Doctoral Thesis in Mathematics

Large Deviations and Related Topics in Random Conformal Geometry

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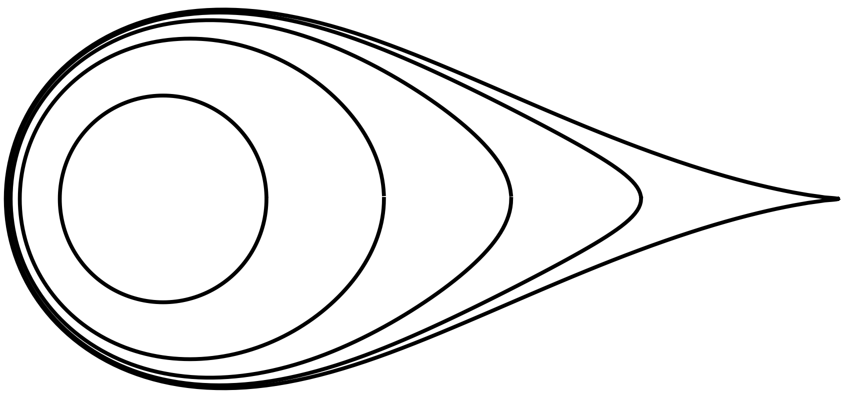

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Abstract

This thesis explores three models of random processes in the complex plane: Schramm–Loewner evolution, the Hastings-Levitov model, and Dyson Brownian motion. A common theme throughout the thesis is the large deviation principle (LDP), which gives rise to functionals, called rate functions, which have intrinsic connections with the geometry of the models.

Paper A presents a proof of the LDP for chordal Schramm–Loewner evolution, SLE_κ, in the upper half-plane, as $\kappa \to 0+$, in the topology of locally uniform convergence. The Loewner energy functional controls large deviations and is shown to be a good rate function.

Paper B studies large deviations of the Hastings-Levitov HL(0) model in the small-particle limit, i.e., when the number of particles tends to infinity and the one-particle capacity vanishes while their product remains constant. In particular, the growing cluster of particles attached to the unit disk is described via Loewner evolution, and we prove the LDP for the corresponding family of driving measures, with the rate function equal to the relative entropy. The LDP at the level of conformal maps is obtained via the contraction principle and leads to an interesting minimization problem of finding a driving measure with minimal relative entropy that produces a given cluster shape. We show that the class of shapes generated by finite-entropy Loewner evolution contains all Weil-Petersson and Becker quasicircles, a non-simple curve, and a Jordan curve with a cusp.

Paper C proposes a rigorous definition of Dyson Brownian motion on a rectifiable Jordan curve. We show that the process can be constructed for inverse temperatures $\beta \geq 1$, and that the transition probability function satisfies the Fokker–Planck–Kolmogorov equation. Under additional smoothness assumptions on the curve, we prove convergence to the stationary Coulomb gas distribution on the curve, study large deviations at low temperature, and derive a mean-field McKean–Vlasov equation in the hydrodynamical limit.

Paper D defines Dyson Brownian motion on a circular arc and is complementary to Paper C. The process exists for all $\beta > 0$, and its transition probability function satisfies the Fokker–Planck–Kolmogorov equation with reflecting boundary conditions. The process is ergodic and its stationary distribution is given by the Coulomb gas density on the circular arc.

Sammanfattning

Denna avhandling utforskar tre modeller av stokastiska geometri i det komplexa planet: SLE-kurvor, Hastings-Levitov-modellen och Dyson Brownsk rörelse. Ett genomgående tema i avhandlingen är stora avvikelser, vilken ger upphov till stora avvikelse-funktionaler med inneboende kopplingar till modellernas geometri.

I Artikel A presenteras ett bevis av en stora avvikelser-sats för *Schramm–Loewner evolution*, SLE_κ, i det övre halvplanet, då $\kappa \to 0+$, i topologin för lokalt likformig konvergens. Stora avvikelse-funktionalen ges i detta fall av Loewnerenergin.

I Artikel B studerar vi stora avvikelser hos Hastings-Levitov-modellen HL(0) i en viss skalgräns när antalet partiklar går mot oändligheten och kapaciteten för varje enskild partikel går mot noll medan deras produkt förblir konstant. Speciellt studeras partikelkluster via Loewners differentialekvation och vi bevisar en stora avvikelser-sats för tillhörande drivmått, med stora avvikelse-funktionalen given av den relativa entropin. Stora avvikelser-satsen på nivån för konforma avbildningar erhålls via kontraktionsprincipen och leder till ett intressant minimeringsproblem som går ut på att hitta ett drivmått med minimal relativ entropi som producerar ett givet kluster. Vi visar att klassen av kluster som genereras av Loewnerevolution med ändlig entropi innehåller alla Weil-Petersson- och Becker-kvasicirklar, en självskärande kurva samt en Jordankurva med en spets.

I Artikel C ges en rigorös definition av Dyson Brownsk rörelse på en rektifierbar Jordankurva. Vi visar att processen kan konstrueras för inversa temperaturer $\beta \geq 1$ och att övergångssannolikhetsfunktionen uppfyller Fokker–Planck–Kolmogorov-ekvationen. Under ytterligare antaganden om kurvans släthet bevisar vi konvergens mot den stationära Coulomb-gasfördelningen på kurvan, studerar stora avvikelser vid låg temperatur och härleder en medelfälts-McKean–Vlasov-ekvation i den hydrodynamiska gränsen.

I Artikel D definieras Dyson Brownsk rörelse på en cirkelbåge. Processen existerar för alla värden $\beta > 0$ och dess övergångssannolikhetsfunktion uppfyller Fokker–Planck–Kolmogorov-ekvationen med reflekterande randvillkor. Processen är ergodisk och dess stationära fördelning ges av Coulomb-gastätheten på cirkelbågen.

Popular science summary

This thesis comprises four research papers (A, B, C, and D) and explores models in probability theory. In particular, we study (A) random fractal curves, called Schramm–Loewner evolution, that can, for example, describe the boundary of a fluid going through a porous material; (B) the Hastings–Levitov model that describes how particles aggregate into a cluster, similar to how bacterial shapes grow in a Petri dish. In Papers C and D, we introduce and study a model that describes a collection of electric charges, undergoing random fluctuations, restricted to a curve in the plane.

A common theme throughout the thesis is the study of rare events that can occur in these models. To quantify how rare an event is, we employ an approach called the *large deviation principle*, which describes events that can happen only with exponentially small probability. Once it is proved that a model satisfies the large deviation principle, it yields a functional that ”controls” large deviations and essentially quantifies how unlikely an event is, giving it a number between 0 and $+\infty$, ranging from rare to almost impossible events. This functional is sometimes referred to as ”energy” — the more unlikely an event is, the more energy it requires to happen.

Paper A establishes a large deviation principle for a family of random fractal curves living in two dimensions. These curves arise as limiting objects of different statistical physics models on lattices as the mesh size of a lattice goes to zero, that is, in the limit of going from a discrete model on a lattice to a continuous model in the plane. The family of random curves obtained this way is described by a positive parameter κ that controls how fractal the curve is — the bigger values of κ correspond to more irregular curves. For small values of κ it is very unlikely that the random curve will take a prescribed shape, and the large deviation principle quantifies the probability of such events.

Paper B proves a large deviation principle for a model, introduced by Hastings and Levitov, that describes the growth of a random cluster in two dimensions. We also answer the question of whether the cluster can take a certain predetermined shape and how much energy it costs. In particular, an event that the shape of a growing random cluster will be very close to the outline depicted on the cover of the thesis requires a finite amount of energy.

The last two papers introduce a random process that models a collection of repelling Brownian-like particles living on a curve in the plane at a given temperature. Paper C defines the model on a closed non-self-intersecting contour. Paper D studies it on a semi-circle — an example of a curve that is not closed, and thus the effects due to the presence of the left and right ends should be taken into account. In these papers, we study different properties of the model. First, we show how the probabilistic distribution of particles evolves over time, and that no matter where the particles are placed on the curve at the initial time moment, its distribution converges in time to a distribution corresponding to a *Coulomb gas*. The temperature in the model controls how prominent the random fluctuations of particles are, and as it approaches zero the dynamics becomes influenced more and more by the repulsion between particles rather than randomness in the system. In Paper C, we show that the large deviation principle holds in the low temperature regime, that is, the probability that the system of particles follows a prescribed dynamics is exponentially small with respect to the temperature.

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List of included papers

This doctoral dissertation consists of two parts. The first part provides an overview of the research field on which I focused and a summary of my research results as well as possible future directions. The second part comprises the contributions that I have made through research articles.

Paper A A large deviation principle for the Schramm–Loewner evolution in the uniform topology
V. Guskov
Annales Fennici Mathematici vol. 48 (no. 1), pp. 389-410, 2023
https://doi.org/10.54330/afm.130997

Paper B Loewner–Kufarev entropy and large deviations of the Hastings–Levitov model
N. Berestycki, V. Guskov and F. Viklund
arXiv:2512.02855, 2025. *To be submitted.*

Paper C Dyson Brownian motion on a Jordan curve
V. Guskov, M. Liu and F. Viklund
To be submitted.

Paper D Dyson Brownian motion on a circular arc
V. Guskov, M. Liu and F. Viklund
To be submitted.

Contents

Part I: Introduction and Summary

Part II: Research Papers

Part I
Introduction and Summary

Introduction

In this chapter, we recall definitions and results necessary for understanding the research papers in the second part of the thesis. We begin, in Section 1, with a short introduction to the theory of large deviations and state the main theorems that will be helpful throughout the thesis. The section ends with a list of resources on large deviations that the author found useful when studying the subject. Then, in Section 2, we present a small part of an extensive field of Loewner evolution: the Loewner differential equation and Schramm–Loewner evolution in the upper half-plane, the definition of Loewner energy, and the Loewner–Kufarev equation. Section 3 presents the Hastings-Levitov model of a random growth process in the complex plane. In particular, we describe the HL(0) model and its construction via the Loewner-Kufarev equation. The final Section 4 gives definitions, basic properties and stochastic techniques in the context of Dyson Brownian motion.

1 Large Deviations

We start with a simple example which demonstrates the notion of large deviations. Let $(X_i)_{i=1}^n$ be a sequence of independent identically distributed (i.i.d.) random variables that follow the normal distribution $\mathcal{N}(0,1)$. By the law of large numbers, the empirical mean $\frac{1}{n}S_n = \frac{1}{n}(X_1 + \cdots + X_n)$ converges to zero almost surely. In fact, a weaker statement suffices: a probability measure $\mathbb{Q}_n(\bullet) = \mathbb{P}[\frac{1}{n}S_n \in \bullet]$ converges weakly to a point mass at zero $\delta_0(dx)$ as $n \to \infty$. We can do an explicit computation to determine how fast it converges to zero. Since $\frac{1}{n}S_n \sim \mathcal{N}(0, \frac{1}{n})$

$$\mathbb{P}\left[\frac{1}{n}S_n > a\right] = \int_{a\sqrt{n}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

For $a > 0$, the gaussian integral can be estimated from below and above by

$$\frac{2}{a\sqrt{n} + \sqrt{a^2 n + 4}} \frac{1}{\sqrt{2\pi}} e^{-n\frac{a^2}{2}} \leq \mathbb{P}\left[\frac{1}{n}S_n > a\right] \leq \frac{1}{a\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-n\frac{a^2}{2}}.$$

We are interested in the rate of exponential convergence to zero, and, as stated, this bound gives more information than many problems would allow for. Introducing the function $I(x) = x^2/2$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n}S_n \in (a, +\infty)\right] = -I(a).$$

The probability measure $\mathbb{Q}_n(\bullet) = \mathbb{P}[\frac{1}{n}S_n \in \bullet]$ is said to converge to zero on any set $V \subset \mathbb{R}$ such that $0 \notin \overline{V}$ with the *rate* n and the *rate function* $I(x) = x^2/2$.

A small comment on the name is in order. The event $\{S_n \geq a\sqrt{n}\}$ represents a deviation of order $\sqrt{n}$ of S_n from zero. Due to the central limit theorem, the probability $\mathbb{P}[S_n > a\sqrt{n}]$ converges to $\int_a^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, i.e., is non-zero. Such deviations are also called "fluctuations". Contrary to this, the probability of a deviation of order n, $\mathbb{P}[S_n \geq an]$, not only converges to zero but does so exponentially fast. Thus, $\{S_n \geq an\}$ is considered to be a "large deviation" event.

1.1 Definitions and classic results

Let E be a Polish space, i.e., a complete separable metric space. Denote by $\mathcal{B}(E)$ the Borel σ-algebra on E.

Definition 1.1 (Rate function). *A function $I : E \to [0, +\infty]$ is a rate function if*

(1.) $I \not\equiv +\infty$,

(2.) I is lower semi-continuous.

It is a good rate function if additionally to 1. and 2. it satisfies

(3.) $\{x \in E : I(x) \leq c\}$ is compact for every $c \in [0, +\infty)$.

Compactness of level sets in (3.) already implies lower semi-continuity in (2.), so saying that a good rate function should satisfy (1.), (2.), and (3.) is excessive but customary. The following short-hand notation will be used

$$I(V) = \inf_{x \in V} I(x),$$

with the convention that the infimum over the empty set is $+\infty$. A good rate function achieves the infimum over closed sets.

Definition 1.2 (Large deviation principle). *A family of probability measures $\mathbb{Q}_\varepsilon$ on $(E, \mathcal{B}(E))$ satisfies the large deviation principle with the rate ε and the rate function I if*

(1.) $\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{Q}_\varepsilon(F) \leq -I(F)$ for all closed $F \subset E$,

(2.) $\underline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{Q}_\varepsilon(G) \geq -I(G)$ for all open $G \subset E$.

If the upper bound in Definition 1.2 holds for all compact sets (instead of closed), then we say that the *weak* large deviation principle holds.

The definition of large deviation principle (LDP) is formulated as a collection of inequalities as to allow "stating asymptotic results that, on the one hand, are accurate enough to be useful and, on the other hand, are loose enough to be correct" [18].

Those sets $V \subset E$ for which $I(V^\circ) = I(\overline{V})$ are called I-continuous, and it follows from the definition that for such sets the large deviation limit exists and equals

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{Q}_\varepsilon(V) = -I(V).$$

Let $a_\varepsilon^i \geq 0$. The following simple asymptotic result is very useful when proving large deviation upper bounds:

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log(a_\varepsilon^1 + \cdots + a_\varepsilon^N) = \max_{i=1,\ldots,N} \overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log(a_\varepsilon^i).$$

Contraction principle. Continuous mappings preserve LDP in the following sense.

Theorem 1.1 (Contraction principle)**.** *Let E and S be Hausdorff topological spaces, $f : E \to S$ is a continuous mapping. Let $I_E : E \to [0, +\infty]$ be a good rate function, and define $I_S : S \to [0, +\infty]$ by*

$$I_S(y) = \inf_{x \in f^{-1}(y)} I_E(x), \quad y \in S,$$

with the convention that the infimum over the empty set is $+\infty$. Then,

1. *I_S is a good rate function on S.*
2. *If a family of probability measures (μ_ε) on E satisfies the LDP with the good rate function I_E, then the family of probability measures $(\mu_\varepsilon \circ f^{-1})$ on S satisfies the LDP with the good rate function I_S.*

The notion of being exponentially close. Oftentimes the mapping at hand fails to be continuous, and so the contraction principle is not directly applicable. However, the points of discontinuity might be inconsequential to the LDP and one can formulate a generalized (or approximate) contraction principle. This leads to the notion of being "exponentially close".

Definition 1.3 (Exponential equivalence)**.** *Let (S, d) be a metric space and $(\mu_\varepsilon), (\nu_\varepsilon)$ be two families of probability measures on S. If there exist a probability space $(\Omega, \mathcal{F}_\varepsilon, P_\varepsilon)$ and two families of random variables (ξ_ε) and (η_ε) on it with the joint law P_ε such that μ_ε and ν_ε are its marginals, and for all $\delta > 0$ the set $\{d(\xi_\varepsilon, \eta_\varepsilon) > \delta\}$ is $\mathcal{F}_\varepsilon$-measurable and*

$$\overline{\lim_{\varepsilon \downarrow 0}}\, \varepsilon \log P_\varepsilon(d(\xi_\varepsilon, \eta_\varepsilon) > \delta) = -\infty,$$

then the families $(\mu_\varepsilon), (\nu_\varepsilon)$ are exponentially equivalent.

Theorem 1.2. *Let two families of probability measures $(\mu_\varepsilon), (\nu_\varepsilon)$ be exponentially equivalent. If any of them satisfies an LDP with a good rate function, then the same LDP holds for the other family (with the same rate function).*

Definition 1.4 (Exponentially good approximation)**.** *Let (S, d) be a metric space and $(\mu_\varepsilon), (\nu_{\varepsilon,n})$ be two families of probability measures on S. If there exist a probability space $(\Omega, \mathcal{F}_\varepsilon, P_{\varepsilon,n})$ and two families of random variables (ξ_ε) and on $(\eta_{\varepsilon,n})$ on it with the joint law $P_{\varepsilon,n}$ such that μ_ε and $\nu_{\varepsilon,n}$ are its marginals, and for all $\delta > 0$ the set $\{d(\xi_\varepsilon, \eta_{\varepsilon,n}) > \delta\}$ is $\mathcal{F}_\varepsilon$-measurable and*

$$\lim_{n \to \infty} \overline{\lim_{\varepsilon \downarrow 0}}\, \varepsilon \log P_{\varepsilon,n}(d(\xi_\varepsilon, \eta_{\varepsilon,n}) > \delta) = -\infty,$$

then the family $(\nu_{\varepsilon,n})$ is an exponentially good approximation of (μ_ε).

Theorem 1.3. *Let two families of probability measures (μ_ε), $(\nu_{\varepsilon,n})$ be such that*

- *$(\nu_{\varepsilon,n})$ is an exponentially good approximation of (μ_ε),*
- *for every $n \in \mathbb{N}$, $(\nu_{\varepsilon,n})$ satisfies an LDP with the rate function I_n.*

Then,

(1.) (μ_ε) *satisfies the weak LDP with the rate function*

$$I(y) = \sup_{\delta>0} \varliminf_{n\to\infty} I_n(B(y,\delta)).$$

(2.) If I *is a good rate function and for every closed set* F

$$I(F) \leq \varlimsup_{n\to\infty} I_n(F),$$

then (μ_ε) *satisfies the LDP with the rate function* I*.*

LDP for empirical measures. An empirical measure of a collection of i.i.d random variables $(X_i)_{i=1}^n$ is

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}(dx).$$

This measure is a random element in the space of probability measures. If it satisfies an LDP, the rate function that controls it is given by the relative entropy.

Definition 1.5 (Relative entropy)**.** *For* $\nu, \mu \in \mathcal{P}(E)$*, the relative entropy of* ν *with respect to* μ *is*

$$H(\nu|\mu) = \begin{cases} \int f \log f d\mu & \text{if } \nu \ll \mu, f = \frac{d\nu}{d\mu}, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 1.4 (Sanov)**.** *Let* $(X_i)_{i\geq 1}$ *be i.i.d. random variables in a Polish space* E *with Law*$(X_1) = \mu$*. Then, the empirical measures* $\frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ *satisfy the LDP in* $\mathcal{P}(E)$ *with the rate* n *and the good rate function given by the relative entropy*

$$I(\nu) = H(\nu|\mu), \quad \nu \in \mathcal{P}(E).$$

1.2 Literature

The theory of large deviations is extensive and a number of monographs was written on the topic. The following non-extensive list includes some of the sources that were consulted during my study of large deviations.

Books

1. A. Bidhiraja and P. Dupuis, *Analysis and Approximation of Rare Events.* Springer 2019.
2. J. M. Bismut, *Large Deviations and the Malliavin Calculus.* Progress in Mathematics 45, Birkhauser 1984.
3. J. A. Bucklew, *Large Deviations Techniques in Decision, Simulations, and Estimation.* John Wiley & Sons, Inc. 1990.
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2. O. Kallenberg, Large Deviations, in *Foundations of Modern Probability.* 3rd edition, Springer, 2019.
3. S.R.S. Varadhan, Large Deviations and Entropy, Chapter 9 in *Entropy.* Princeton University Press 2004.

Lecture notes

1. A.A. Puhalskii, *Large Deviations for Stochastic Processes*, 2006.
2. J.M. Swart, *Large Deviation Theory*, 2021.

2 Loewner Evolution

Loewner evolution is a technique, based on differential equations, used to describe a ”time” evolution of continuously increasing (or decreasing) family of domains in the complex plane. It was first introduced by Loewner in 1923 and was extended by Kufarev in 1940s and later by Pommerenke. For historical development of the theory see [1].

In the following sections we state the main definitions and results that are used in Papers A and B. For an in-depth introduction to the theory we refer to [38, 9, 7, 36] and [45, 20].

2.1 Loewner differential equation in $\mathbb{H}$

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ denote the upper half-plane. We call a set $K \subset \overline{\mathbb{H}}$ a *hull* if K is compact and its complement is simply-connected. By the Riemann mapping theorem, there exists a unique conformal map $g : \mathbb{H} \setminus K \to \mathbb{H}$ with the *hydrodynamic* normalization at ∞:

$$g(z) = z + \frac{a_1(K)}{z} + \frac{a_2(K)}{z^2} + \dots ,$$

with real coefficients $a_k(K)$. The first coefficient $a_1(K)$ is called the *half-plane capacity* of K. By $f = g^{-1}$ we denote the inverse conformal map from $\mathbb{H}$ to $\mathbb{H}\setminus K$.

A family $(K_t)_{t\geq 0}$ of hulls is said to be *growing* if $K_s \subsetneq K_t$ for $s < t$ and for every $\varepsilon > 0$ and $T > 0$ one can find $\delta > 0$ such that $K_{t+s} \subset K_t^{\varepsilon}$ for all $t \in [0, T - \delta]$, where K^{ε} is an ε-neighborhood of K. The family is said to be parametrized by the half-plane capacity if $a_1(K_t) = 2t$ for all $t > 0$.

In the case when the family of hulls is given by $(\gamma([0,t]))_{t\geq 0}$, where γ is a simple curve in $\mathbb{H}$, its evolution is encoded through a family of conformal maps $g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$ which satisfy the Loewner differential equation. Let $\gamma : [0, +\infty) \to \overline{\mathbb{H}}$ be a simple curve that starts on the real line $\gamma(0) \in \mathbb{R}$, $\gamma((0, +\infty)) \subset \mathbb{H}$ and is parametrized by the half-plane capacity, i.e., $a_1(\gamma([0,t])) = 2t$. Then, for any $t \geq 0$, the limit

$$\lambda(t) = \lim_{z\to\gamma(t)} g_t(z)$$

exists, is real-valued, and is a continuous function. Denote $\dot{g}_t = \partial_t g_t$. The family $g_t : \mathbb{H}\setminus\gamma([0,t]) \to \mathbb{H}$ of conformal maps satisfies the *Loewner differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad z \in \mathbb{H} \setminus \gamma([0,t]),\ t > 0,$$

with the initial condition $g_0(z) = z$. Moreover, the inverse map $f_t = g_t^{-1}$ satisfies the Loewner partial differential equation

$$\dot{f}_t(z) = -\frac{2f_t'(z)}{z - \lambda(t)}, \quad z \in \mathbb{H},\ t > 0,$$

with the initial condition $f_0(z) = z$.

In this context, the real-valued function λ is called the *driving function.* One can think of it as a rule that encodes the evolution of the family of conformal maps $(g_t)_{t\geq 0}$, called the *Loewner chain*, or, equivalently, the family of decreasing simply-connected domains $(\mathbb{H} \setminus \gamma([0,t]))_{t\geq 0}$.

One can, of course, consider the reverse problem: given a real-valued continuous function $\lambda : [0, +\infty) \to \mathbb{R}$, find a family of conformal maps that solves the Loewner differential equation. Not all real-valued continuous functions give rise to a solution to the Loewner equation, for elementary examples see [34]. It is an open problem to describe those functions that do give rise to a solution of the Loewner equation, i.e., the set of all driving functions. However, there are sufficient conditions that guarantee the existence of a solution, e.g., see [23, 57, 32, 50, 40].

2.2 Schramm-Loewner evolution

Schramm-Loewner evolution, abbreviated SLE or SLE_κ, is a one-parameter family of random fractal curves in a simply-connected domain that enjoys conformal invariance and satisfies the domain Markov property[1]. SLEs arise as scaling limits of interfaces of many discrete statistical mechanics models.[2] There are many variants of SLE, but here we focus on *chordal* SLE in the upper half-plane $\mathbb{H}$ as it is the setup of Paper A. *Chordal* means that an SLE curve connects two boundary point of $\mathbb{H}$. In particular, 0 and ∞ is the canonical choice for the reference points.

Chordal SLE in $\mathbb{H}$ can be realized as a Loewner evolution driven by a scaled Brownian motion $\lambda = \sqrt{\kappa}B$, $\kappa > 0$. Let g be a solution to the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B(t)}, \quad t > 0,$$

with the initial condition $g_0(z) = z$, and where $B = (B(t))_{t\geq 0}$ is the standard Brownian motion. For any $\kappa > 0$, SLE_κ is a random curve $\gamma^\kappa = (\gamma^\kappa(t))_{t\geq 0}$ given by

$$\gamma^\kappa(t) = \lim_{y\to 0+} g_t^{-1}(\sqrt{\kappa}B(t) + iy).$$

It is a non-trivial fact that the above limit defines a curve, see [47] for details.

The parameter $\kappa > 0$ controls the geometry of the curve, in particular how windy it is. There are three phases, depending on κ, where the behavior of SLE_κ is qualitatively different: almost surely,

(1) for $0 \leq \kappa \leq 4$, γ^κ is a simply curve;

(2) for $4 < \kappa < 8$, γ^κ hits itself and the real line;

(3) for $\kappa \geq 8$, γ^κ is space-filling.

The boundary case $\kappa = 0$ corresponds to the deterministic Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z)}, \quad t > 0,$$

which has an explicit solution

$$g_t(z) = \sqrt{4t + z^2}, \quad z \in \mathbb{H} \setminus [0, i2\sqrt{t}].$$

That is, the driving function $\lambda \equiv 0$ generates the curve $(\eta(t))_{t\geq 0}$ given by

$$\eta(t) = i2\sqrt{t}, \quad t \geq 0.$$

It is known, see [33], that for small enough κ the SLE curves are continuous in κ. In particular, if we explicitly denote the dependence of $\gamma^\kappa(t,\omega)$ on elementary events $\omega \in \Omega$, then there exists $\kappa_0 > 0$ such that for all $\omega \in \Omega \setminus \mathcal{N}$, where $\mathcal{N}$ is a set of probability measure zero, the function $\kappa \to \sup_{t\in[0,1]} |\gamma^\kappa(t,\omega)|$ is continuous

[1]For precise formulation of conformal invariance and domain Markov property see [36, Section 5.1.1]

[2]loop-erased random walk, Ising model, Fortuin-Kasteleyn-Ising model, Gaussian Free Field, critical percolation and uniform spanning trees.

on $[0, \kappa_0)$. Therefore, as $\kappa \to 0+$, the SLE curve $(\gamma^\kappa(t))_{t\geq 0}$ converges to $(\eta(t))_{t\geq 0}$ uniformly on $[0, T]$ for every $T > 0$.

Let S denote the subspace of $C([0, +\infty), \overline{\mathbb{H}})$, continuous curves in the upper half-plane started at the origin, that are simple and parametrized by the half-plane capacity. Equip S with the topology generated by the locally uniform convergence, and denote by $\mathcal{B}(S)$ the Borel σ-algebra. Then, the continuity in κ implies that the law of SLE_κ converges to a point mass at η:

$$\lim_{\kappa \to 0+} \mathbb{P}[\gamma^\kappa \in V] = \mathbb{1}_{\{\eta \in V\}}, \quad V \in \mathcal{B}(S).$$

In fact, for any measurable set V, whose closure does not contain η, the probability $\mathbb{P}[\gamma^\kappa \in V]$ decays to zero exponentially fast as $\kappa \to 0+$. In Paper A we study this convergence and prove that the large deviation principle holds with a good rate function given by the *Loewner energy*. See Section 4.6 for a summary.

2.3 Loewner energy

Let γ be a simple curve in the upper half-plane connecting 0 and ∞, parametrized by the half-plane capacity, with the driving function λ. The Loewner energy of $\gamma([0, T])$ is defined as

$$I_T(\gamma) = \frac{1}{2} \int_0^T \lambda'(t)^2 dt$$

if λ is absolutely continuous on $[0, T]$ and set to $+\infty$ otherwise. The Loewner energy of the curve γ is $I(\gamma) = \lim_{T \to +\infty} I_T(\gamma)$, that is, the Dirichlet energy of the driving function.

The definition of the Loewner energy is not restricted to curves connecting 0 and ∞ in the upper-half plane and can be extended to other domains and boundary points using conformal mappings. The Loewner energy was originally introduced in [57] in the context of large deviations of SLEκ as $\kappa \to 0+$. Moreover, it can be written down in terms independent of Loewner evolution of the curve, see [56, 48].

2.4 Loewner-Kufarev equation

Let $\Delta = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ denote the exterior disk. A family of conformal maps $f_t : \Delta \to \Delta_t$, $\Delta_t := f_t(\Delta)$, is called the Loewner chain if it satisfies the Loewner-Kufarev equation

$$\dot{f}_t(z) = z f_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \mu_t(d\theta), \quad f_0(z) = z, \quad z \in \Delta,$$

where $(\mu_t)_{t\geq 0}$ is a family of positive finite measures on $[0, 2\pi]$. If the equation admits a solution, then it generates a family of growing compact clusters $K_t = \hat{\mathbb{C}} \setminus \Delta_t$ that has logarithmic capacity equal to

$$c(t) = \int_0^t \mu_s([0, 2\pi]]) ds$$

and the maps have the normalization

$$f_t(z) = e^{c(t)}z + o(1) \quad \text{as } z \to \infty.$$

The Loewner-Kufarev equation admits a unique solution if, for instance, $(\mu_t)_{t\geq 0}$ is a family of probability measures and the integral

$$p_t(z) = \int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} \mu_t(d\theta)$$

defines a measurable, in t, function. In this case, the logarithmic capacity $c(t) \equiv 1$, and the capacity of K_t equals e^t. For more details, see [45]. Numerous examples of Loewner chains driven by the probability measures with densities are described in [53].

Remark 2.1. *Taking a point mass probability measure $\mu_t(d\theta) = \delta_{\lambda(t)}(d\theta)$ reproduces the radial Loewner equation*

$$\dot{f}_t(z) = z f_t'(z) \frac{z+e^{i\lambda(t)}}{z-e^{i\lambda(t)}}, \quad f_0(z) = z, \quad z \in \Delta.$$

Example 2.1. *For $\mu_t(d\theta) = \frac{1}{2\pi} d\theta$, the Loewner-Kufarev equations admits an explicit solution $f_t(z) = e^t z$. In particular, the growing cluster equals $K_t = e^t \overline{\mathbb{D}}$.*

3 Growth Models

The aim of the section is to describe the Hastings-Levitov model of planar growth. We consider a growing cluster as a family of simply-connected compact subsets of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. *Growing* is understood in a sense that any point of the complex plane which becomes a part of the cluster must remain such for all time. This restriction does not allows the cluster to wobble and allows to employ tools such as the Loewner–Kufarev evolution to describe the dynamic of the cluster.

Let (K_n) be a family of increasing compact sets $K_n \subsetneq K_{n+1}$, that represent a growing cluster. Assume they are connected and include the origin, to avoid needless complications. Denote by $\Delta = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ the extended complex plane and by $\Delta_n = \hat{\mathbb{C}} \backslash K_n$ the complement of the cluster K_n. By the Riemann mapping theorem, there exists a unique conformal map $\Phi_n : \Delta \to \Delta_n$ with the normalization at infinity

$$\Phi_n(z) = C(K_n)z + O(1),$$

where the coefficient $C(K_n)$ is called the *capacity* of K_n and is comparable to the diameter of the set K_n.

One can start the other way around – with a construction of the family of conformal maps (Φ_n), and then obtain a growing cluster (K_n). Hastings and Levitov proposed in 1998, see [27], a model for the family (Φ_n) that we now describe.

3.1 Hastings–Levitov model

The building block of the Hastings–Levitov model is the *one-particle mapping*, that is, a conformal mapping that takes Δ to $\Delta\backslash P$, where P is compact set that represents a physical particle. For example, one can take $P = (1, 1+d]$, a slit of length d, or a small disk tangent to Δ, or any other bump one can think of.

By the Riemann mapping theorem there exists a unique conformal map $\varphi_P : \Delta \to \Delta\backslash P$ with the expansion at infinity $\varphi_P(z) = C(P)z + O(1)$. The coefficient $C(P)$ is the capacity of the set $\overline{\mathbb{D}} \cup P$. The map φ_P attaches the particle P at the point $z = 1$, while the rotated map $\varphi_{P,\theta}(z) = e^{i\theta}\varphi_P(e^{-i\theta}z)$ attaches it at $z = e^{i\theta}$ at the angle θ.

Remark 3.1. *Usually it is more convenient to work with the logarithmic capacity* $c(P) = \log C(P)$. *From time to time, we denote the particle mapping as* φ_P *or* φ_d *or* φ_c, *with the subscripts referring to the set* P, *the diameter* d, *and the logarithmic capacity* c, *correspondingly.*

Given a collection of one-particle maps (φ_{P_k,θ_k}), consider the composition of thereof

$$\Phi_n \overset{\text{def}}{=} \varphi_{P_1,\theta_1} \circ \ldots \circ \varphi_{P_n,\theta_n}. \tag{3.1}$$

A growing sequence of compact sets (K_n) is then encoded through the conform mappings $\Phi_n : \Delta \to \hat{\mathbb{C}}\backslash K_n$. The capacity of K_n equals $\prod_{i=1}^n C(P_i)$ or $C(P)^n$ in the case of identical particles. Note that in the construction of K_{n+1} from K_n via $\Phi_{n+1} = \Phi_n \circ \varphi_{P_{n+1},\theta_{n+1}}$ we are not just attaching the new particle P_{n+1} to the existent cluster K_n, but P_{n+1} gets distorted by the map Φ_n. This leads to the consequence that although we can control the size of the particles we are not in control of their shape.

We choose the particle to be $P = (1, 1+d]$, a slit of length d. The one-particle conformal map $\varphi_d : \Delta \to \Delta\backslash(1, 1+d]$ is explicit:

$$\varphi_d(z) = (1+\lambda^2)\frac{z+1}{2z}\left(z + 1 + \sqrt{z^2 + 1 - 2z\frac{1-\lambda^2}{1+\lambda^2}}\right) - 1, \tag{3.2}$$

$$d = 2\lambda(\sqrt{1+\lambda^2} + \lambda) \text{ or } \lambda = d/2\sqrt{1+d}. \tag{3.3}$$

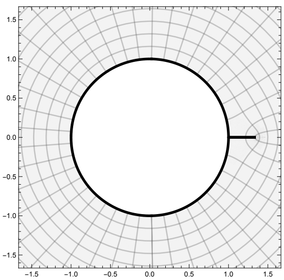

Figure 3.1: Conformal image of the exterior disk Δ under the slit mapping φ_d.

It is also elementary to grow a straight slit in the framework of the Loewner evolution. From now on the slit mapping $\varphi_{d,\theta} : \Delta \to \Delta\backslash e^{i\theta}(1, 1+d]$ is the one we choose to work with.

In the Hastings–Levitov model, the angles (θ_k) are independent identically distributed (i.i.d.) random variables which follow the uniform distribution on the interval $[0, 2\pi)$, while the lengths (d_k) of the slits are encoded through their logarithmic capacities (c_k):

$$e^{c_k} = 1 + \frac{d_k^2}{4(1+d_k)}.$$

In turn, the logarithmic capacity of k:th particle is chosen to be

$$c_k \stackrel{\text{def}}{=} c \left| \Phi'_{k-1}(e^{i\theta_k}) \right|^{-\alpha}, \tag{3.4}$$

where the constant $c > 0$ is responsible for the overall size of attached particles, while the parameter $\alpha > 0$ scales the newly arrived particles compared to the existent cluster.

Given this construction, the Hastings–Levitov model with parameter $\alpha > 0$, abbreviated as HL(α), is a sequence (Φ_k) of random conformal maps, given by (3.1).

Comments on literature

The HL(α) model was introduced in 1998 by physicists Hastings and Levitov in [27]. The idea was to come up with a continuous analogue for the lattice-based dielectric breakdown model DBM(η). The latter is a one-parameter family which includes the Eden model (1961) for bacterial growth and diffusion-limited aggregation (DLA, 1981). For the early numerical analysis of the Hastings–Levitov model, see [17].

Introduction of a deterministic analogue of DLA described with the help of Loewner–Kufarev equation is presented in [11], where the authors also obtain a Kesten-type estimate for the growth of the cluster.

In [49] the scaling limit of HL(0) cluster, scaled by its capacity, as $n \to +\infty$ is studied. The authors proved that the Hausdorff dimension of HL(0) cluster equals 1 almost surely and gave an upper bound for the Hausdorff dimension of HL(α). Moreover, the first regularized version of HL(α) is introduced.

In [29] the authors introduce a Loewner evolution driven by unimodular Lévy processes, a model driven by a compound Poisson process similar to HL(0), and study the Hausdorff dimension of the cluster.

Anisotropic Hastings–Levitov model is introduced in [30], where the attachment angles can follow a general distribution, not necessarily a uniform one. It presents a proof of a continuity of the Loewner–Kufarev map with respect to the driving measure; a proof of a shape theorem for the Loewner hulls, and studies the evolution of the harmonic measure of the cluster's boundary.

A proof that $\Phi_n(z) \approx e^{cn} z$ is presented in [41], which illustrates, on the level of conformal maps, that the small particle scaling limit of HL(0) cluster is a disk. The branching structure is related to the Brownian web.

Regularized HL(α) model is studied in [31], with emphasis on the scaling limit of the cluster and the study of the harmonic measure.

Aggregate Loewner evolution is introduced in [54], where the scaling limit as well as the phase transition are studied.

In [8] a new variation of the Hasting-Levitov model, called the constrained Hastings–Levitov model, is introduced. It is shown that the model exhibits explosive growth.

3.2 Hastings–Levitov HL(0) model

The HL(0) model is the most studied one among the HL(α) family. Many properties were derived in [41, 49, 51] to name but a few, as well as variations and generalizations of the model were introduced in [30, 8, 29, 10].

The capacity parameters, defined by (3.4) with $\alpha = 0$, are identical $c_k \equiv c$ for all particles. The angles (θ_k) are i.i.d., uniformly distributed on $[0, 2\pi)$, random variables. The conformal map $\Phi_n : \Delta \to K_n$ is given by

$$\Phi_n = \varphi_{c,\theta_1} \circ \dots \circ \varphi_{c,\theta_n}, \tag{3.5}$$

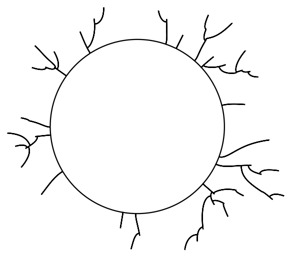

Figure 3.2: HL(0) cluster with $n = 50$ and $d \approx 0.2$ demonstrating the distortion of slit particles.

and the logarithmic capacity of the cluster at step n equals $c(K_n) = nc$, where $c = c(P)$ is the logarithmic capacity of the slit (more precisely, of the slit union the disk).

Although the length parameter $d = d(c)$ is fixed for all particles this does not mean that the arriving particles all have the same Euclidean size. The newly arrived particles get distorted by the composition of conformal maps in the construction of Φ_n. In fact, the distortion is quite prominent: the particles that have arrived at the later steps of the process are much larger than the ones attached in the begging, with the diameter of the n:th particle being of order de^{cn}, see [41]. On the one hand, this might lead to a conclusion that the model is not physically relevant, but on the other hand, the model is mathematically trackable and offers solid ground for rigorous analysis.

3.3 HL(0) model via Loewner evolution

The Loewner–Kufarev equation can be used to study random growth problems. In the work of Carleson and Makarov [11] it was used to define a deterministic analogue of the DLA model. In connection with the Hastings–Levitov model the Loewner–Kufarev equation was employed in [49, 31, 54].

In this section, we formulate the HL(0) model in terms on the Loewner evolution. The sequence of conformal maps (Φ_k) can be realized as a subsequence of the Loewner chain $(f_t)_{t\geq 0}$ that satisfies the Loewner–Kufarev equation with the driving probability measure

$$\mu_n(d\theta, dt) = \sum_{k=1}^{n} \delta_{\theta_k}(d\theta) \mathbb{1}_{[c(k-1),ck)}(t)dt, \quad \theta_k \sim U([0, 2\pi)). \tag{3.6}$$

In fact, with this choice of the driving measure, the Loewner–Kufarev equation is reduced to the Loewner differential equation in the exterior disk with a piece-wise constant driving function:

$$\dot{f}_t(z) = z f_t'(z) \frac{z + e^{i\theta(t)}}{z - e^{i\theta(t)}}, \quad z \in \Delta, \quad t > 0,$$

where

$$\theta(t) = \sum_{k=1}^{n} \theta_k \mathbb{1}_{[c(k-1),ck)}(t), \quad \theta_k \sim U([0, 2\pi)). \tag{3.7}$$

The following lemma, see [49, Lemma 4.1], demonstrates that a driving term given by a sum has the effect of composing the Loewner chains corresponding to separate summands.

Lemma 3.1 ([49]). *For $i = 1, 2$, let $\lambda^{(i)} : [0, t_i] \to [0, \pi)$ be piece-wise continuous. If $f_t^{(i)}$ is the solution to the Loewner equation with the initial condition $f_0^{(i)}(z) = z$ and the driving function $\lambda^{(i)}$, then $f_{t_1}^{(1)} \circ f_{t_2}^{(2)}$ is the solution to the Loewner equation driven by*

$$\lambda(t) = \lambda^{(1)}(t)\mathbb{1}_{[0,t_1)}(t) + \lambda^{(2)}(t)\mathbb{1}_{[t_1,t_1+t_2]}(t).$$

The Loewner equation driven by the constant function $\theta(t) \equiv 0$

$$\dot{f}_t(z) = z f_t'(z) \frac{z+1}{z-1}$$

has an explicit solution, see [53, Equation 3.2], given by the conformal map

$$f_t(z) = e^t \frac{z+1}{2z}\left(z + 1 + \sqrt{z^2 + 1 + 2z(1 - 2e^{-t})}\right) - 1;$$

compare it with (3.2). At time t this conformal map attaches a horizontal slit at $z = 1$ with logarithmic capacity $c(t) = t$ and of length $d(t) = 2\sqrt{e^t - 1}\left(e^{t/2} + \sqrt{e^t - 1}\right)$. The radial slit at the point $z = e^{i\theta}$ is obtained by rotating the map: $f_{t,\theta}(z) = e^{i\theta} f_t(e^{-i\theta} z)$.

Therefore, the piece-wise constant driving function (3.7) generates the Loewner chain (f_t) which at time $t = cn$ is given by

$$f_{cn} = f_{c,\theta_1} \circ \ldots \circ f_{c,\theta_k}.$$

Comparing it to the map (3.5), the relation between the conformal maps in the construction of the HL(0) model and the Loewner chain driven by piece-wise constant function (3.7) follows:

$$\Phi_n = f_{cn}.$$

Scaling limit

The following theorem quantifies the convergence of the cluster K_n to a disk in the scaling limit as $n \to +\infty$ and $c \to 0$, while $nc \to T > 0$.

Theorem 3.1 (Norris, Turner 2012). *For any $R > 1$, $\varepsilon > 0$, and $T > 0$*

$$\lim_{\substack{n\to+\infty \\ c\to 0 \\ nc\to T}} \mathbb{P}\left[\sup_{|z|\geq R} |\Phi_n(z) - e^{nc} z| > \varepsilon\right] = 0.$$

One can also look at the scaling limit through the lens of Loewner evolution. In [30] it was established that the driving measure μ_n, defined in (3.6), converges (as $n \to +\infty$, $c \to 0$, $nc \to T$) weakly almost surely to $\lambda(d\theta) \otimes dt$, where λ is the uniform measure on $[0, 2\pi]$, and the corresponding Loewner chain converges locally uniformly almost surely to the conformal map $f(z) = e^T z$.

Consider the following space of holomorphic functions

$$\Sigma' = \{f \in H(\Delta) : \text{omits a neighborhood of zero and } f(z) = z + O(1/z)\},$$

equipped with the locally uniform topology, i.e., the topology generated by the uniform convergence on compact subset of Δ. The Loewner chain (f_t) has the logarithmic capacity parameterization, i.e., $f_t(z) = e^t z + O(1/z)$, so that $e^{-t} f_t \in \Sigma'$ for all $t \geq 0$. Similarly, $e^{-cn}\Phi_n \in \Sigma'$ for all $n \geq 1$. Let $\mathbb{P}_n$ be the probability distribution on Σ' induced by $e^{-cn}\Phi_n$. Let

$$\mathcal{L}_T : (\mu_t)_{t\in[0,T]} \mapsto e^{-T} f_T$$

denote the Loewner mapping from the space on driving measures to capacity-normalized Loewner chains at the final time T. Then, since $\Phi_n = f_{cn}$,

$$\mathbb{P}_n = (\mathbb{P}\mu_n^{-1}) \circ \mathcal{L}_{cn}^{-1}.$$

Almost sure convergence in Σ' of f_n to $e^T z$ implies that the law $\mathbb{P}_n$ induced by $e^{-cn} f_{cn}$ converges to a point mass:

$$\mathbb{P}_n[F] \to \mathbb{1}_{\{id \in F\}}, \quad F \in \mathcal{B}_{\Sigma'},$$

where $id \equiv z$ is the identity map.

In turn, this leads to the question of the speed of converges, and, hence, to the problem of large deviations of the probability measure $\mathbb{P}_n$. Heuristically, we would like to see whether for any $f \in \Sigma'$, away from the identity map, it is true that

$$\mathbb{P}\left[\Phi_n \approx e^{cn} f\right] \approx e^{-nI(f)}.$$

In Paper B we prove the large deviation principle and study the geometric properties of clusters with finite rate function. See the summary in Section 4.6 for more details.

4 Dyson Brownian Motion

In his seminal 1962 paper [21] Dyson introduced a model of *Brownian motion gas* as a model for the eigenvalues of random matrices which have standard Brownian motions as matrix elements. The model in [21] is treated in two settings: on the real line and on the unit circle. Moreover, certain properties of the long-time behavior are derived on the physical level of rigor. Both models, on the real line and on the circle, received considerable attention in physics as well as in mathematics. A presentation of Dyson Brownian motion can be found in numerous textbooks and articles. For example, see [13, 14], for an approach based on multivalued stochastic differential equations, or [5] which uses standard techniques of stochastic calculus. The definitions and basic properties of the models are recalled in Sections 4.2-4.3.

Recently, in [58], Zabrodin proposed a generalization of the Dyson Brownian motion to the model on a smooth Jordan curve. In Paper C we give a mathematical definition of the model on a rectifiable Jordan curve, and in the smooth enough setting study its properties: Fokker-Planck-Kolmogorov equation, convergence of the transition probabilities to the stationary distribution in the long time limit, large deviations as well as the hydrodynamical limit of the empirical measure.

In a nutshell, Dyson Brownian motion is a diffusion process under the assumptions that

(1) the particles are mutually repellent with Coulomb-type interaction;

(2) the particles are reflective, i.e., the order of particles on the curve is preserved for all time.

Before we give examples of Dyson Brownian motion the real line and on the unit circle, and comment on its construction on a general Jordan curve, it is illustrative to first take a quick look on how *Brownian motion* can be defined on a Jordan curve.

4.1 Brownian Motion on a Jordan curve

In this section we review several viewpoints on how Brownian motion can be defined on a (regular enough) Jordan curve. For example, the curve can be seen as an embedded submanifold of $\mathbb{C}$, or, the notion that we later adapt to the Dyson model, as a parametrized curve.

Parametric representation

Let Γ be a Jordan curve in the plane. The curve is assumed to be rectifiable, that is, it admits a Lipschitz continuous parametrization. In particular, one can parametrize Γ by the arc-length, thus obtaining the natural parametrization $(\gamma(s))_{s\in[0,l]}$, where $l = l(\Gamma)$ is the length of the curve. We extend the parametrization to $\mathbb{R}/l(\Gamma)\mathbb{Z}$, making it an l-periodic function on the real line. By Rademacher's theorem, the natural parametrization is differentiable almost everywhere, which means that the tangent vector $\gamma'(s)$ can be assigned to almost every point of the curve. The natural parametrization has the property that $|\gamma'| = 1$ almost everywhere.

Definition 4.1. *The process $(\gamma(B_t))_{t\geq 0}$, where B is the standard Brownian motion, is a Brownian motion on the curve Γ.*

If we assume that $\gamma \in C^2$, then by Itô formula

$$d\gamma(B_t) = \gamma'(B_t)dB_t + \frac{1}{2}\gamma''(B_t)dt.$$

Taking into account that $\gamma'(s) = \tau(\gamma(s))$, where τ is the unit tangent vector, and $\gamma''(s) = -\nu(\gamma(s))k(\gamma(s))$, where ν is the unit outer normal vector and k is the plane curvature, we see that the process $Z(t) = \gamma(B_t)$ satisfies the following SDE in the complex plane

$$dZ_t = \tau(Z_t)dB_t - \nu(Z_t)\frac{1}{2}k(Z_t)dt.$$

Remark 4.1. *Øksendal gives an example of "Brownian motion on the ellipse" in [42, Exercise 5.2]. There, Brownian motion on the ellipse* $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ *is defined to be the process* $Z(t) = a\cos(B_t) + ib\sin(B_t)$. *Note that it does not coincides with our definition because* $\gamma(s) = a\cos(s) + b\sin(s)$ *is not the natural parametrization of the ellipse.*

Conformal deformation

Let $w : \mathcal{N}(\Gamma) \to \mathcal{N}(\partial\mathbb{D})$ be a conformal map defined in a neighborhood of Γ that sends the curve to the unit circle. Let $Z = (Z_t)$ be a Brownian motion on Γ defined as a solution to the SDE

$$dZ_t = \tau(Z_t)dB_t - \nu(Z_t)\frac{1}{2}k(Z_t)dt.$$

Applying Itô formula to $w(Z_t)$ we obtain

$$dw(Z_t) = \tau(Z_t)w'(Z_t)dB_t - \frac{1}{2}\nu(Z_t)w'(Z_t)\left[\nu(Z_t)\frac{w''(Z_t)}{w'(Z_t)} + k(Z_t)\right]dt.$$

The tangent and normal vectors change to

$$\tau(w) = \tau(z)\frac{w'(z)}{|w'(z)|}, \quad \nu(w) = \nu(z)\frac{w'(z)}{|w'(z)|}, \quad z \in \Gamma.$$

The curvature changes to

$$k(w) = \frac{1}{|w'(z)|}\left[\operatorname{Im}\left\{\tau(z)\frac{w''(z)}{w'(z)}\right\} + k(z)\right], \quad z \in \Gamma.$$

The new process $W_t = w(Z_t)$ is now confined to the unit circle and satisfies the following SDE in the complex plane:

$$dW_t = \tau(W_t)|w'(w^{-1}(W_t))|\left(dB_t + \frac{1}{2}\operatorname{Re}\left\{\tau(W_t)\frac{w''(w^{-1}(W_t))}{w'(w^{-1}(W_t))^2}\right\}|w'(w^{-1}(W_t))|dt\right)$$
$$- \frac{1}{2}\nu(W_t)k(W_t)|w'(w^{-1}(W_t))|^2dt.$$

One can remove the factor $|w'(w^{-1}(W_t))|$ by making a random time reparametrization. Let $\tilde{W}_t = W_{\alpha_t}$, where $\alpha_t = \beta_t^{-1}$ and $\beta_t = \int_0^t |w'(w^{-1}(W_s))|^2 ds$, then

$$d\tilde{W}_t = \tau(\tilde{W}_t)\left(d\tilde{B}_t + \frac{1}{2}\operatorname{Re}\left\{\tau(\tilde{W}_t)\frac{w''(w^{-1}(\tilde{W}_t))}{w'(w^{-1}(\tilde{W}_t))^2}\right\}dt\right) - \frac{1}{2}\nu(\tilde{W}_t)k(\tilde{W}_t)dt.$$

This is a Brownian motion on the circle plus a drift in the tangential direction.

Brownian Motion on a Riemannian Manifold

Let (M, g) be a Riemannian manifold.

Intrinsic construction *Brownian motion on M* can be defined as a Markov process whose transition probability density function $p(t, p, q)$ is the heat kernel (minimal fundamental solution) of the Laplace-Beltrami operator Δ_M. In general, it might happen that $\int_M p(t,p,q)dq < 1$, but assuming M to be compact avoids this problem. Let X be a Brownian motion on M started at $p \in M$. For a smooth compact supported function f on M, the following Itô formula holds

$$f(X_t) = f(X_0) + \frac{1}{2}\int_0^t \Delta_M f(X_s)ds + M_t^f,$$

where M^f is a local martingale with the quadratic variation

$$\langle M^f \rangle_t = \int_0^t |\nabla f(X_s)|^2 ds.$$

For example, if $M = \mathbb{R}^d$, the martingale is explicit and equals

$$M_t^f = \int_0^t \langle \nabla f(X_s), dX_s \rangle.$$

Extrinsic construction Here we consider the Riemannian manifold M as a sub-manifold of an ambient Euclidean space $\mathbb{R}^m$ which is always possible due to Nash embedding theorem. *Extrinsic* construction of Brownian Motion on M uses an embedding of the manifold into an Euclidean space in order to represent the Laplace-Beltrami operator as a ”sum of squares”:

$$\Delta_M = \sum_{i=1}^m P_i^2,$$

where P_i is the projection of the unit vector e_i in $\mathbb{R}^m$ on the tangent space T_pM.

Let $(B_i)_{i=1}^m$ be a collection of independent standard Brownian motions, and consider the following Stratonovich SDE

$$dX(t) = \sum_{i=1}^m P_i(X(t)) \circ dB_i(t), \quad X(0) = p \in M.$$

Itô’s formula reads

$$f(X(t)) = f(X(0)) + \frac{1}{2}\int_0^t \Delta_M f(X(s))ds + \sum_{i=1}^m \int_0^1 \langle P_i f(X(s)), dB_i(s) \rangle.$$

Brownian motion on the circle $\mathbb{S} \subset \mathbb{R}^2$

Stroock's representation Let $\xi = (\xi_1, \xi_2) \in \mathbb{S}$ be a point on the circle, and $x = (x_1, x_2)$ be an arbitrary point in $\mathbb{R}^2$. The projection of x on the tangent space $T_\xi \mathbb{S}$ is given by $x - \langle x, \xi \rangle \xi$. If $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then the projections $P_i \xi$ are given by

$$P_1\xi = \begin{pmatrix} 1-\xi_1^2 \\ -\xi_1\xi_2 \end{pmatrix}, \quad P_2\xi = \begin{pmatrix} -\xi_2\xi_1 \\ 1-\xi_2^2. \end{pmatrix},$$

The Stratonovich SDE for $X(t) = (X_1(t), X_2(t))$ becomes

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 - X_1(t)^2 \\ -X_1(t)X_2(t) \end{pmatrix} \circ dB_1(t) + \begin{pmatrix} -X_2(t)X_1(t) \\ 1 - X_2(t)^2 \end{pmatrix} \circ dB_2(t),$$

here B_1, B_2 are independent standard Brownian motions. In the matrix form:

$$dX(t) = (id - X(t)X(t)^T) \circ dB(t).$$

In the Itô form:

$$dX(t) = (id - X(t)X(t)^T)dB(t) - \frac{1}{2}X(t)dt.$$

This representation depends on the embedding of the circle $\mathbb{S}$ into $\mathbb{R}^2$, since the equation for the Brownian motion on the 1-dimensional circle arises from 2-dimensional standard Brownian motion $B = (B_1, B_2)$.

Parametric representation The unit circle can be parametrized by $\gamma(s) = e^{is}, 0 \le s < 2\pi$. Then, according to Definition 4.1, a Brownian motion on the circle is given by $Y(t) = \exp(iB(t))$, where $B = (B(t))_{t \ge 0}$ is the standard 1-dimensional standard Brownian motion. The SDE for $Y(t)$ is

$$dY(t) = iY(t)dB(t) - \frac{1}{2}Y(t)dt.$$

More generally, if $q : \partial\mathbb{D} \to \mathbb{R}$ is real-valued, then the process

$$dY_t = iq(Y_t)\, dB_t - \tfrac{1}{2}q(Y_t)^2\, dt$$

is a time-changed Brownian motion on the circle.

If $X = (X_t^1, X_t^2)_{t \ge 0}$ is the Brownian motion in the Stroock's representation, then its connection to the parametric representation is given by

$$\begin{cases} Y_t = X_t^1 + iX_t^2 \\ dB_t = -X_t^2 dB_t^1 + X_t^1 dB_t^2. \end{cases}$$

4.2 Dyson Brownian motion on the real line

Dyson Brownian motion on the real line was introduced by Dyson in [21], a mathematical treatment of the model can be found, for example, in [13, 5]. The latter reference puts the process in the context of random matrix theory.

Denote

$$D = \{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_n\},$$

and let $B = (B_1(t), \dots, B_N(t))_{t \geq 0}$ be the standard N-dimensional Brownian motion.

Theorem 4.1. *For any $\beta > 0$ and $x \in \overline{D}$, there exists a unique strong solution $X = (X_1(t), X_2(t), \dots, X_N(t))_{t \geq 0} \in C([0,\infty), \overline{D})$ to the system of stochastic differential equations*

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{j:j \neq i} \frac{1}{X_i - X_j} dt, \quad i = 1, 2, \dots, N,$$

such that $X(0) = x$.

Comment on the proof: The proof of the existence of a unique strong solution can be found in many articles and textbooks. For example, in [5] a cut-off of the drift is introduced which makes it a Lipschitz continuous function, which allows the use of standard theorems of SDE theory, and then limiting arguments yield the result. In contract, the same problem of existence is treated in [13] using the theory of stochastic variational inequalities, also referred to as multivalued SDEs. Yet another approach, used in [25], is to map the process to a new space, so that the singularity of the drift disappears, establish the existence by standard means, and map the process back to the original state space. Last but not least, Dyson Brownian motion can be obtained as a collection of N Brownian motions conditioned never to collide, see [24].

4.3 Dyson Brownian motion on a circle

The model is originally proposed by Dyson in [21] under the name *Unitary Brownian motion.* It is also referred to as *Circular Dyson Brownian motion.*

We start with some physics heuristics. Consider a collection of $N \geq 2$ point charges on the unit circle in the complex plane, with positions $z = (z_1, z_2, \dots, z_N)$. The *electric potential* at the point z_i is given by the Coulomb interaction

$$V(z_i) = -\frac{\beta}{2} \sum_{j:j \neq i} \log |z_i - z_j|,$$

where $\beta > 0$ represents the intensity of the interaction and is called the *inverse temperature*; the division by 2 is a convention that we adapt. The potential creates an *electric field* at z_i given by $-\nabla V$. If the particles are restricted to the unit circle, then only the projection of the electric field on the tangent to the circle at the point z_i affects its dynamics. If we denote $z_i = e^{i\theta_i}$, then the component, tangential to the circle, of the electric field produced at z_i by all other unit charges equals

$$E(\theta_i) = -\frac{\partial V(z_i)}{\partial \theta_i} = \frac{\beta}{2} \sum_{j:j\neq i} \frac{1}{2} \cot\left(\frac{\theta_i - \theta_j}{2}\right).$$

The dynamics of the point charges $z_i(t) = e^{i\theta_i(t)}$ is descried by the system of SDEs:

$$d\theta_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{j:j\neq i} \frac{1}{2} \cot\left(\frac{\theta_i(t) - \theta_j(t)}{2}\right) dt, \quad i = 1, 2, \dots, N.$$

Next theorem shows the existence of a strong solution the SDE above. Denote

$$D = \{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_n < x_1 + 2\pi\},$$

and let $B = (B_1(t), \dots, B_N(t))_{t\geq 0}$ be the standard N-dimensional Brownian motion.

Theorem 4.2. *For any $\beta > 0$ and $x \in \overline{D}$, there exists a unique strong solution $X = (X_1(t), X_2(t), \dots, X_N(t))_{t\geq 0} \in C([0,\infty), \overline{D})$ to the system of stochastic differential equations*

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{j:j\neq i} \frac{1}{2} \cot\left(\frac{X_i(t) - X_j(t)}{2}\right) dt, \quad i = 1, 2, \dots, N,$$

such that $X(0) = x$.

See the proof of the theorem, for example, in [14].

Note that the process X stays in $\overline{D}$ for all time almost surely, that is, any two "angles" satisfy $0 \leq |X_i(t) - X_j(t)| \leq 2\pi$ for all $t \geq 0$ almost surely. The repulsion due to cotangent term in the drift does not allow particles to go through each other, this is the reflective nature of the model.

Definition 4.2. *The collection $Z = (Z_1(t), Z_2(t), \dots, Z_N(t))_{t\geq 0}$, given by $Z_i(t) = e^{iX_i(t)}$, where X is the process of Theorem 4.2, is called Dyson Brownian motion on the unit circle.*

Remark 1. *The point $\frac{1}{N}(X_1 + \cdots + X_N)$ can move along $\mathbb{R}$ unrestricted. In fact, it performs a standard Brownian motion started at $\frac{1}{N}(x_1 + \dots x_N)$ since*

$$d\left(\frac{X_1 + \cdots + X_N}{N}\right) = d\left(\frac{B_1 + \cdots + B_N}{N}\right) + \underbrace{\frac{\beta}{2} \sum_{i\neq j} \frac{1}{2} \cot\left(\frac{X_i - X_j}{2}\right)}_{0} dt = d\tilde{B}.$$

The process $\tilde{X}_i = X_i - (X_1 + \cdots + X_N)/N$, centered with respect to the center of the mass, lives on $[-\pi, \pi]$.

The following list presents some basic properties of the "angle" process X and the process Z on the circle.

- For $\beta \geq 1$, the particles $(Z_1, Z_2, \dots, Z_N)$ do not collide almost surely, that is

$$\mathbb{P}\left[Z_i(t) \neq Z_j(t) \text{ for all } t > 0 \text{ and } i \neq j\right] = 1.$$

For $0 < \beta < 1$, the particles collide almost surely, that is, the first collision time $\tau = \inf\{t > 0 : Z_i(t) = Z_j(t) \text{ for any } i \neq j\}$ is finite almost surely.

- $X = X^x$ is a strong Markov-Feller process with the transition probability function

$$P(x, t, B) = \mathbb{P}[X^x(t) \in B], \qquad B \in \mathcal{B}(\overline{D}).$$

- The generator L of the process is given by

$$L\varphi = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 \varphi}{\partial x_i^2} + \frac{\beta}{2}\sum_{i\neq j} \frac{1}{2}\cot\left(\frac{x_i - x_j}{2}\right)\frac{\partial \varphi}{\partial x_i}, \qquad \varphi \in C_0^\infty(D).$$

The formal adjoint operator L^* is given by

$$L^*\varphi = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 \varphi}{\partial x_i^2} - \frac{\beta}{2}\sum_{i\neq j} \frac{\partial}{\partial x_i}\left(\frac{1}{2}\cot\left(\frac{x_i - x_j}{2}\right)\varphi\right).$$

- $P(x, t, dy)$ has a density $p(x, t, y)$ which satisfies Fokker-Planck-Kolmogorov equation $\partial_t p = L^* p$ pointwise, i.e., for $t > 0$ and $x, y \in D$

$$\frac{\partial p(x,t,y)}{\partial t} = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 p(x,t,y)}{\partial y_i^2} - \frac{\beta}{2}\sum_{i\neq j} \frac{\partial}{\partial y_i}\left(\frac{1}{2}\cot\left(\frac{y_i - y_j}{2}\right)p(x,t,y)\right).$$

- The stationery distribution of X is given by the density

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}}\prod_{i\neq j}|e^{ix_i} - e^{ix_j}|^{\frac{\beta}{2}}, \quad x \in D,$$

which solves the stationary Fokker-Planck-Kolmogorov equation $L^*\rho_{\beta,N} = 0$.

- Let $\mu_t^{(N)}(dx)$ be the empirical measure of Z, i.e.,

$$\mu_t^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{N} \delta_{Z_i(t)}(dx),$$

which can be considered as a probability measure on $[-\pi, \pi)$ after a homeomorphic identification of the unit circle with this interval. For any measurable function f, denote by $\mu_t^{(N)}(f)$ the integral of f with respect to $\mu_t^{(N)}(dx)$, i.e.,

$$\mu_t^{(N)}(f) = \frac{1}{N}\sum_{i=1}^{N} f(Z_i(t)).$$

Let $\beta = 4\lambda/N$ and the initial distribution of $Z(0) = (Z_1(0), \dots, Z_N(0))$ be such that $\mu_0^{(N)}$ converges weakly, as $N \to +\infty$, to μ_0 . Then, the empirical measure

$\mu_t^{(N)}$ converges weakly, as $N \to +\infty$, to a deterministic limit μ_t, where $\mu = (\mu_t)_{t\geq 0}$ is the unique continuous probability-measure-valued function satisfying the McKean-Vlasov equation

$$\mu_t(f) = \mu_0(f) + \frac{1}{2}\int_0^t \mu_s(f'')ds$$
$$+ \lambda \int_0^t \left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (f'(x) - f'(y))\frac{1}{2}\cot\left(\frac{x-y}{2}\right)\mu_s(dx)\mu_s(dy) \right) ds$$

for all 2π-periodic functions $f \in C^2(\mathbb{R})$.

4.4 Dyson Brownian motion on a Jordan curve

Preliminary discussion

Dyson Brownian motion on a Jordan curve Γ is thought of as a collection of $N \geq 2$ particles $Z = (Z_1(t), \dots, Z_N(t))_{t\geq 0}$, where each Z_i lives on the curve Γ, and the collection satisfies two a priori imposed conditions:

(1) The particles are repulsive. In particular, they interact via planar Coulomb potential

$$-\beta \sum_{i<j} \log|z_i - z_j|, \quad \beta > 0. \tag{4.8}$$

(2) The particles are reflective. If any two particles collide, they reflect of each other. This ensures that the ordering of particles on the curve is preserved for all time.

In defining the process we adapt the parametric approach as in Definition 4.1. First, we need to construct an $\mathbb{R}^N$-valued diffusion process $X = (X_1(t), \dots, X_N(t))_{t\geq 0}$ such that, for all $t \geq 0$, almost surely,

$$X_1(t) \leq X_2(t) \leq \dots \leq X_N(t) \leq X_1(t) + l.$$

Then, we map it to the curve via the arc-length parametrization. The inequality insures that the particles are ordered in the counter clock-wise direction on the curve, and can be equivalently stated as the requirement for the process to never leave the closure of the domain

$$D = \{x \in \mathbb{R}^N : x_1 < x_2 < \dots < x_N < x_1 + l\}.$$

One way to construct the diffusion process X is to solve the stochastic differential equation with the drift given by $\frac{1}{2}\nabla \log \rho_{\beta,N}(x)$, where

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}} \prod_{i\neq j} \left|\gamma(x_i) - \gamma(x_j)\right|^{\frac{\beta}{2}}, \quad x \in D, \quad \beta > 0, \tag{4.9}$$

where the normalizing constant $Z_{\beta,N}$ is called the partition function and equals

$$Z_{\beta,N} = \int_D \prod_{i \neq j} \left|\gamma(x_i) - \gamma(x_j)\right|^{\frac{\beta}{2}} dx_1 \dots dx_N.$$

This drift is singular as it divergences whenever any of the particles collide. Moreover, a priori one has to consider the SDE with reflecting boundary conditions to insure that the order of particles is preserved. Hence, the process lays at the intersection of such classes of stochastic processes as distorted Brownian motion, reflected diffusions and singular diffusions.

Definition

Let Γ be a rectifiable Jordan curve of length $l = l(\Gamma)$. It admits a Lipschitz continuous parametrization $(\gamma(s))_{s\in[0,l)}$ by the arc-length. It is convenient to extend the domain of γ to $\mathbb{R}/l(\Gamma)\mathbb{Z}$, making it an l-periodic function on the real line.

By Rademacher's theorem, the arc-length parametrization is differentiable almost everywhere, so that the partial derivatives of the density $\rho_{\beta,N}$ exist almost everywhere, and the gradient of $\frac{1}{2}\log\rho_{\beta,N}$ is a measurable locally bounded function.

Definition 4.3 (Parametrization process)**.** *Let Γ be a rectifiable Jordan curve. The parametrization process for Dyson Brownian motion on Γ, with inverse temperature parameter $\beta > 0$, is a continuous strong Markov process $X = (X_1(t), \dots, X_N(t))_{t\geq 0}$, with diffusion matrix equal to the identity and the drift equal to the weak gradient of $\frac{1}{2}\log\rho_{\beta,N}$, where $\rho_{\beta,N}$ depends on Γ and β via* (4.9)*, and such that for all $t \geq 0$, almost surely,*

$$X_1(t) \leq X_2(t) \leq \dots \leq X_N(t) \leq X_1(t) + l.$$

The parametrization process can be obtained as a solution to the system of SDEs

$$dX_i(t) = dB_i(t) + \sum_{j:j\neq i} \mathrm{Re}\left(\frac{\gamma'(X_i)}{\gamma(X_i) - \gamma(X_j)}\right) dt, \quad i = 1, \dots, N, \tag{4.10}$$

where $B = (B_1(t), \dots, B_N(t))_{t\geq 0}$ is a standard N-dimensional Brownian motion. The existence of a unique strong solution is discussed below.

Given the parametrization process, we obtain the process on the curve by transferring it via arc-length parametrization.

Definition 4.4 (Dyson Brownian motion on a Jordan curve)**.** *Let Γ be a rectifiable Jordan curve. Dyson Brownian motion on Γ is the stochastic process $Z = (\gamma(X_1), \dots, \gamma(X_N))$, where $X = (X_1, \dots, X_N)$ is the parametrization process of Definition 4.3.*

Construction problem

The parametrization process X can be constructed as a solution to an SDE with reflecting boundary conditions. Since we want the particles $X = (X_1, X_2, \dots, X_N)$

to be ordered for all times, i.e., for all $t \geq 0$, almost surely,

$$X_1(t) \leq X_2(t) \leq \dots X_N(t) \leq X_1(t) + l,$$

we require the $\mathbb{R}^N$-valued process X to be restricted to the closure of the domain

$$D = \{x \in \mathbb{R}^N : x_1 < x_2 < \dots < x_N < x_1 + l(\Gamma)\}.$$

This leads to the SDE with reflecting boundary conditions

$$dX(t) = dB(t) + \frac{1}{2}\nabla \log \rho_{\beta,N}(X)dt + n(X)dL(t), \tag{4.11}$$

where $n(x)$ is an inward-pointing normal unit vector at $x \in \partial D$ (at the part of the boundary that does not admit a normal vector $n(x)$ is assumed to be any unit vector in the normal cone $\mathcal{N}_x$ at $x \in \partial D$), and L is the boundary process, i.e., a continuous non-decreasing process of finite variation such that $L(0) = 0$ and

$$L(t) = \int_0^t \mathbb{1}_{\{X(s)\in\partial D\}} dL(s).$$

Problem 4.3. *Does there exist a unique strong solution* (X, L) *to the SDE* (4.11) *with reflecting boundary conditions? More precisely, for a given standard* N*-dimensional Brownian motion* $B = (B_1, \dots, B_N)$ *on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and a given starting point* $x \in \overline{D}$*, one needs to find a pair* (X, L) *of* $(\mathcal{F}_t)$*-adapted processes, where* X *is a* $\overline{D}$*-valued continuous process and* L *is an* $\mathbb{R}^N$*-valued continuous non-deceasing process of finite variation such that* $L(0) = 0$ *and*

$$L(t) = \int_0^t \mathbb{1}_{\{X(s)\in\partial D\}} dL(s),$$

and they satisfy the equation (4.11)*. Moreover, pathwise uniqueness holds.*

Problem 4.4. *Let* (X, L) *be a solution to Problem 4.3. Does* $L \equiv 0$ *for any* $\beta > 0$*?*

In the cases of the real line and the unit circle Problems 4.3 and 4.4 are resolved in [13, 14]. There, the existence of a unique strong solution is established with the help of stochastic variational inequalities, also known as multivalued SDEs. However, it is important that the drift is the gradient of a convex function, which is not the case for more general curves Γ.

The result of Krylov and Röckner, see [37], provides the existence of a unique strong local solution for any $\beta > 0$ and $x \in D$, that is, a solution defined up to collision time. Additionally, they provide a condition on the drift which insures that the process never hits the boundary ∂D, thus yielding a global solution. The Krylov–Röckner condition imposes certain differentiability restrictions on the parametrization γ of the curve.

The following table summarizes what we know about the existence of a unique strong global solution depending on the regularity of the curve, the values of β, and the staring point.

Regularity of the curve	β	Initial data	Reference
Γ is the real line or a circle	$\beta > 0$	$x \in \overline{D}$	[13, 14]
Γ satisfies Krylov–Röckner condition	$\beta > 1$	$x \in D$	[37]
Γ is rectifiable	$\beta \geq 1$	$x \in D$	[26] ([37, 28, 4])
Γ is rectifiable	$\beta > 0$	$x \in \overline{D}$	Open problem

Table 4.1: This table shows instances when a unique strong global solution exists given the regularity of the curve, the parameter β, and the initial data.

Now we explain with more details the condition in [37], that ensures the existence of global solution, and how it applies to our problem. Is is assumed that the drift is of gradient form, i.e., $b = -\nabla\psi$.

Definition 4.5. *ψ is said to satisfy Krylov–Röckner condition if there exist $\varepsilon \in [0, 2)$ and a continuous function $h : D \to \mathbb{R}$ satisfying*

$$\forall\sigma > 0\ \exists r > 1 : \int_D h^r(x)e^{-\sigma|x|^2}dx < +\infty.$$

such that the inequality $\Delta\psi \leq he^{\varepsilon\psi}$ holds weakly, that is

$$\int_D \psi\Delta f dx \leq \int_D fhe^{\varepsilon\psi}dx \quad \forall f \in C_0^2(D, \mathbb{R}_+)$$

Theorem 4.5. *[37, Theorem 2.7] If ψ satisfies Krylov–Röckner condition, then*

$$\mathbb{P}_x\left[\tau_{\partial D}(X) = +\infty\right] = 1 \quad \forall x \in D.$$

In our case $\psi = -\log\rho_{\beta,N}$. Let us see what curve geometries work well with the Krylov–Röckner condition. First, assume there exist almost everywhere first and second derivatives γ', γ'' of the arc-length parametrization. Then,

$$\begin{aligned}\Delta\psi &= -\frac{\beta}{2}\sum_{i\neq j}\mathrm{Re}\left\{\frac{\gamma''(x_i)}{\gamma(x_i)-\gamma(x_j)}\right\} + \frac{\beta}{2}\sum_{i\neq j}\mathrm{Re}\left\{\left(\frac{\gamma'(x_i)}{\gamma(x_i)-\gamma(x_j)}\right)^2\right\}\\ &\leq \frac{\beta}{2}\sum_{i\neq j}\frac{|\gamma''(x_i)|}{|\gamma(x_i)-\gamma(x_j)|} + \frac{\beta}{2}\sum_{i\neq j}\frac{1}{|\gamma(x_i)-\gamma(x_j)|^2}.\end{aligned}$$

Let $\alpha > 0$, if $\int_{0+}|\gamma(x+h)-\gamma(x)|^{-\alpha}dh < +\infty$ for all $x \in [0, l]$, then one can take

$$h(x) = C\prod_{j<i}|\gamma(x_i)-\gamma(x_j)|^{-\alpha},$$

so that

$$he^{\varepsilon\psi} = C\prod_{j<i}|\gamma(x_i)-\gamma(x_j)|^{-(\varepsilon\frac{\beta}{2}+\alpha)}.$$

If $||\gamma''||_\infty < +\infty$, there exists $C > 0$ such that $\Delta\psi \le he^{\varepsilon\psi}$ whenever $\varepsilon\frac{\beta}{2} + \alpha \ge 2$, that is, when $\beta > 2 - \alpha$. The inequality

$$\int_{0+} |\gamma(x+h) - \gamma(h)|^{-\alpha} dh \ge \int_{0+} h^{-\alpha} dh$$

shows that the condition $\int_{0+} |\gamma(x+h) - \gamma(x)|^{-\alpha} dh < +\infty$ for all $x \in [0, l]$ forces $\alpha < 1$. Hence, we have the following

Example 4.1. *Let* Γ *be a rectifiable Jordan curve with an arc-length parametrization* $(\gamma(s))_{s\in[0,l]}$. *Assume* γ' *and* γ'' *exist almost everywhere,* $||\gamma''||_\infty < +\infty$ *and* $\beta > 1$. *Then,* $\psi = -\log \rho_{\beta,N}$, *where* $\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}(\gamma)} \prod_{i\neq j} |\gamma(x_i) - \gamma(x_j)|^{\beta/2}$, *satisfies Krylov–Röckner condition. Consequently, for any* $\beta > 1$ *and any starting point* $x \in D$, *there exists a unique strong solution to* $dX_t = dB_t + \frac{1}{2}\log \rho_{\beta,N}(X_t)dt$ *such that* $X(0) = x$ *and for all* $t \ge 0$, *almost surely,* $X(t) \in D$.

4.5 Reflected diffusions

The following approaches are used to construct diffusion processes constrained to a given domain:

- Submartingale problem (Stroock, Varadhan 1971)
- Dirichlet forms (Fukushima 1967)
- Controlled martingale problem (Kurtz, Stockbridge 2001)
- Reflected stochastic differential equations via Skorokhod problem (Tanaka, Dupius, Ishi, Williams, Ramanan)
- Stochastic variational inequalities / Multivalued SDEs (Cepa, Lepingle).

Different approaches offer its own advantages when studying properties of reflected diffusions. For example, existence and uniqueness results, large deviations can be established when working with SDEs with reflections; while certain boundary properties, characterization of stationary distributions are more established within the submartingale problem.

Is having a unique weak solution to an SDE with reflections equivalent to a well-posed submartingale problem? In [35] an affirmative answer was given when the coefficients are continuous and locally bounded in $\mathbb{R}^N$, and assuming uniform ellipticity of the diffusion coefficient. In the following section we define these notions.

Submartingale problem

As in the whole space setting the main characteristics of a reflected diffusion are the drift coefficient $b : D \to \mathbb{R}^{\mathbb{N}}$ and the dispersion coefficient $\sigma : D \to \mathbb{R}^{N\times N}$ (which gives rise to the diffusion coefficient $a = \sigma\sigma^T$). With these coefficients one associates a second order differential operator

$$(Lf)(x) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b_i(x)\frac{\partial f}{\partial x_i}(x), \quad f \in C_0^2(D).$$

Definition 4.6 (Submartingale problem). *A family of probability measures* $\{\mathbb{P}_x, x \in \overline{D}\}$ *on* $C([0,\infty))$ *is a solution to the submartingale problem associated with the domain* D*, the family of reflection directions* $\{N_x, x \in \partial D\}$*, the exceptional set* $E \subset \partial D$*, the drift* b *and the dispersion* σ *if for each Borel measurable* $A \subset C([0,\infty))$ *the mapping* $x \to \mathbb{P}_x[A]$ *is* $\mathcal{B}(\overline{D})$*-measurable and for each point* $x \in \overline{D}$*, the measure* $\mathbb{P}_x$ *satisfies*

(1.) $X_0 = x$ *and* $X_t \in \overline{D}$*, for all* $t \geq 0$*,* $\mathbb{P}_x$*-almost surely;*

(2.) for every $f \in C_0^2(\mathbb{R}^{\mathbb{N}})$ *such that* f *is constant in a neighborhood of* E*, and* $(n, \nabla f) \geq 0$ *for* $n \in N_x$*,* $x \in \partial D$ *the process*

$$f(X_t) - \int_0^t (Lf)(X_s)ds, \quad t \geq 0$$

is a $\mathbb{P}_x$*-submartingale on* $C([0,\infty))$*;*

(3.) for every $x \in \overline{D}$

$$\mathbb{E}_x\left[\int_0^\infty \mathbb{1}_E(X_s)ds\right] = 0.$$

That is, $\mathbb{P}_x$*-almost surely, Leb* $(\{s \geq 0 : X_s \in E\}) = 0$.

Definition 4.7 (Well-posed problem). *The submartingale problem associated with* $(D, N_\cdot, E, b, \sigma)$ *is said to be well-posed if there exists exactly one solution to the submartingale problem.*

Given a law $\mathbb{P}_x$, $x \in \overline{D}$, one can construct a stochastic process $X \in C([0,\infty))$ with $X_0 = x$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}_x[A] = \mathbb{P}[X \in A | X_0 = x]$ for $A \in \mathcal{F}$.

Definition 4.8 (Reflected diffusion). *A stochastic process* X *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a reflected diffusion associated with* $(D, N_\cdot, E, b, \sigma)$ *if its distribution laws* $\{\mathbb{P}_x, x \in \overline{D}\}$*, where* $\mathbb{P}_x[A] = \mathbb{P}[X \in A | X_0 = x]$*, is the unique solution to the corresponding submartingale problem.*

Skorokhod problem

The Skorokhod problem dates back to A. Skorokhod's work, see [52], where he constructed a 1-dimensional diffusion restricted to the positive half-line. The setup of the problem was later extended by many authors: [16, 55, 6, 19, 39, 46]. Skorokhod problem is deterministic, and when applied to stochastic differential equations with boundary conditions it enables the construction of solutions on the pathwise basis. Naturally, extra work is required to show that the resulted process is adapted to the filtration.

Definition 4.9 (Skorokhod problem). *Let* D *be a connected domain in* $\mathbb{R}^N$ *and* $\{N_x, x \in \partial D\}$ *be a collection of reflection directions. Assume* $\psi \in C([0,\infty), \mathbb{R}^N)$*,* $\psi(0) \in \overline{D}$*. Then, a pair* (ϕ, η) *of continuous functions solves the Skorokhod problem for* $(D, \{N_\cdot\}, \psi)$ *if*

(1.) $\phi(0) = \psi(0)$ *and* $\phi(t) = \psi(t) + \eta(t), \quad t \geq 0;$

(2.) $\phi(t) \in \overline{D}, \quad t \geq 0$;

(3.) η *is of bounded variation on every interval* $[0,t]$, *there exists a measurable function* $n : \partial D \to S^1$ *such that* $n(x) \in N_x \ \forall x \in \partial D$, *and*

$$\eta(t) = \int_0^t \mathbb{1}_{\{\phi(s)\in\partial D\}} n(\phi(s)) d|\eta|(s).$$

There is a generalization of this definition, called *extended Skorokhod problem*, introduced in [46]. There, the third property in Definition 4.9 is replaced by

$$\eta(t) - \eta(s) \in \overline{\text{co}} \left[\cup_{r\in[s,t]} N_{\phi(r)}\right], \quad s \in [0,t],$$

where $\overline{\text{co}}\,[A]$ is the closure of the convex hull generated by the set A. This definition does require the boundary process η to be of bounded variation. This weaker constrain allows to construct reflected diffusions which are not necessarily semimartingales.

4.6 Fokker-Planck-Kolmogorov equation

Let $\Omega \subset \mathbb{R}^N$ be a domain, $b = (b_1, \dots, b_N)$ be a Borel vector field on Ω, $A = (a_{ij})$ be matrix-valued mapping on Ω such that a_{ij} are Borel measurable, $a_{ij} = a_{ji}$, and $A \geq 0$. For $\varphi \in C_0^\infty((0,T) \times \Omega)$ define a differential operator

$$L_{A,b}\varphi = \sum_{i,j=1}^N a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial \varphi}{\partial x_i},$$

and denote by $L^*_{A,b}$ its formal adjoint.

Definition 4.10. *(a) A locally finite Borel measure* μ *on the domain* $(0,T) \times \Omega \subset \mathbb{R}^{N+1}$ *satisfies Fokker-Planck-Kolmogorov equation* $\partial_t \mu = L^*_{A,b}\mu$ *if* $a_{ij}, b_i \in L^1_{loc}(|\mu|)$ *and*

$$\int_{(0,T)\times\Omega} (\partial_t \varphi + L_{A,b}\varphi) d\mu = 0 \quad \forall \varphi \in C_0^\infty((0,T) \times \Omega).$$

(b) A locally finite Borel measure μ *on a domain* $\Omega \subset \mathbb{R}^N$ *satisfies stationary Fokker-Planck-Kolmogorov equation* $L^*_{A,b}\mu = 0$ *if* $a_{ij}, b_i \in L^1_{loc}(|\mu|)$ *and*

$$\int_\Omega (L_{A,b}\varphi)(x)\mu(dx) = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

In the rest, we choose $A = \frac{1}{2}I$ and denote by $L = L_{\frac{1}{2}I,b}$, where $b = \frac{1}{2}\nabla \log \rho$. The differential operator L is given by

$$Lf = \frac{1}{2}\Delta f + b \cdot \nabla f, \qquad f \in C_0^\infty(\Omega),$$

and L^* denotes its formal adjoint

$$L^* f = \frac{1}{2}\Delta f - \nabla \cdot (bf).$$

In the special case $d\mu = \mu_t(dy)dt$, one can consider a Cauchy problem with initial condition $\mu_t|_{t=0} = \nu$, where ν is a locally bounded measure on Ω:

$$\int_0^T \int_\Omega (\partial_t \varphi + L\varphi)\mu_t(dy)dt = 0 \quad \forall \varphi \in C_0^\infty((0,T) \times \Omega) \tag{4.12}$$

and

$$\int_\Omega \varphi(x)\nu(dx) = \lim_{t \to 0+, t \in I_\varphi} \int_\Omega \varphi(x)\mu_t(dx) \quad \forall \varphi \in C_0^\infty(\Omega), \tag{4.13}$$

where the set $I_\varphi \subset (0,T)$ has full measure $|I_\varphi| = T$ and, in general, depends on the test function. However, when $t \to \int_\Omega \varphi d\mu_t$ is continuous for all test functions, which is the case when $\mu_t(dy)$ is a transition probability of a Feller process, one can take $I_\varphi = (0,T)$.

An equivalent definition of the Cauchy problem (4.12), (4.13) is

$$\int_\Omega \varphi d\mu_t = \int_\Omega \varphi d\nu + \int_0^t \int_\Omega L\varphi d\mu_s ds \quad \forall \varphi \in C_0^\infty(\Omega),$$

which for the initial distribution $\nu = \delta_x$, i.e., the point mass at $x \in \Omega$, becomes

$$\int_\Omega \varphi d\mu_t = \varphi(x) + \int_0^t \int_\Omega L\varphi d\mu_s ds \quad \forall \varphi \in C_0^\infty(\Omega).$$

Summary of Results and Future Work

Paper A

A large deviation principle for the Schramm-Loewner evolution in the uniform topology

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Schramm-Loewner evolution (abbrev. SLE_κ) is a one-parameter family, indexed by $\kappa > 0$, of random fractal curves in a simply connected domain $D \subset \mathbb{C}$ connecting two boundary points $a, b \in \partial D$ that satisfies Schramm's principle: conformal invariance and domain Markov property. In this work the domain D is the upper half-plane $\mathbb{H}$, the starting point $a = 0$ and the terminal point $b = \infty$.

SLE_κ in $\mathbb{H}$ is referred to as chordal SLE_κ and is a particular instance of Loewner evolution described by a family of conformal maps $g_t : \mathbb{H}\backslash\gamma([0,T]) \to \mathbb{H}$ which satisfy the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)},\ g_0(z) = z.$$

It is a non-trivial result of Schramm, see [47], that the choice $\lambda(t) = \sqrt{\kappa}B(t)$ of the driving function, where B is a standard one-dimensional Brownian motion, produces the SLE_κ curve. One can think of Loewner evolution as a map $\mathcal{L} : \mathcal{D} \to S$ from the space $\mathcal{D}$ of driving functions to a space S of curves in $\mathbb{H}$. In this notation the SLE_κ curve is given by

$$\gamma^\kappa = \mathcal{L}(\sqrt{\kappa}B).$$

There are several perspectives on how one can see an SLE_κ curve. The most popular choices are to view γ^κ as:

1. a subset $\gamma^\kappa \subset \mathbb{H}$,
2. a continuous mapping $\gamma^\kappa : [0,T] \to \mathbb{H}$
    - in the half-plane capacity parametrization,
    - in the natural parametrization,
3. a p-variation path.

In this paper we adopt the view of looking at a curve as a continuous mapping in the half-plane capacity parametrization. The distance between two curves γ and η, run up to time $T > 0$, is measured by the supremum norm

$$d(\gamma, \eta) = \sup_{t\in[0,T]} |\gamma(t) - \eta(t)|.$$

Curves converging under this distance are said to converge uniformly on $[0,T]$. The topology τ we impose on the space S, making it a topological space (S, τ), is generated by the uniform convergence on $[0, T]$ for every $T > 0$. The resulting topology τ is called the uniform topology.

As $\kappa \downarrow 0$ the law of SLE_κ converges to a point mass

$$\mathbb{P}[\gamma^\kappa \in \bullet] \to \delta_\eta(\bullet),$$

where η is a geodesic curve in $\mathbb{H}$ connecting 0 and ∞, i.e., the imaginary axis. If we consider any measurable family $V \subset S$ of curves such that $\eta \notin V$, then as $\kappa \downarrow 0$

$$\mathbb{P}[\gamma^\kappa \in V] \to 0.$$

Roughly speaking, a large deviation principle establishes that this probability converges exponentially fast and identifies the rate of the convergence.

In [43] Peltola and Wang proved a large deviation principle for SLE_κ in the Hausdorff topology. This topology is generated by viewing curves as subsets of the plane and measuring the distance between them with the Hausdorff distance. On a space of curves run up to a finite time $T > 0$ the uniform topology is strictly stronger than the Hausdorff topology. However, if curves are run all the way to ∞, our uniform topology τ and the Hausdorff topology are not comparable, as the former lacks the information at ∞ so to speak. The setting was improved in a recent paper [3], where the topology τ was strengthened to a topology generated by the uniform convergence on $[0, +\infty)$, as opposed to the uniform convergence on $[0, T]$ for every $T > 0$.

The Loewner energy of a curve γ generated by the driving function λ can be defined as

$$I_L(\gamma) = \frac{1}{2}\int_0^{+\infty} |\lambda'(t)|^2 dt.$$

Theorem 1. *The chordal SLE_κ curve γ^κ in the space (S, τ) of simple curves in the upper half-plane equipped with the uniform topology satisfies the large deviation principle with the good rate function $I_L : S \to [0, \infty]$:*

(1.) $\overline{\lim_{\kappa\downarrow 0}}\, \kappa \log \mathbb{P}[\gamma^\kappa \in F] \leq -I(F)$ *for any closed subset F of S,*

(2.) $\underline{\lim_{\kappa\downarrow 0}}\, \kappa \log \mathbb{P}[\gamma^\kappa \in G] \geq -I(G)$ *for any open subset G of S,*

where $I(V) = \inf_{\gamma \in V} I_L(\gamma)$ *for any subset V of S.*

In short, Paper A presents a proof of the large deviation principle, as $\kappa \to 0+$, for the law of chordal SLE_κ in $\mathbb{H}$ in the topology generated by the locally uniform convergence. The Loewner energy functional $I_L : S \to [0, +\infty]$ is proved to be a good rate function that governs these large deviations.

Future work

One of the possible directions for future research is to consider a different topology. For example, it would be interesting to prove an LDP using *natural parametrization*, i.e., the d-dimensional Minkowski content for SLE_κ curves and the arc-length for finite Loewner energy curves. Another possible choice is to consider a topology generated by the p-variation norm

$$||\gamma||_{p-\text{var},\ [0,1]} = \left(\sup_{\mathcal{P}} \sum_{i=1}^{|\mathcal{P}|} |\gamma(t_i) - \gamma(t_{i-1})|^p \right)^{1/p},$$

where $\mathcal{P}$ is a partition of $[0,1]$. This topology offers a parametrization-free study of SLE curves since it includes all parametrization-dependent topologies. In [22] it was proved that SLE_κ satisfies p-variation regularity $||\gamma^\kappa||_{p-\text{var},\ [0,1]} < +\infty$ for all $p > \min(1 + \kappa/8, 2)$, $\kappa \neq 8$.

Paper B

Loewner–Kufarev entropy and large deviations of the Hastings–Levitov model

N. Berestycki, V. Guskov and F. Viklund
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In this paper we consider the Hastings-Levitov HL(0) model, introduced in Section 3, and study the small-particles scaling limit $n \to +\infty$, $c \to 0+$, and $nc \to T > 0$.

We prove that the large deviation principle holds at the level of driving measures with the good rate function given by the relative entropy of the driving measure ρ for the Loewner-Kufarev equation:

$$\mathcal{H}(\rho) = \frac{1}{2\pi} \iint \bar{\rho}_t(\theta) \log \bar{\rho}_t(\theta) d\theta dt,$$

whenever $\rho = \bar{\rho}_t d\theta dt/2\pi$ with $\int_{S^1} \bar{\rho}_t d\theta/2\pi = 1$. Let μ_ε be the HL(0) driving measure, seen as an element of

$$\mathcal{N}_T = \{\rho \in \mathcal{M}_+(S^1 \times [0,T]) : \rho(S^1 \times I) = |I| \text{ for any interval } I \subset \mathbb{R}_+\},$$

that is

$$\mu_\varepsilon(d\theta, dt) = \sum_{k=1}^{n} \delta_{\theta_k}(d\theta) \mathbb{1}_{[c(k-1),ck)}(t) dt,$$

where $(\theta_k)_{k\geq 1}$ are i.i.d. random variables with uniform distribution over $[0, 2\pi]$.

Theorem 1. *The family $(\mu_\varepsilon)_{\varepsilon>0}$ of HL(0) driving measures satisfies an LDP in $\mathcal{N}_T$, equipped with the weak topology, with rate $4/\varepsilon^2$, and the good rate function given by the relative entropy. In other words, for any closed set C and any open set O of $\mathcal{N}_T$*

$$\text{(1.)} \quad \limsup_{\varepsilon\to 0+} \frac{\varepsilon^2}{4} \log \mathbb{P}(\mu_\varepsilon \in C) \leq - \inf_{\rho \in C} \mathcal{H}(\rho),$$

$$\text{(2.)} \quad \liminf_{\varepsilon\to 0+} \frac{\varepsilon^2}{4} \log \mathbb{P}(\mu_\varepsilon \in O) \geq - \inf_{\rho \in O} \mathcal{H}(\rho).$$

Applying the contraction principle, see Section 1, we obtain the LDP on the level of the random conformal maps Φ_n, as elements of

$$\Sigma'_T = \{f \in H(\Delta) : \text{omits a neighborhood of 0 and } f(z) = e^T z + O(1/z)\}.$$

Theorem 2. *The family $(\Phi_n)_{n\geq 1}$ of HL(0) conformal maps satisfies an LDP in Σ'_T, equipped with the topology of locally uniform convergence, with rate n and the good rate function*

$$\mathcal{H}^*_T(\Phi) = \inf_{\rho} \mathcal{H}(\rho),$$

where the infimum is taken over all driving measures in $\mathcal{N}_T$ generating the Loewner chain $(f_t)_{t\in[0,T]}$ such that $f_T = \Phi$.

Next we investigate the class of shapes that can be generated by driving measures with $\mathcal{H}(\rho) < +\infty$. Although we do not know the complete description of this class, we show that it contains the following shapes.

Theorem 3. *The set of boundaries of compact hulls that can be generated by Loewner-Kufarev evolution driven by a finite entropy measure contains:*

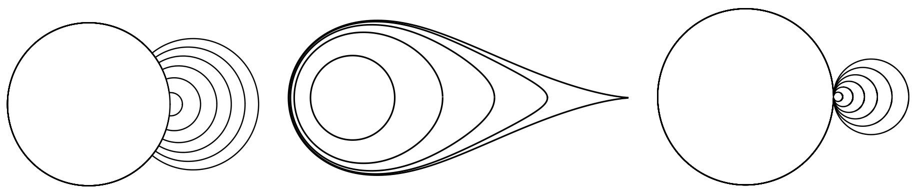
(i) every Weil-Petersson quasicircle separating 0 *from* ∞*,*

(ii) every finite time Becker quasicircle,

(iii) a Jordan curve with a corner,

(iv) a Jordan curve with a cusp,

(v) a non-simple curve.



(a) a Jordan curve with a corner (b) a Jordan curve with a cusp (c) a non-simple curve

Figure 0.3: The hulls illustrating items $(iii) - (v)$ of Theorem 3.

Finally, we address the problem of finding the measure of minimal entropy $\inf_\rho \mathcal{H}(\rho)$ generating a given shape Φ and propose a simplified version for a related transport equation.

Future work

The question of a complete geometric description of shapes that can be generated by driving measures with finite relative entropy remains open.

Another direction one could investigate is to try finding a measure ρ^* that solves the minimization problem

$$\mathcal{H}^*_T(\Phi) = \inf_\rho \mathcal{H}(\rho)$$

for a given shape Φ. In other words this problem can be formulated as follows.

Problem 1. *Let* γ *be a Jordan curve separating* 0 *from* ∞*. Assume that the capacity of* γ *equals* e^T *for* $T > 0$*.*

$$\textit{Minimize} \quad \mathcal{H}(\rho) = \frac{1}{2\pi} \int_{-\infty}^{T} \int_{S^1} \bar{\rho}_t(\theta) \log \bar{\rho}_t(\theta) d\theta \; dt$$

where $\rho = \bar{\rho}_t(\theta) d\theta dt / 2\pi$ *is a driving measure subject to the constraint*

$$\partial f_T(\Delta) = \gamma,$$

where (f_t) *is the Loewner chain of* ρ*.*

Paper C

Dyson Brownian motion on a Jordan curve

V. Guskov, M. Liu and F. Viklund
To be submitted.

This work is motivated by a recent paper of Zabrodin [58], where he proposed a generalization of Dyson Brownian motion. In [58], the collection of $N \geq 2$ complex-valued processes $\boldsymbol{Z} = (Z_1(t), \dots, Z_N(t))_{t\geq 0}$ on a smooth Jordan curve Γ is said to satisfy the system of SDEs in the complex plane

$$dZ_i(t) = \tau(Z_i)dB_i(t) + \tau(Z_i)\partial_{s_i}E(\boldsymbol{Z})dt - \frac{1}{2}\nu(Z_i)k(Z_i)dt,$$

where $\tau(z)$, $\nu(z)$, and $k(z)$ are the unit tangent vector, the unit outward-pointing normal vector, and the curvature of the curve at the point $z \in \Gamma$ correspondingly; $\partial_{s_i}E$ is the tangential derivative of the interaction potential E at the point $z_i \in \Gamma$; $B = (B_1(t), \dots, B_N(t))_{t\geq 0}$ is N-dimensional Brownian motion.

Paper C rigorously defines Dyson Brownian motion on any rectifiable Jordan curve Γ. By Rademacher's theorem, Γ admits a parametrization by arc-length; we write $(\gamma(s))_{s\in[0,l)}$ for it, where $l = l(\Gamma)$ is the length of Γ. It will be convenient to extend the domain of γ to $\mathbb{R}/l(\Gamma)\mathbb{Z}$, making γ an l-periodic function on the real line.

First, we introduce an $\mathbb{R}^N$-valued *parametrization process* X as a solution to the system of SDEs. Second, the parametrization process is mapped to the curve with the help of the arc-length parametrization and its image defines a continuous diffusion process which we call *Dyson Brownian motion on* Γ.

Definition 1 (Parametrization process)**.** *Let Γ be a rectifiable Jordan curve. The parametrization process for Dyson Brownian motion on Γ, with inverse temperature parameter $\beta > 0$, is a continuous strong Markov process $X = (X_1(t), \dots, X_N(t))_{t\geq 0}$, with diffusion matrix equal to the identity and the drift equal to the weak gradient of $\frac{1}{2}\log \rho_{\beta,N}$, where*

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}} \prod_{i\neq j} \left|\gamma(x_i) - \gamma(x_j)\right|^{\beta/2}, \quad x_i \in \mathbb{R}, \quad \beta > 0,$$

and such that for all $t \geq 0$, almost surely,

$$X_1(t) \leq X_2(t) \leq \dots \leq X_N(t) \leq X_1(t) + l.$$

Given this, we define Dyson Brownian motion on Γ by transplanting the parametrization process using the arc-length parametrization.

Definition 2 (Dyson Brownian motion on a Jordan curve)**.** *Let Γ be a rectifiable Jordan curve. Dyson Brownian motion on Γ is the stochastic process $\boldsymbol{Z} = (\gamma(X_1), \dots, \gamma(X_N))$, where $X = (X_1, \dots, X_N)$ is the parametrization process of Definition 1.*

Theorem 1 (Existence for $\beta \geq 1$)**.** *Let Γ be a rectifiable Jordan curve of length $l > 0$. For any collection $\boldsymbol{z} = \{z_1, \dots, z_N\}$ of points on Γ such that $z_i \neq z_j$, and $\beta \geq 1$, there exists a parametrization process $X = (X_1(t), \dots, X_N(t))_{t \geq 0}$, with $z_i = \gamma(X_i(0))$. This process is the unique strong solution of the SDE*

$$dX(t) = dB(t) + \frac{1}{2}\nabla \log \rho_{\beta,N}(X(t))dt,$$

such that for all $t \geq 0$, almost surely,

$$X_1(t) < X_2(t) < \cdots < X_N(t) < X_1(t) + l,$$

and the process

$$\boldsymbol{Z} = (\gamma(X_1(t)), \dots, \gamma(X_N(t)))_{t \geq 0}$$

defines a continuous strong Markov process taking values in $\Gamma^N = \Gamma \times \cdots \times \Gamma$.

Transition probabilities. Let

$$D = \{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N < x_1 + l\}.$$

We next discuss the Fokker-Planck-Kolmogorov equation corresponding to the process constructed in Theorem 1. Let $b_{\beta,N} = \frac{1}{2}\nabla \log \rho_{\beta,N}$, define the following differential operator

$$Lf = \frac{1}{2}\Delta f + b_{\beta,N} \cdot \nabla f, \qquad f \in C_0^\infty(D), \tag{0.14}$$

and denote by L^* its formal adjoint

$$L^* f = \frac{1}{2}\Delta f - \nabla \cdot \left(b_{\beta,N} f\right). \tag{0.15}$$

Theorem 2 (Fokker-Planck-Kolmogorov equation)**.** *Let Γ be a rectifiable Jordan curve of length $l > 0$. The transition probability function $P(x,t,dy)$ of the parametrization process X^x in the Weyl chamber D has a positive locally Hölder continuous density $p(x,t,y)$ for every $t > 0$ which satisfies the Fokker-Planck-Kolmogorov equation with reflecting boundary conditions, that is,*

$$\begin{aligned} &\partial_t p(x,t,\bullet) = L^* p(x,t,\bullet) \text{ weakly in } D, \\ &n \cdot \left(-\frac{1}{2}\rho_{\beta,N} \frac{\nabla p(x,t,\bullet)}{\rho_{\beta,N}} \right) = 0 \text{ weakly on } \partial D, \end{aligned}$$

where n is the unit inward-pointing normal vector.

Proposition 1. *Suppose $\gamma \in C^2$. The parametrization process X^x is a diffusion process in the sense of Kolmogorov, that is,*

(1.) $\lim\limits_{h \to 0+} \frac{1}{h} \int_{B(x,\delta)^c} p(x,h,y)dy = 0,$

(2.) $\lim\limits_{h \to 0+} \frac{1}{h} \int_{B(x,\delta)} (y-x)p(x,h,y)dy = b_{\beta,N}(x),$

(3.) $\lim_{h\to 0+} \frac{1}{h}\int_{B(x,\delta)}((y-x)\cdot z)^2 p(x,h,y)dy = |z|^2.$

The mapping $f : D \to \Gamma^N$ given by $f(x) = (\gamma(x_1), \dots, \gamma(x_N))$ takes the parametrization process back to the curve Γ. The process $\boldsymbol{Z} = (\gamma(X_1), \dots, \gamma(X_N))$, started at $\boldsymbol{z} = (z_1, \dots, z_N)$, is a strong Markov process in the state space $S = f(D) \subset \Gamma^N$ with the transition probability function $P(\boldsymbol{z}, t, |d\boldsymbol{w}|)$.

Theorem 3 (Convergence to stationary distribution.)**.** *Suppose $\gamma \in C^\infty$. The transition probability function $P(\boldsymbol{z}, t, |d\boldsymbol{w}|)$, of the Dyson Brownian motion $\boldsymbol{Z} = (Z_1(t), \dots, Z_N(t))_{t\geq 0}$ started at $\boldsymbol{z} = (z_1, \dots, z_N)$, converges weakly, exponentially fast, as $t \to +\infty$, to the unique stationary distribution on $(S, \mathcal{B}_S)$ given by the Coulomb gas density*

$$\rho_{\beta,N}(\boldsymbol{z}) = \frac{1}{Z_{\beta,N}}\prod_{i\neq j}\left|z_i - z_j\right|^{\beta/2}, \quad z_i \in \Gamma, \quad \beta > 0.$$

Large deviations. The next result concerns large deviations of (time- reparametrized) Dyson Brownian motion $\boldsymbol{Z} = (Z_1(t), \dots, Z_N(t))_{t\geq 0}$, started at $\boldsymbol{z} = \{z_1, \dots, z_N\}$, given by $Z_i(t) = \gamma(X_i(t))$, where the parametrization process $X = (X_1, \dots, X_N)$ is a strong solution to the SDE

$$dX_i(t) = \sqrt{\kappa}dB_i(t) + \frac{1}{2}\nabla \log \rho_{2,N}(X)dt.$$

Theorem 4 (Large deviations as $\kappa \to 0+$)**.** *Suppose $\gamma \in C^3$. The process $\boldsymbol{Z}$ satisfies a large deviation principle, as $\kappa \to 0+$, with the good rate function*

$$J(\boldsymbol{w}) = \frac{1}{2}\int_0^\infty \sum_{i=1}^N \mathrm{Re}\left\{\left(\dot{w}_i - \sum_{j:j\neq i}\frac{w_i - w_j}{|w_i - w_j|^2}\right)\overline{\tau(w_i)}\right\}^2 dt,$$

for absolutely continuous $\boldsymbol{w} = (w_1(t), \dots, w_N(t))_{t\geq 0}$, and set to $+\infty$ otherwise.

The proof of Theorem 4 is a direct application of the contraction principle, given a large deviation principle for the parametrization process. The main result of [2] applies in our setting, and gives a large deviation principle for the parametrization process restricted to any compact time interval $[0, T]$. Building on [2] we give the following extension to the time interval $[0, +\infty)$ in the topology of locally uniform convergence.

Proposition 2. *Suppose $\gamma \in C^3$. The time-reparametrized parametrization process X^x, started at $x \in D$, satisfies a large deviation principle with the good rate function*

$$I(u) = \frac{1}{2}\int_0^\infty \sum_{i=1}^N \left|\dot{u}_i(t) - \sum_{j:j\neq i}\mathrm{Re}\left\{\frac{\gamma'(u_i(t))}{\gamma(u_i(t)) - \gamma(u_j(t))}\right\}\right|^2 dt,$$

for absolutely continuous $u = (u_1(t), \dots, u_N(t))_{t\geq 0}$ and set to $+\infty$ otherwise.

Hydrodynamical limit. The final result of this paper concerns the hydrodynamical limit of the process. Let Z be Dyson Brownian motion on Γ as in Theorem 1. For $t \geq 0$, denote by

$$\mu_t^{(N)}(dz) = \frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i(\frac{t}{N})}(dz)$$

the empirical probability measure on Γ. We allow the initial condition to be random and chosen according to some distribution $\mu_0^{(N)}$ which is assumed to converge weakly to some limiting distribution μ_0 such that $\mu_0^{(N)} \to \mu_0$ as the number of particles $N \to +\infty$. Let $\tau(z)$ be the unit tangent vector to Γ at the point $z \in \Gamma$, oriented counterclockwise. The following theorem shows that the system approaches a mean-field, curve-dependent, McKean–Vlasov equation in the many-particle limit $N \to +\infty$.

Theorem 5 (Hydrodynamical limit)**.** *Suppose $\gamma \in C^2$. Let μ_0 be a probability measure on Γ and suppose that the initial distribution of particles is chosen such that $\mu_0^{(N)} \Rightarrow \mu_0$. Then, the family $\{(\mu_t^{(N)})_{t\geq 0}\}_{N\geq 1}$ is tight, and any subsequential limit $\mu = (\mu_t)_{t\geq 0}$ satisfies*

$$d\mu_t(f) = \frac{\beta}{4} \int_\Gamma \int_\Gamma \left(\partial_s f(z) \operatorname{Re} \frac{\tau(z)}{z-w} - \partial_s f(w) \operatorname{Re} \frac{\tau(w)}{z-w} \right) \mu_t(dz)\mu_t(dw)dt,$$

for all functions f smooth in a neighborhood of Γ.

Future work

A possible continuation of this work can be done in several directions. Here we point out the most obvious ones.

Existence in the high temperature regime. Theorem 1 gives the existence of the parametrization process for any $\beta \geq 1$. The main reason is that in this regime the particles never collide almost surely, or in other words, the process X never hits the boundary ∂D where the drift is singular. However, for $0 < \beta < 1$ the process almost surely hits ∂D, and one has to consider the stochastic differential equation with singular drift and reflecting boundary conditions in the non-smooth domain D. In the case of a circle or the real line this problem was resolved in [12, 14] with the help of multivalued SDEs. Giving a proof for the existence in the case of any rectifiable Jordan curve is one of the directions for future research.

The boundary of D consists of hyperplanes and their intersections, where the intersections correspond to a collision of three or more particles, which we refer to as *multiple collisions*. The idea is to show that multiple collisions do not occur almost surely. Hence, the reflection problem is effectively reduced to the setting of a half-space.

Problem A. *Prove that for any rectifiable Jordan curve and any $\beta > 0$ there exists a strong solution to the stochastic differential equation*

$$dX(t) = dB(t) + \frac{1}{2}\nabla \log \rho_{\beta,N}(X(t))dt,$$

such that for all $t \geq 0$, almost surely,

$$X_1(t) \leq X_2(t) \leq \ldots \leq X_N(t) \leq X_1(t) + l.$$

Sketch of a solution. 1°. Show that multiple collision do not occur almost surely. That is, the process cannot hit the part of the boundary of D which consists of the intersections of the hyperplanes. This can be done by showing that for all $3 \leq n \leq N$ and $1 \leq q \leq N - n + 1$, the stochastic process

$$R(t) = \sum_{i,j \in I} |\gamma(X_i(t)) - \gamma(X_j(t))|^2,$$

where $I = \{q, \ldots, q + n - 1\}$, is strictly bigger than 0 almost surely. See [15].

2°. Show that there exists a strong solution to the stochastic differential equation with reflecting boundary condition

$$dX(t) = dB(t) + b(X(t))dt + d\varphi(t), \quad b = \frac{1}{2}\nabla \log \rho_{\beta,N},$$

where φ is a boundary process. Because of 1°. this can be attempted using a localization approach, [44, Section 2.4], since the process hits only smooth part of the boundary of D. In particular, the process is reflected back into D at one of the hyperplanes defining D, so that locally reflection problem is reduced to a reflection in the half-space $\mathbb{R}_+ \times \mathbb{R}^{N-1}$.

3°. Show that $\varphi \equiv 0$ almost surely. □

Improved regularity assumptions. The proofs of Proposition 1, Theorems 3, 4, and 5 all assume some regularity of the curve for technical reasons. One can naturally try to improve these assumptions and find the optimal ones.

Fekete points. Recall that N:th level Fekete points of Γ is a collection $\{z_1^*, \ldots, z_N^*\}$ of points on Γ that minimizes the logarithmic energy

$$-\sum_{i<j} \log |z_i - z_j|, \qquad z_i \in \Gamma.$$

Note that a configuration of Fekete points is not necessarily unique, e.g., a circle admits infinitely many configurations on any level. The probability measure, given by the density

$$\rho_{\beta,N}(\boldsymbol{z}) = \frac{1}{Z_{\beta,N}} \prod_{i \neq j} |z_i - z_j|^{\beta/2}, \quad z_i \in \Gamma, \quad \beta > 0$$

converges weakly, as $\beta \to +\infty$, to a probability measure supported on the Fekete configurations. It is interesting to see if an analysis of Dyson Brownian motion at low temperatures can offer any insights into the description of Fekete points of a given curve.

Paper D

Dyson Brownian motion on a circular arc

V. Guskov, M. Liu and F. Viklund
To be submitted.

This work is an companion paper to Paper C. It presents a definition of Dyson Brownian motion on a rectifiable Jordan arc, constructs the process on a circular arc and studies its properties such as uniqueness of the stationary distribution and convergence of the transition probabilities of the process to the stationary distribution in the long-time limit.

Let Γ be a rectifiable Jordan arc. By Rademacher's theorem, Γ admits a parametrization by arc-length, which we denote by $(\gamma(s))_{s\in[0,l]}$, where $l = l(\Gamma)$ is the length of the arc. Denote

$$D = \{x \in \mathbb{R}^N : 0 < x_1 < \dots < x_N < l\}.$$

Definition 1 (Parametrization process). *Let Γ be a rectifiable Jordan arc. The parametrization process for Dyson Brownian motion on Γ, with inverse temperature $\beta > 0$, is a continuous strong Markov process $X = (X_1(t), \dots, X_N(t))_{t\geq 0}$ with diffusion matrix equal to the identity and drift equal to the weak gradient of $\frac{1}{2}\log \rho_{\beta,N}$, where*

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}} \prod_{i\neq j} \left|\gamma(x_i) - \gamma(x_j)\right|^{\beta/2}, \quad x \in D, \quad \beta > 0,$$

is a probability density on D, and such that for all $t \geq 0$, almost surely,

$$0 \leq X_1(t) \leq \dots \leq X_N(t) \leq l.$$

Definition 2 (Dyson Brownian motion on a Jordan arc). *Let Γ be a rectifiable Jordan arc. Dyson Brownian motion on Γ is the stochastic process $\boldsymbol{Z} = (\gamma(X_1), \dots, \gamma(X_N))$, where $X = (X_1, \dots, X_N)$ is the parametrization process of Definition 1.*

In the case of a circular arc, the following theorem gives the existence of Dyson Brownian motion for any $\beta > 0$.

Theorem 1 (Existence for $\beta > 0$). *Let Γ be a circular arc. For any collection $\boldsymbol{z} = \{z_1, \dots, z_N\}$ of points on Γ and $\beta > 0$, there exists a parametrization process $X = (X_1(t), \dots, X_N(t))_{t\geq 0}$, with $z_i = \gamma(X_i(0))$. This process is the unique strong solution of the SDE with reflecting boundary conditions*

$$dX(t) = dB(t) + b(X(t))dt + d\varphi(t), \quad b = \frac{1}{2}\nabla \log \rho_{\beta,N},$$

such that for all $t \geq 0$, almost surely,

$$0 \leq X_1(t) \leq \dots \leq X_N(t) \leq l,$$

and the process

$$\boldsymbol{Z} = (\gamma(X_1(t)), \dots, \gamma(X_N(t)))_{t\geq 0}$$

defines a continuous strong Markov process taking values in $\Gamma^N = \Gamma \times \cdots \times \Gamma$.

In the coordinate form the system is

$$\begin{cases} dX_1(t) = dB_1(t) + b_1(X(t))dt + dL(t), \\ dX_2(t) = dB_2(t) + b_2(X(t))dt, \\ \dots \\ dX_{N-1}(t) = dB_{N-1}(t) + b_{N-1}(X(t))dt, \\ dX_N(t) = dB_N(t) + b_N(X(t))dt - dR(t), \end{cases}$$

where L, R *are non-decreasing continuous processes of finite total variation with* $L(0) = R(0) = 0$ *and*

$$L(t) = \int_0^t \mathbb{1}_{\{X_1(s)=0\}} dL(s),$$
$$R(t) = \int_0^t \mathbb{1}_{\{X_N(s)=l\}} dR(s).$$

Theorem 2. *The parametrization process* X^ξ, *with random initial distribution* ξ *independent of the Brownian motion, is a continuous Feller, strong Markov process. The transition probability* $P(x,t,dy)$ *of the parametrization process* X^x, *started at* $x \in D$, *has a density,*

$$P(x,t,dy) = p(x,t,y)dy,$$

which satisfies the Fokker-Planck-Kolmogorov equation

$$\partial_t p(x,t,\bullet) = L^* p(x,t,\bullet) \text{ weakly in } D$$

with reflecting boundary condition

$$n \cdot \left(-\frac{1}{2}\nabla p(x,t,\bullet) + \left(\frac{1}{2}\nabla \log \rho_{\beta,N}\right) p(x,t,\bullet)\right) = 0 \text{ weakly on } \partial D.$$

Moreover, the Revuz measure associated with the boundary process is $\frac{1}{2}\sigma$, *where* σ *is the surface measure on* ∂D.

Theorem 3 (Stationary distribution). *A probability measure* $\mu_{\beta,N}(dx) = \rho_{\beta,N}(x)dx$ *on* $(\overline{D}, \mathcal{B}(\overline{D}))$, *with the density*

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}(\gamma)} \prod_{i\neq j}^{N} |\gamma(x_i) - \gamma(x_j)|^{\beta/2}, \quad x \in \overline{D}, \quad \beta > 0,$$

is the unique stationary distribution for the parametrization process X.

Theorem 4 (Convergence to the stationary distribution)**.** *The transition probability function $P(x, t, dy)$ of the parametrization process X, started at $x \in \overline{D}$, converges weakly, exponentially fast, as $t \to +\infty$ to the unique stationary distribution on $(\overline{D}, \mathcal{B}(\overline{D}))$ given by the Coulomb gas density*

$$\rho_{\beta,N}(x) = \frac{1}{Z_{\beta,N}(\gamma)} \prod_{i \neq j}^{N} |\gamma(x_i) - \gamma(x_j)|^{\beta/2}, \quad x \in \overline{D}, \quad \beta > 0.$$

Future work

The most natural future direction to explore is to prove existence of the process in the case of more general arcs, preferably a rectifiable Jordan arc. Other questions involve similar analysis as in Paper C and proposed future work after its summary above.

Contributions

Paper A is a single-author paper. The problem was proposed by Viklund. The article was written by the candidate.

Paper B is a collaboration between Nathanaël Berestycki, Fredrik Viklund, and the candidate. The project was initiated by Berestycki and Viklund who proposed the problems studied in the paper. All co-authors participated equally in discussions about the problems and proof strategies. The candidate worked out the details of the proof of the large deviation principle and geometric analysis of finite entropy shapes, in particular, the proof of the existence of a finite energy cusp. The writing of the paper was carried out by all co-authors equally.

Paper C is a collaboration between Mingchang Liu, Fredrik Viklund, and the candidate. The project was proposed by Viklund. All co-authors participated equally in discussions about the problems and proof strategies. The candidate worked out the details in Sections 2-5. Liu worked out the details of Section 6. The major part of the writing of the paper was done by the candidate.

Paper D is a collaboration between Mingchang Liu, Fredrik Viklund, and the candidate. The project was proposed by Viklund. All co-authors participated equally in discussions about the problems and proof strategies. The candidate wrote the current version of the paper.

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Part II
Research Papers