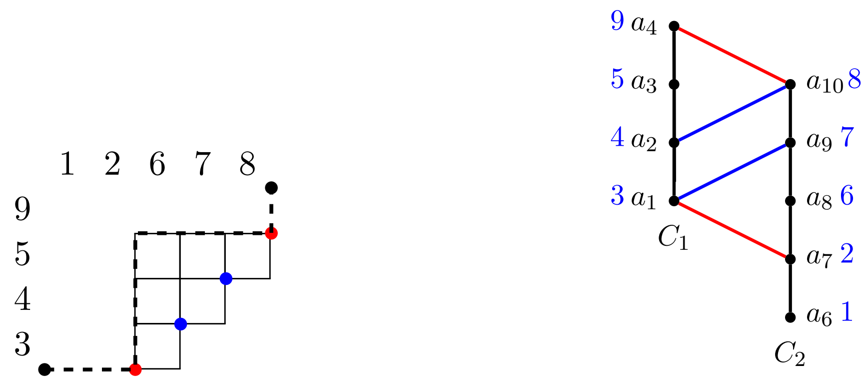
$$\begin{aligned} f(R(\rho)) &= \psi(\{\, \texttt{n} \text{ steps of outer corners of lattice path } L = L_\rho\}) \\ &= \{\sigma_i^L : \sigma_{i-1}^L > \sigma_i^L\} \\ &= \mathrm{DB}(\sigma), \end{aligned}$$

and

$$\begin{aligned} f(C(\rho)) &= \psi(\{\, \texttt{e} \text{ steps of outer corners of lattice path } L = L_\rho\}) \\ &= \{\sigma_i^L : \sigma_i^L < \sigma_{i+1}^L,\ \sigma_i \text{ is a column variable}\} \\ &= \{\sigma_i^L : \sigma_i^L < \sigma_{i+1}^L, \sigma_i^L \in C_2\} \quad (\text{since } C_2 \text{ is labeled by column variables}) \\ &= \mathrm{CA}(\sigma). \end{aligned}$$

This completes the proof of the assertion that $\widetilde{W}_{P,\omega}$ can be written as a multivariate generating polynomial of non-nesting rook placements on λ/μ. To finish, we note that after reindexing the row and columns of the skew shape in accordance with the matroid labeling, this latter polynomial is precisely the basis-generating polynomial of $\mathcal{R}_{\lambda/\mu}$. $\square$

Remark 5.6. Note that if skew shape λ/μ has empty rows and columns (for e.g. 5543/5222) then the resulting poset may have a unique minimal element and/or unique maximal element. In particular the decomposition into chains might not be unique.



(A) Skew shape 5543/5222 with innermost path marked with dashes.

(B) Naturally labeled width two poset P corresponding to skew shape 54421/31.

FIGURE 20. Skew shape – poset correspondence, where the skew shape has empty columns $1, 2$ and empty rows 9. This corresponds to the poset P having 1 as a unique minimal element and 2, 9 as the unique maximal element and $P \setminus 1$ having 2 as a unique minimal element.

Let $\mathcal{E}$ be the map from Theorem 5.4 that takes a skew shape λ/μ to a width two poset P. From the construction in the proof above, it is implicit that $\mathcal{E}$ is injective. We record the surjectivity of $\mathcal{E}$ in the lemma below.

Lemma 5.7. *For every poset P of width two, there exists a skew shape λ/μ such that $\mathcal{E}(\lambda/\mu) = P$.*

Proof. While the poset P has a unique minimum x, remove x from P and designate it as an empty column index. Similarly, while there is a unique maximum y in P, remove y and designate it as an empty row index. Once the poset — which by a minor abuse of notation we continue to denote by P — has two minimal and two maximal elements, proceed as follows.

Let $\{t_1, \ldots, t_{r+c}\}$ be the elements of P and let C_1, C_2 be the two chains of P of length r and c respectively. Define λ/μ to be the skew shape with r rows (labeled from bottom to top by elements of C_1) and c columns (labeled from left to right by elements of C_2) such that:

(1) There is an inner corner at (t_i, t_j) for every cover relation of the form $t_i \prec t_j$ where $t_i \in C_2$ and $t_j \in C_1$.

(2) There is an outer corner at (t_i, t_j) for every cover relation of the form $t_i \prec t_j$ where $t_i \in C_1$ and $t_j \in C_2$.

By the correspondence between corners and cover relations detailed in the proof of Theorem 5.4, this is precisely the skew shape λ/μ such that $\mathcal{E}(\lambda/\mu) = P$. □

Remark 5.8. Let (P, ω') be a naturally labeled poset. While Theorem 5.4 and Lemma 5.7 together yield a skew shape λ/μ such that $(\mathcal{E}(\lambda/\mu), \omega)$ is a naturally labeled poset satisfying $\mathcal{E}(\lambda/\mu) = P$, it might be possible that $\omega \neq \omega'$. This will not matter for us however, since the polynomial W_P is independent of the natural labeling.

Theorem 5.4 and Lemma 5.7 together allow one to draw mutually enriching connections between P-Eulerian polynomials and non-nesting rook polynomials. In one direction, one can use the matroidal properties of $M_{\lambda/\mu}$ to make the strongest possible conclusion for the P-Eulerian polynomial of width two posets; namely that its coefficient sequence is ultra log-concave. We collect a few other corollaries below, beginning with the precise relation between Brändén and Leander's multivariate P-Eulerian polynomial and our modified multivariate width two W-polynomial. Recall the definition of $A_{P,\omega}$ from equation 9.

By Lemma 5.7, every width two poset can be given the natural labeling induced from the skew shape labeling in the proof of Theorem 5.4. We refer to this labeling hereafter as the *canonical labeling*.

Corollary 5.9. *Let P be a poset of width two and let ω be the canonical labeling of P. Then $A_{P,\omega}$ is Lorentzian and its support is equal to the collection of bases of a rook matroid.*

Proof. Since the product of Lorentzian polynomials is Lorentzian, $A_{P,\omega}$ being Lorentzian would follow from Theorem 5.4 and Lemma 5.7 if we can prove the following identity:

$$A_{P,\omega}(\mathbf{x},\mathbf{y}) = x_1 \prod_{i\in C_1} y_i \cdot \widetilde{W}_{P,\omega}(\mathbf{x}^*,\mathbf{y}) \tag{13}$$

where $\mathbf{x}^*$ denotes the vector $\mathbf{x}$ after the change of variables $x_1 \to x_{\min C_2}$. To see why this holds, recall that

$$A_{P,\omega}(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathcal{L}(P,\omega)} \prod_{i\in \mathrm{DB}'(\sigma)} x_i \prod_{j\in \mathrm{AB}'(\sigma)} y_j$$

and that 1 must always lie in $\mathrm{DB}'(\sigma)$. Further, $C_1 \subseteq \mathrm{AB}'(\sigma)$ for the following reason. For every $a \in C_1$ and $\sigma \in \mathcal{L}(P,\omega)$, let b be the element succeeding a in σ: that is, $\sigma = \sigma_1 \dots ab \dots \sigma_n$. If $b \in C_1$, we must necessarily have $a \prec b$ which by the natural labeling ω implies $a < b$. Similarly, if $b \in C_2$ and $a \prec b$, it follows that $a < b$. On the other hand, if $b \in C_2$ and is incomparable with a then (a,b) must be a cell in the corresponding skew-shape. By claim 1 in the proof of Theorem 5.4, it follows that $a < b$. Hence $a \in \mathrm{AB}'(\sigma)$ and $C_1 \subseteq \mathrm{AB}(\sigma)$. Since every ascent bottom must either be a row or column ascent, we must then have

$$\mathrm{AB}'(\sigma) = C_1 \sqcup \mathrm{CA}(\sigma)$$

for every $\sigma \in \mathcal{L}(P,\omega)$. Similarly for every $\sigma \in \mathcal{L}(P,\omega)$, we have

$$\mathrm{DB}'(\sigma) = \mathrm{DB}(\sigma) \sqcup \{\sigma_1\}.$$

Note that $1 \in \mathrm{DB}(\sigma)$ precisely when $\sigma = q\,1\dots$ where $q = \min(C_2)$. This implies that q plays the role of 1 whenever 1 is in $\mathrm{DB}(\sigma)$, from which (13) follows.

The second statement also follows from (13), Theorem 5.4 and the fact that a coloop is an element that is present in every basis of a matroid. □

The next corollary shows that the ultra-log-concavity part of the Neggers–Stanley conjecture is true for naturally labeled width two posets; in particular this holds for Stembridge's counterexample, shown in Figure 21 and computed in Corollary 5.13. We can deduce an analogous result for the polynomial $E_P(t) = \sum_{j=1}^{|P|} e_j(P)t^j$ where $e_j(P)$ denotes the number of surjective order–preserving maps from P to $[j]$. An alternative formulation of the Neggers–Stanley conjecture for naturally labeled posets P asserts the real-rootedness of E_P.

Corollary 5.10. *Let P be a naturally labeled poset of width two and λ/μ be the skew shape obtained from Lemma 5.7. Then the W-polynomial of P equals the non-nesting rook polynomial of λ/μ:*

$$W_P(t) = M_{\lambda/\mu}(t), \tag{14}$$

and hence W_P is ultra-log-concave. Consequently, E_P is also ultra-log-concave.

Proof. Since every natural labeling of the poset P gives rise to the same W_P, assume P has the canonical labeling ω. By Theorem 5.4, the polynomial $\widetilde{W}_{P,\omega}$ is equal, up to a relabeling of the variables, to the basis-generating polynomial of the rook matroid on λ/μ. The latter polynomial is Lorentzian, and this property is preserved under a linear change of variables [BH20, Theorem 2.10]. The aforementioned relabeling maps $\mathbf{x}$ variables (resp. $\mathbf{y}$ variables) labeled by elements of C_1 (resp. C_2) to $\mathbf{x}$ variables labeled by rows $1, \dots, r$ (resp. columns $r+1, \dots, r+c$) of the skew shape. Thus, by specialising the $\mathbf{x}$ variables to t and the $\mathbf{y}$ variables to 1 in (11), we obtain (14). The ultra-log-concavity statement of W_P then follows from Corollary 4.1. Finally, the ultra-log-concavity of E_P follows from [BFJ24, Lemma 2.5 (ii)]. □

Two simple but immediate consequences of this correspondence are that (a) $M_{\lambda/\mu}$ has a discrete geometric interpretation — as the h^*-polynomial of the order polytope of a naturally labeled width two poset; and (b) M_λ can be seen as the f-polynomial of a (flag) simplicial complex, that is, the generating polynomial of the f-vector of a simplicial complex. The context for (a) is that in [DN23], it was shown that (regular) rook polynomials of a large class of boards are h-polynomials of the coordinate rings associated to the boards. The context for (b) is a recent question raised by Mu–Welker on whether the P-Eulerian polynomial of a poset can be written as the f-polynomial of a simplicial complex [MW25]. The question of realizability of (real-rooted) polynomials as f-polynomials of a simplicial complex goes back to Bell and Skandera [BS07].

The poset–skew shape correspondence allows us to deduce a stronger result for non-nesting rook polynomials in the context of skew shaped boards.

Corollary 5.11. *Let λ/μ be a skew shape, P be the corresponding naturally labeled poset obtained from Theorem 5.4, and $\mathcal{O}(P)$ be the order polytope of P. Then*

$$M_{\lambda/\mu}(t) = h^*_{\mathcal{O}(P)}(t).$$

In particular, the h^*-polynomial of the order polytope of a naturally labeled width two poset is ultra-log-concave.

Proof. The proof is immediate from Theorem 5.4 together with the well-known fact that for a naturally labeled poset P, its P-Eulerian polynomial coincides with the h^*-polynomial of its order polytope; see, for example, [BS18, Theorem 6.3.11]. □

The following can be seen as evidence for an affirmative answer to [MW25, Question 3.16].

Corollary 5.12. *Let P be a naturally labeled poset of width two. Then W_P is the f-polynomial of a flag simplicial complex.*

Proof. Let λ/μ be the skew shape corresponding to P, from Lemma 5.7. By Corollary 5.10, $W_P(t) = M_{\lambda/\mu}(t)$. Note that $\mathrm{NN}_{\lambda/\mu}$ can also be interpreted as the set of antichains of the cell poset of λ/μ; that is, the poset on the cells of λ/μ with the product partial order: $(i,j) \preceq (k,\ell)$ if $i \leq j$ and $k \leq \ell$. The antichains of a poset are in turn the independent sets of a graph. Since independence complexes of graphs are flag, the result follows. □

In the other direction of the poset–skew shape correspondence, one can use known results on P-Eulerian polynomials to deduce distributional properties of the non-nesting rook numbers, extending the results in Section 4. We mention two such applications below.

Stembridge's width two counterexample to the Neggers–Stanley conjecture [Ste07] shows that real-rootedness of $M_{\lambda/\mu}$ can fail in general. This counterexample is illustrated in Figure 21 together with the corresponding skew shape. The following corollary gives a rook theoretic and matroid theoretic context within which Stembridge's counterexample can be understood.

Corollary 5.13. *The non-nesting rook polynomial $M_{\lambda/\mu}$ is not real-rooted in general. Concretely, $M_{\lambda/\mu}$ is not real-rooted for the skew shape $\lambda/\mu = 888888765/76654321$.*

Proof. Let $\lambda/\mu = 888888765/76654321$. Applying the skew shape – poset correspondence Corollary 5.10, we obtain the poset on the left in Figure 21, which we recognize as Stembridge's counterexample to Neggers' formulation of the Poset conjecture. Thus $M_{\lambda/\mu}$ is not real-rooted.

Alternatively, using SageMath, we computed $M_{\lambda/\mu}$ to be:

$$M_{\lambda/\mu}(t) = 3t^8 + 86t^7 + 658t^6 + 1946t^5 + 2534t^4 + 1420t^3 + 336t^2 + 32t + 1.$$

This polynomial has non-real roots near $-1.85884 \pm 0.14976i$. □

Remark 5.14. By the experimental evidence gathered in [Ste07] and the poset–skew shape correspondence above, it follows that $\lambda/\mu = 888888765/76654321$ is the smallest skew shape for which $M_{\lambda/\mu}$ fails to be real-rooted. It would be interesting to find a large class of skew shapes for which real-rootedness holds; the conjecture in the next section suggests that Ferrers shapes might be natural candidates for this.

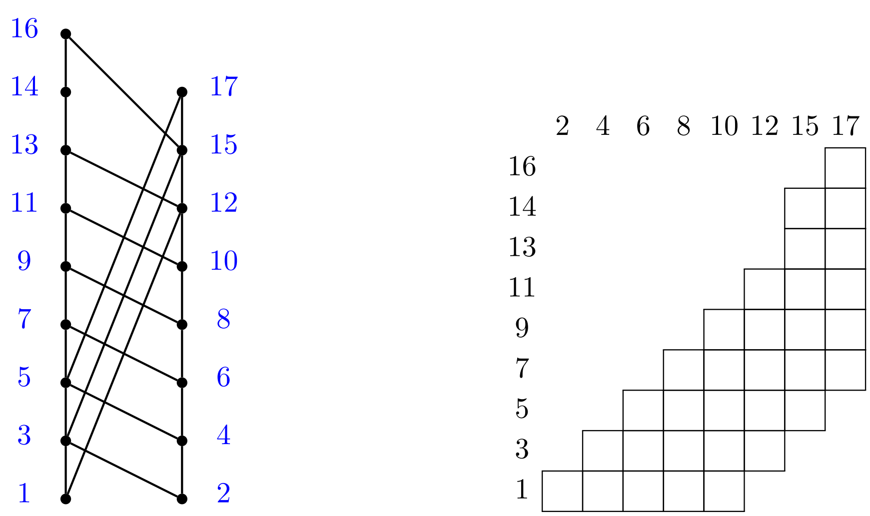
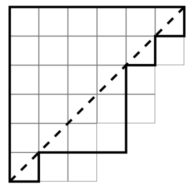


Figure 21. On the left: Stembridge's counterexample [Ste07] to the Neggers–Stanley conjecture (albeit with a different natural labeling). On the right: the corresponding skew shape, with the poset labeling of rows and columns.

Using the skew shape – poset correspondence, we can also deduce precisely when the non-nesting rook polynomial is palindromic, a fact not immediately obvious from the combinatorics of rook placements alone. The partitions that satisfy this symmetry are defined below.

Definition 5.15. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is a *squarecase* if each of its outer corners lie exactly on the northwest diagonal of the shape.

For an example, see the figure below.



Corollary 5.16. *Let λ/μ be a skew shape. Then $M_{\lambda/\mu}$ is palindromic if and only if λ/μ is the direct sum of partitions ν_i each of which is a squarecase. Moreover, when the shape is a squarecase with n rows, then*

(1) $M_{\lambda/\mu}$ is γ-positive, that is $M_{\lambda/\mu}$ can be written as:

$$M_{\lambda/\mu}(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k},$$

where each $\gamma_i \geq 0$.

(2) In particular, $(-1)^{\lfloor \frac{n}{2} \rfloor} M_{\lambda/\mu}(-1) \geq 0$.

Proof. It suffices to show that the result holds for partition shapes with the squarecase property. This is because $M_{\nu_1 \oplus \nu_2} = M_{\nu_1} \cdot M_{\nu_2}$ is palindromic if and only each M_{ν_i} is palindromic.

Label the rows of $\nu = (\nu_1, \dots, \nu_n)$ with 1 to n from bottom to top, and the columns $n+1$ to $2n$ from left to right. Let Q be the poset corresponding to ν. By Theorem 5.4, and the reciprocity theorem for P-partitions, M_ν is symmetric if and only if W_Q is symmetric, which in turn occurs if and only if Q is graded; that is, all maximal chains of Q have the same length n.

This in turn holds if and only if the two chains of Q are $C_1 = \{1, \dots, n\}$ and $C_2 = \{n+1, \dots, 2n\}$ and the only non-trivial cover relations are of the form $i_j \prec n+1+i_j$ for $j = 1, \dots, k$. The latter condition on Q is equivalent to the outer corners of ν being of the form $(i_j, n+1+i_j)$, that is the outer corners lie exactly on the North-West diagonal of ν, which is precisely the definition of a squarecase.

The γ-positivity of $M_{\lambda/\mu}$ then follows from Theorem 5.4 and [Brä06, Theorem 4.2] while the statement about the sign of $M_{\lambda/\mu}$ at -1 follows from γ-positivity. □

6. Future Work

One of the main results of this paper was the ultra-log-concavity of the polynomial $M_{\lambda/\mu}$. In the previous section, we noted that real-rootedness holds for special families of λ/μ including all skew shapes with r rows and c columns with $r + c \leq 16$. Here we make a conjecture about the real-rootedness of M_λ.

Conjecture 6.1. *Let λ be a Ferrers shape and let $\lambda - \mathbf{1}$ denote $(\lambda_1 - 1, \dots, \lambda_n - 1)$ and let $(\lambda)^{(i)}$ denote its truncation to its i^{th} part $(\lambda_1, \dots, \lambda_i)$. The non-nesting rook polynomial M_λ is real-rooted. Moreover, the following interlacing relations hold:*

$$M_\lambda \ll M_{\lambda - \mathbf{1}} \quad \textit{and} \quad M_\lambda \ll \sum_{i=0}^{n-1} M_{(\lambda)^{(i)}}.$$

See [Brä15] for the definition and applications of interlacing polynomials. Apart from the points mentioned above, we note the following two things in support of the conjecture. One, computational evidence supports this conjecture for all partitions λ fitting inside a 7×7 box. Two, a simple interlacing argument shows that for partitions of the form $\lambda = (m, \delta_n) = (m, n, n-1, \dots, 1)$, M_λ is real-rooted.

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