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Symmetric and Asymmetric Funding Charges

ERIK ILDRING

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Abstract

This thesis investigates funding value adjustment (FVA), focusing on differences between symmetric and asymmetric FVA in derivatives pricing. Symmetric FVA assumes a single funding rate for both lending and borrowing, while asymmetric FVA accounts for two distinct rates, the more common scenario in practice. A challenge with asymmetric FVA is the reliance on computationally intensive Monte Carlo estimates, which require extensive sampling to achieve accuracy for portfolios with many derivatives. Therefore, it would be advantageous if symmetric FVA could effectively approximate asymmetric FVA.

Through analytical methods and numerical simulations, this study explores whether symmetric FVA can approximate asymmetric FVA under certain conditions. A simplified model is used to formulate the asymmetric FVA in closed form, used to derive an asymptotic result showing that asymmetric FVA converges to symmetric FVA as the expected portfolio value goes to infinity. Additionally, an approximation formula is derived for asymmetric FVA. Furthermore, Monte Carlo simulations within a foreign exchange model further evaluate the feasibility of using symmetric FVA as a substitute for asymmetric FVA. The findings suggest that symmetric FVA is a viable alternative when the expected portfolio value exceeds three times the portfolio's annual volatility.

The thesis also contributes by more clearly explaining the derivation of the value adjustments by Bugard and Kjaer [1].

Keywords Symmetric Funding Value Adjustment, Asymmetric Funding Value Adjustment, Derivatives Pricing, Value Adjustments

Sammanfattning

I denna rapport utforskas finansieringsvärdejustering (FVA) med fokus på skillnaderna mellan symmetrisk och asymmetrisk FVA när det gäller prissättning av derivat. För den symmetriska FVA antas en ränta både för utlåning och inlåning, medan den asymmetriska metoden använder två skilda räntor, vilket oftare förekommer i verkliga situationer. En utmaning med den asymmetriska metoden är att den kräver beräkningstunga Monte Carlo-estimat. Det krävs en omfattande mängd slumpdragningar för att uppnå tillräcklig noggrannhet för portföljer med många derivat. Det hade därför varit fördelaktigt om symmetrisk FVA kunde användas istället för asymmetrisk FVA.

Vidare undersöks med analytiska metoder och numeriska simuleringar om den symmetriska FVA värdet kan approximera den asymmetriska FVA värdet under vissa förutsättningar. Vi använder därför en förenklad modell för att utrycka den asymmetriska FVA i sluten form, och från detta härleds ett asymptotiskt resultat som visar att asymmetriska FVA konvergerar till en symmetrisk FVA när portföljvärdet går mot oändlighten. Utifrån den förenklade modellen härleder vi även en approximationsformel för asymmetrisk FVA. Därefter utförs Monte Carlo-simuleringar i en valutamodell för att undersöka möjligheten att ersätta den asymmetriska FVA med den symmetriska FVA. Resultaten indikerar att den symmetriska metoden fungerar bra när portföljväntevärdet är minst tre gånger större än portföljens årliga volatilitet.

Dessutom bidrar rapporten genom att tydligöra Bugards och Kjears [1] härledning av värdejusteringarna.

Nyckelord Symmetrisk Finansieringsvärdejustering, Asymmetrisk Finansieringsvärdejustering, Derivatprissättning, Värdejusteringar

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Erik Ildring

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AFVA	Asymmetric Funding Value Adjustment
ATM	At-The-Money
CVA	Credit Value Adjustment
COLLVA	Credit Value Adjustment
DVA	Debit Value Adjustment
FBA	Funding Benefit Adjustment
FCA	Funding Cost Adjustment
FVA	Funding Value Adjustment
FX	Foreign Exchange
GBM	Geometric Brownain Motion
ITM	In-The-Money
KVA	Capital Value Adjustment
OTC	Over-The-Counter
SFVA	Symmetric Funding Value Adjustment
VA	Value Adjustment

1 Introduction

1.1 Background

A financial derivative is a contract whose value is determined by the value of an underlying asset, index, or rate [2]. Common types of derivatives include options, futures, forwards, and swaps. These instruments are used for various purposes, including hedging against risk, speculating on price movements, or gaining access to assets or markets without owning them directly. The price and value of a derivative fluctuate based on the changes in price of the underlying asset, which could be anything from commodities and currencies to interest rates and stocks.

The financial crisis of 2007-2008 highlighted significant weaknesses in the global financial system, particularly in the regulation and management of risk associated with complex financial instruments like derivatives in the over-the-counter (OTC) market [3]. In OTC market, the participants trade directly with their respective counterparties without an intermediary. Before the crisis, derivatives were often priced without fully accounting for the potential risk of counterparty default, i.e. the counterparty not fulfilling its contractual obligations associated with the derivative, and the costs associated with funding the required capital for these trades. As a result, post-crisis, there was a push from policy makers for stricter regulations and enhanced risk management practices. This led to the implementation of additional charges, included in the price by issuers (sellers) of derivatives. These charges are collectively known as value adjustments (VA). Some of these charges include:

- **Credit value adjustment (CVA):** A charge that accounts for the counterparty's default risk, compensating the issuer for the possibility of the counterparty's default.
- **Debit value adjustment (DVA):** A charge that accounts for the issuer's default risk, compensating the counterparty for the possibility of the issuer's default.
- **Collateral value adjustment (COLLVA):** A charge that reflects the value of the collateral agreement between the issuer and the counterparty. Collateral and collateralized trades will be discussed in more detail in section 2.
- **Capital value adjustment (KVA):** A charge that accounts for the capital requirements imposed on the issuer. Following the crisis, issuers have been under regulatory pressure (see, for instance, the Basel III framework [4]) to hold capital (funds) against their derivatives, as an insurance/security.
- **Funding value adjustment (FVA):** A charge that accounts for possible funding costs arising from the derivative trade. We will discuss how funding costs can arise in subsection 2.3.

In this thesis, we will focus on the FVA charge. Unlike the other value adjustments there exist several common ways to calculate FVA, and they are therefore of interest for SEB to analyze.

The two approaches that we will consider in this thesis are symmetric FVA (SFVA) and asymmetric FVA (AFVA). The SFVA is based on the assumption that there is a single funding rate that the issuer of a financial derivative can both lend and borrow money at. Contrary to the symmetric case, the AFVA assumes that the issuer of a financial derivative uses one rate for lending and one rate for borrowing money, which is often the case in practice.

For SFVA one often uses analytical methods, but one sometimes resorts to Monte Carlo estimates that in practice perform well. However, for AFVA one has to use Monte Carlo estimates, which do not perform well. Obtaining a Monte Carlo estimate of the AFVA requires the whole derivatives portfolio to be sampled, which is, for a bank such as SEB with the amount of derivatives being on the order of hundreds of thousands, computationally expensive and might take a long time to compute. However, the derivative trading market is very competitive and a bank might only have a couple of minutes to quote a price for a derivative, in which a charge for funding has to be computed (among other charges). Thus, if the calculation is too slow, the bank might not win the trade. On the other hand, if the estimate of the funding charge is not accurate enough, the bank does not win the trade either, or wins the trade at a loss. Therefore, using a Monte Carlo method to calculate an AFVA charge for a new uncollateralized derivative is not so practical, since to achieve both a fast and accurate Monte Carlo estimate of AFVA is practically infeasible.

With the infeasibility of a Monte Carlo estimate of the AFVA in mind, we would like to understand if we can instead use the computationally more tractable SFVA with a certain rate as an approximation of AFVA. To investigate scenarios where SFVA might be used instead of AFVA, when the issuer of a derivative has two different funding rates, we will formulate a simplified model where the AFVA can be computed analytically. Using this, we perform an asymptotic analysis on the analytical AFVA formula, showing that the approximate AFVA formula converges to a SFVA formula when the issuer's expected total portfolio value goes to infinity. Inspired by the asymptotic argument, we then investigate in a foreign exchange (FX) model if Monte Carlo estimates of AFVA and SFVA seem to coincide when the expected portfolio value is increased.

1.2 Purpose and Research question

This thesis aims to analyze the differences between SFVA and AFVA through numerical comparisons using a simplified model. We further demonstrate that in a Gaussian model, AFVA converges to SFVA as the expected portfolio value approaches infinity. The numerical comparison and asymptotic argument will aid the external project provider Skandinaviska Enskilda Banken (SEB) in determining how FVA should be calculated efficiently. Therefore, we aim to investigate the main research question

> *How large is the charge for* AFVA *compared to the charge of* SFVA *and can we substitute* SFVA *for* AFVA *under some assumptions on the derivative portfolio?*

1.3 Related work

The inclusion of a funding value adjustment (FVA) in the pricing of derivatives has been a topic of debate. Burgard and Kjaer, in [5], derive a partial differential equation (PDE) framework that provides a PDE representation of the derivative price, incorporating both funding costs and counterparty credit risk. Furthermore, Hull and White argue in [6] that an FVA should not be included in the fair value of a derivative. An argument that Hull and White give is that, to align with corporate finance theory, "pricing should be kept separate from funding". However, Burgard and Kjaer, in their paper [1], show that the inclusion of an FVA, which they refer to as a funding cost adjustment (FCA), should depend on the funding strategy (the method of receiving funding) employed by the issuer. Burgard and Kjaer further demonstrate that excluding an FVA would require a funding strategy that may not be feasible from a regulatory standpoint.

In this thesis, we will consider symmetric FVA and asymmetric FVA. A symmetric funding value adjustment is discussed by Albanese and Andersen [7] as well as by Burgard and Kjaer [1]. Both papers demonstrate that SFVA can be decomposed into a funding cost adjustment and a funding benefit adjustment (FBA). Additionally, Ruiz [8] shows how different funding rates for FCA and FBA can be applied to calculate an AFVA. Moreover, Gregory presents an AFVA in discrete form in his book [3], which is consistent with the AFVA discussed by Ruiz.

1.4 Outline

The thesis consists of seven sections. Section 1 presents an introduction of the thesis and the research question. Section 2 introduces some background knowledge needed to understand the necessity of funding charges. Thereafter, in section 3, the mathematical derivation of the VAs is presented together with the definitions of SFVA and AFVA. In section 4, we consider a simplfied model and derive an analytical expression for AFVA. In addition, we derive some asymptotic results with this expression, which helps in understanding the properties of AFVA. In section 5, we consider a foreign exchange model, consisting of five geometric Brownian motions (GBM), and compute and present Monte Carlo estimates of AFVA and SFVA charges. Furthermore, in section 6, we analyze our results. We find that for a portfolio with large enough expected value the AFVA is close in value to a SFVA with a certain funding rate. The limitations of the model are also discussed. Additionally, possible future work is proposed and social and ethical aspects are outlined. Lastly, in section 7, the conclusions are stated.

2 Exposure, Collateral and Funding

In this section, we will begin by presenting the concept of credit exposure (subsection 2.1), a metric that quantifies the losses that can be realized from a derivative trade in case of default of one of the trading parties. Thereafter, collateral and collateralized trades, a method for reducing the credit exposure, are presented in subsection 2.2. We illustrate in subsection 2.3 how the combination of collateralized and non-collateralized trades might lead to funding costs for an issuer of a financial derivative. The funding costs are what drives an issuer of a financial derivative to calculate and charge an FVA. Lastly, we discuss in subsection 2.4 how, at default of one of the parties, the derivative value is determined.

But first, we illustrate with an example why one might want to do derivative trades. Assume that there is a wheat farmer. This farmer has many costs associated with farming wheat. However, since the wheat price may fluctuate over time, the farmer is not sure if he can cover his costs if the wheat price draws down. To secure the price in the future the farmer can do a derivatives contract with another party known as a forward contract on wheat. A forward contract on wheat is a financial agreement where the farmer can agree to sell wheat at a set price on a future date. Such a contract helps the farmer lock in a price now, which protects against future market fluctuations. This approach stabilizes revenue by ensuring a predictable income, regardless of how wheat prices change. Additionally, the other party in the trade also benefits from the wheat forward because it is able to lock in the purchase price in advance.

Throughout this section, let A be a bank and B a counterparty to A having an ongoing derivative trade with value $\mathcal{V}$ and some predetermined date of expiration, i.e. date when the derivative trade is no longer active. When we use the phrase "from the point of view of A" or similar, we mean that if $\mathcal{V} > 0$, then B would have to pay A if the two parties want to prematurely close the trade. To prematurely close a trade means that the two parties agree to terminate the derivative contract before its scheduled expiration. Conversely, if $\mathcal{V} < 0$, then A would have to pay B in case of prematurely closing the trade. If nothing else is stated, the value $\mathcal{V}$ will be considered from the point of view of A throughout the section.

2.1 Credit exposure

Default occurs when a borrower fails to fulfill their debt obligations, such as missing payments on a loan or bond. Defaults can arise from various factors such as economic downturns, poor financial management, or industry-specific challenges. The amount of money that A can lose if B goes default at some time t is known as credit exposure [3] (or simply exposure), it will be shown in subsection 3.5 that the credit exposure is needed for defining a funding value adjustment. A characterizing property of exposure is that a positive portfolio value corresponds to us having to do a claim on the defaulted counterparty, and thus, we may not receive the full portfolio value back. On the contrary, if the counterparty defaults and the portfolio value is negative, we are still obliged to honor the contractual payments associated with the portfolio. Mathematically, we can define the concepts.

Definition 2.1. *Let $\mathcal{V}_t$ be the value/price of a financial derivative (or a portfolio of financial derivatives) at time t. Then the positive exposure*[1] *and negative exposure are given by* $\mathcal{V}_t^+ :=$

[1] Also known as credit exposure or exposure

$\max(\mathcal{V}_t, 0)$ *and* $\mathcal{V}_t^- := \min(\mathcal{V}_t, 0)$*, respectively. Moreover, the expected positive exposure and expected negative exposure are given by* $\mathrm{EPE}_t^{\mathcal{V}} = \mathbb{E}\left[\mathcal{V}_t^+\right]$ *and* $\mathrm{ENE}_t^{\mathcal{V}} = \mathbb{E}\left[\mathcal{V}_t^-\right]$*, respectively.*

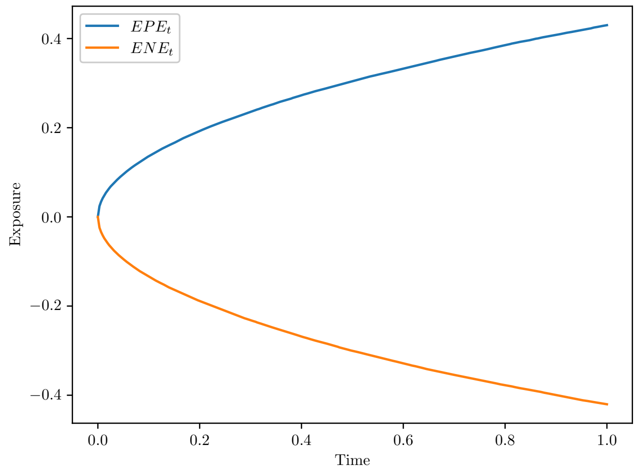

Figure 1: Example expected positive and negative exposure for a derivative

Figure 1 displays an example of the expected positive and negative exposures for a portfolio with only one FX forward (see section 5 table 11 for details).

From the point of view of bank A, the expected positive exposure represents the expected loss that A would make if B defaulted at time t and the expected negative exposure represents the expected loss that B would make if A defaulted (with a minus sign in front). Since both exposures represent potential losses they can be seen as measurements of risk.

A deterministic simple example is: Assume that the bank A lends the counterparty B the amount \$2 at day 0 ($t = 0$) and B promises to pay \$1 each day until its debt is paid. If we assume that the payments occur instantly, that is, A receives the payments on the same day, then in case that B defaults, A would lose,

$$\begin{aligned}
&\$2 \text{ if } B \text{ defaults at time } t = 0,\\
&\$1 \text{ if } B \text{ defaults at time } t = 1,\\
&\$0 \text{ if } B \text{ defaults at time } t = 2.
\end{aligned}$$

In other words, in this example, $\mathrm{EPE}_t = 2 - t$ and $\mathrm{ENE}_t = 0$, for $t \in \{0, 1, 2\}$.

2.2 Collateralized trades

Between the bank and the counterparty, a high credit exposure means a potential significant loss if the counterparty defaults. Therefore, the bank may seek to reduce its exposure to mitigate this risk. There are several methods for reducing the credit exposure. One of the methods is through collateralization [3]. Collateral is something of value that a lender can claim from the borrower in case of the borrower's default [9], and thus act as a "security/insurance" for the lender. For instance, when a person takes out a mortgage, the bank usually requires the person to post their home as collateral. For derivative trades the process is a bit more involved. Usually cash or securities are posted, with an amount based on the value of the derivative. However, since this value can vary with the market, collateral is often transfered back and forth between the parties during the lifespan of the derivative. Simply put: if the derivative value $\mathcal{V}$ goes up (from the point of view of the bank A) the counterparty B might have to transfer more collateral to A and vice versa, which leads to many collateral transfers. How the collateralization is done is governed by a contractual agreement called Credit Support Annex (CSA), which both parties agree upon as part of the process of establishing the derivative contract. The CSA may cover, but is not limited to:

- Method and timings of the underlying valuation of the derivative
  - This means that the two parties determine how the value of the derivative should be established and at what time the valuation should occur prior to the collateral transfer.
- The calculation of the collateral amount that will be posted
  - The two parties can agree upon a minimal transfer amount, which is the least amount of collateral that can be posted. This amount is set in order to avoid having a lot of small transfers, which would be a considerable workload. The parties could also agree on a threshold, which means that no collateral is posted until the value of the derivative reaches the threshold.
- Timing of collateral transfers
  - The two parties determine how often collateral should be transferred. This could, for instance, be every day, once a week or once a month.
- Allowed collateral
  - The two parties determine what allowed collateral is. They could, for example, only allow cash in a specific currency, or allow both cash and bonds (or other derivatives as well).
- Rate for the cash transferred
  - When a party posts collateral in cash, it earns interest from the party receiving the collateral. The interest rate applied is known as the collateral rate, which is specified in the CSA.

Furthermore, the two parties in a collateralized trade can come to an agreement that the party receiving collateral is allowed to reuse (rehypothecate) the collateral. This is the common case.

Let X denote the value of the collateral. Note that, from the point of view of A, if X is positive it means that A has received collateral from B and if X is negative it means that A has given collateral to B. See Figure 2 for an illustration of the collateral transfer.

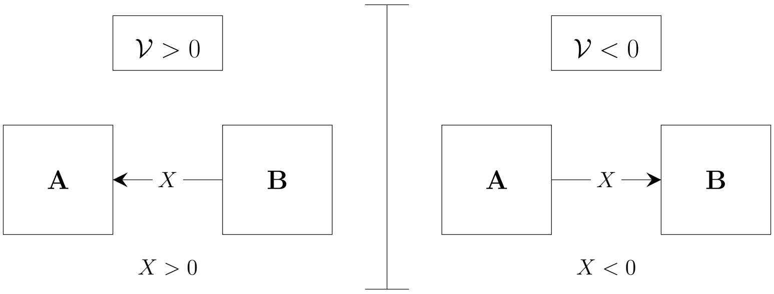

Figure 2: Illustrative example of how the collateral is transferred from the point of view of bank A when having a collateralized trade with counterparty B.

Furthermore, since the collateral X represents a value, we have that the total value of a derivatives trade including collateral is given by $\mathcal{V} - X$. The minus sign is there since $X > 0$ when $\mathcal{V} > 0$ (A is receiving the collateral) and $X < 0$ when $\mathcal{V} < 0$ (A is providing the collateral). The following Figure 3 gives an illustrative view of the value of a derivative in the uncollateralized case, and the total value including the collateral for the collateralized case. The value shows the current exposure to the counterparty, as well as the potential loss if the counterparty defaults.

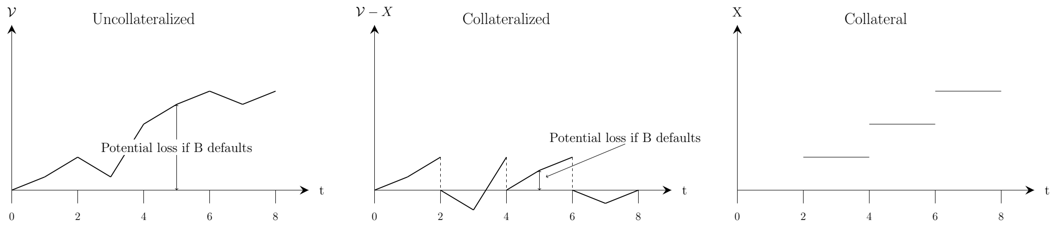

Figure 3: Illustrative example of the total value of the derivative and collateral in both an uncollateralized and collateralized case, demonstrating how exposure is reduced over time.

In the above example (Figure 3), collateral is provided at times $t = 2, 4, 6, \ldots$. As $\mathcal{V}$ is trending upward, the collateral X provided is positive.

2.3 Funding costs and benefits

As we have discussed, collateral reduces the exposure to a counterparty, and therefore one might think that all trades are done with collateralization. However, performing a trade with collateral has its downsides, such as the need for the operational capacity to keep track of the collateral, or having enough assets/cash to be able to post collateral in time. Some market participants, for instance corporates, might neither have the operational capacity nor the amount of assets/cash needed to participate in a collateralized trade. Therefore, when an issuer, such as a bank, and a corporate make a derivative trade, it might be uncollateralized even if this increases both parties' exposure to one another.

In addition to the bank A and counterparty B, assume that C is another counterparty. We will illustrate how a funding cost can occur by an example (see [3] p. 336 for more details). Suppose bank A is doing an uncollateralized derivative trade with a counterparty B. To avoid the market risk of the trade, A does the opposite trade (hedge) with another counterparty C. However, this hedging trade is usually done between two banks, and is thus often collateralized, because banks commonly have both the operational capacity to have the trade collateralized and they often do not want to increase their credit exposure. At some time t, assume that the initial derivative value is positive from the point of view of A. Then the hedging derivative has a negative value against couterparty C resulting in A having to post collateral to C (the hedging derivative is assumed to be collateralized). But the trade between A and B is uncollateralized, meaning that B will not post collateral to A. Therefore, for A to be able to post collateral to C, bank A has to borrow money at some funding rate r_F. This rate is influenced by many factors, such as the duration of the loan, the credit rating of the bank, and central bank or inter-bank rates (rates between banks). On the collateral, A will earn interest at a collateral rate r from B. But at the same time, A has to pay interest at the funding rate r_F, which almost always is greater than the collateral rate r. The bank A will thus incur a net loss, a funding cost, by not having a collateralized trade with B.

On the contrary, if the value of the trade between A and B is negative, then A would receive collateral from C. Therefore, A would have to pay interest at the collateral rate r to C. But, if we assume that the bank A has an agreement with C that allows A to reuse the collateral, which usually is the case, then A would be able to lend out the collateral, earning interest at the funding rate $r_F > r$. Thus A receives a net earning, a funding benefit, from not having a collateralized trade with B.

To account for the possibility of a funding cost/benefit, the bank A might want to charge/compensate the counterparty B. This charge/compensation is what the funding valuation adjustment tries to capture, since the normal market price of a trade is assumed to be a collateralized trade.

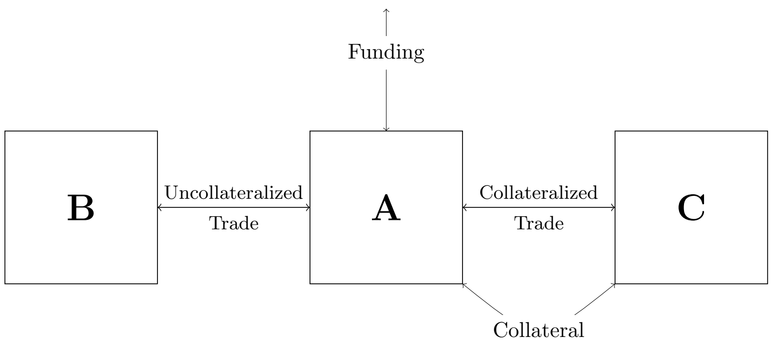

Figure 4: Uncollateralized hedging case with three parties.

2.4 Recovery at default

We now discuss how, in practice, the value of a collateralized derivative is determined when one of the parties involved in the trade defaults, which is important when defining the FVA in section 3.

Furthermore, when a party defaults, the surviving party aims to recover as much of the value owed by the defaulted party as possible. However, they are often unable to recover the full amount, as the defaulting party may lack sufficient funds. The percentage of a derivative's price that can be recovered after default is called the recovery rate. The recovery rate is party specific in the sense that it is determined by how much money that can be reclaimed from a specific defaulting party. The recovery rate is also important in the derivation of FVA presented in section 3.

There are two cases we will consider, A defaults before B and B defaults before A. For simplicity, the case where both parties default at the same time will not be discussed in this thesis. Denote A and B's recovery rates on the derivative by $R_A \in [0, 1]$ and $R_B \in [0, 1]$, respectively. X denotes the collateral, as before.

A defaults before B Suppose A defaults first. Let the value recovered from a collateralized derivative with value $\mathcal{V}$ be a function $g_A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(\mathcal{V}, X) \mapsto g_A(\mathcal{V}, X)$, where X is the collateral.

Since the derivative is collateralized, we know that the amount X has been transferred from A to B or B to A. Therefore it makes intuitive sense that g_A is a function of X. Furthermore, for the remaining value $\mathcal{V} - X$, if it is positive, i.e., $\mathcal{V} - X > 0$ (B owes A money), B can pay the amount $\mathcal{V} - X$ to A. On the contrary, if $\mathcal{V} - X < 0$ (A owes B money), since A has defaulted it can not recover more that the fraction $R_A(\mathcal{V} - X)$. Since A has defaulted it does not have enough money to pay more than the percentage R_A of $\mathcal{V} - X$, which it can pay to B. The above argument gives

$$g_A(\mathcal{V}, X) = (\mathcal{V} - X)^+ + R_A(\mathcal{V} - X)^- + X \tag{1}$$

In the above, we have introduced the shorthand notation $(x)^+$ for the expression $\max(x, 0)$ and $(x)^-$ for the expression $\min(x, 0)$.

B defaults before A Suppose B defaults first. Let the value recovered from a collateralized derivative with value $\mathcal{V}$ be a function $g_B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(\mathcal{V}, X) \mapsto g_B(\mathcal{V}, X)$, where X is the collateral. Just as for the case "A defaults before B", since the derivative is collateralized, we have that the amount X has been transferred between the parties A and B. Therefore it makes intuitive sense that g_B is a function of X. Furthermore, for the remaining value $\mathcal{V} - X$, if it is negative $\mathcal{V} - X < 0$ (A owes B money), A can pay the amount $\mathcal{V} - X$ to B. On the contrary, if $\mathcal{V} - X > 0$ (B owes A money), since B has defaulted it can not recover more that the fraction $R_B(\mathcal{V} - X)$, which it can pay to A. The above argument gives

$$g_B(\mathcal{V}, X) = R_B(\mathcal{V} - X)^+ + (\mathcal{V} - X)^- + X \tag{2}$$

The concepts presented in this section will be used throughout the thesis and are core concepts for understanding FVA.

3 Value Adjustments

We will now go through the derivation of the value adjustments (VAs) as Bugard and Kjaer do in their paper [1], with some modifications for simplicity. This derivation gives a theoretical justification for the fact that the adjustments should be included in the valuation of financial derivatives. Bugard and Kjaer also define the FVA as the sum of the funding cost adjustment (FCA) and debit value adjustment (DVA), which gives us an expression for calculating an FVA. We contribute to the argument by trying to make it more clear, by adding details that that are not explained in the original argument.

In the subsections 3.1 - 3.4, the derivation of the XVAs are presented. Then, in subsection 3.5 the mathematical expression of symmetric FVA for uncollateralized derivatives is given. Lastly, in subsection 3.6 the formula for asymmetric FVA for uncollateralized derivatives is presented.

3.1 The model

Assume that A is a bank, also called an issuer, and B a counterparty to A and that A is to sell a financial derivative to B. Let $\hat{\mathcal{V}}$ denote the value of the derivative contract that incorporates the counterparty's risk between the issuer of a derivative A and counterparty B, and the net funding costs the issuer have before default, and pays out $H(S)$ at maturity T. Let X denote the collateral, which for simplicity, is assumed to be in cash and is assumed to be reusable (this is the usual case). Additionally, assume that the interest paid by the party that is holding the collateral is r_X. Lastly, zero coupon bonds are used in the derivation, the definition now follows.

Definition 3.1. *A zero coupon bond, sometimes just called a bond, with maturity T is a financial derivative that pays out $\mathcal{X} = 1$ to the holder at time T. The price of this derivative at time t is denoted usually $p(t, T)$.*

Note We will throughout the section denote the price of a zero coupon bond with a P ("capital p"). We will also use bond and zero coupon bond interchangeably.

Furthermore, assume that the tradable assets are

- An asset S
  - This could be, for instance, a stock or other asset. Compared to Bugard and Kjaer, we assume for simplicity that the market asset pays no dividend.
- Issuer zero coupon bonds P_i with recovery rate R_i for $i = 1, 2$ and maturity T
  - The issuer bonds are bonds that the issuer A can sell and are used to model two sources of funding (methods for obtaining money) for the issuer A, one for each bond P_1 and P_2. The recovery rates are the percentage of the bond value that can be recovered if A defaults (as described in subsection 2.4).
- Counterparty zero-recovery zero coupon bond P_B with maturity T
  - Zero-recovery means that no value can be recovered from the bond if the counterparty B defaults. The existence of bond P_B serves two purposes. Firstly, it reflects the

reality that counterparty B has a means of obtaining money. Secondly, for the semi-replication argument in subsection 3.2, an asset that depends on J_B is needed to hedge the credit risk of B.

To model the market asset S, we assume a geometric Brownian motion, which is the same assumption as Bugard and Kjaer do (this is a commmon model). To model the bonds P_1, P_2, and P_B, Bugard and Kjaer choose to use jump processes together with the usual risk-free bond dynamics ([10], page 95), "$dB = rBdt$". The jump processes are used to model the decrease in value when a party defaults.

The dynamics of the above assets are assumed to be

$$dS = \mu S dt + \sigma S\, dW, \tag{3}$$

$$dP_i = r_i P_i^- dt - (1 - R_i) P_i^-\, dJ_A, \quad i = 1, 2, \tag{4}$$

$$dP_B = r_B P_B^- dt - P_B^-\, dJ_B, \tag{5}$$

where W is a standard Wiener process and J_A and J_B are independent processes that spike from 0 to 1 in case of default of A and B, respectively. To give a more mathematical definition of J_A and J_B, we need the notion of a Poisson process.

Definition 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, and let λ be a positive real number. A Poisson process N [11] with intensity λ is a stochastic process such that*

1. *N is $\mathbb{F}$-adapted.*
2. *The stochastic variable $N(t) - N(s)$ is independent of $\mathcal{F}_s$, for all $s \leq t$.*
3. *For all $s \leq t$, the conditional distribution of $N(t) - N(s)$ is*

$$\mathbb{P}\left(N(t) - N(s) = n \mid \mathcal{F}_s\right) = e^{-\lambda(t-s)} \frac{\lambda^n (t-s)^n}{n!}$$

for all $s = 0, 1, 2, \ldots$.

More specifically, J_A and J_B are assumed to be capped Poisson processes with intensities λ_A and λ_B, i.e.,

$$J_A = \min(N_A, 1) \quad \text{and} \quad J_B = \min(N_B, 1),$$

where N_A is a Poisson process with intensity λ_A and N_B is a Poisson process with intensity λ_B. Note that the above definition of J_A and J_B ensures that they take values in $\{0, 1\}$. The minus sign in the bond dynamics, e.g. $P_B^- = P_B(t^-)$ is a technical detail and represents the pre-default price of the bond. Additionally, issuer bond P_2 is assumed to have greater seniority than issuer bond P_1. Greater seniority means that buyers of bond P_2 will be prioritized over buyers of bond P_1 in receiving their money if issuer A defaults. Therefore, the rate r_1 for bond P_1 should be higher than the rate r_2 for bond P_2 to compensate for the priority system. Mathematically, this means that we incorporate additional assumptions that $r_1 > r_2$ and $R_1 < R_2$. The assumption that P_2 has greater seniority than P_1 allows for the modeling of two funding sources with different costs.

Definition 3.3. *A cash account, or bank account, $L : \mathbb{R} \to \mathbb{R}$ with rate $R \in \mathbb{R}$ is a (deterministic) function of time, satisfying*

$$\frac{dL}{dt} = RL(t) \tag{6}$$

and represents a bank account from which money can be borrowed or to which money can be deposited, where a rate of R is paid. Solving the ODE gives $L(t) = L(0)e^{Rt}$.

Furthermore, we assume that the issuer has access to the cash accounts β_S and β_B with constant rates q_S and q_B, respectively. By definition 3.3

$$d\beta_S = q_S\beta_S \; dt \quad \text{and} \quad d\beta_B = q_B\beta_B \; dt. \tag{7}$$

The two cash accounts β_S and β_B are intended to model the practical reality that the cost (i.e., the amount of interest paid) of borrowing money may depend on the purpose of the loan. Therefore, we need the model to accommodate two rates, q_S and q_B. We assume that if issuer A borrows money to buy asset S, it can only do so from β_S. Similarly, A can only borrow money from β_B to purchase P_B.

Let r be the risk-free rate, assume for simplicity that it is constant. The assumption of a risk-free rate implies that there exists a cash account (3.3), denoted as B, with rate r, which is the smallest rate among all available cash accounts. We also assume the existence of a hypothetical issuer zero-recovery bond P_z with a spread s_A above r (i.e., the rate is $r + s_A$), a bond from which no value can be recovered if A defaults. The dynamics of this issuer zero-recovery bond are assumed to be

$$dP_z = (r + s_A)P_z^- \; dt - P_z^- \; dJ_A,$$

to match the dynamics (4), this hypothetical bond will be used to show that the intensity of J_A is equal to the spread s_A, i.e. $\lambda_A = s_A$.

Bugard and Kjaer then assume zero basis between P_1, P_2, P_z, and B_t, meaning that the bonds and B_t should have the same expected value. This is a technical assumption. Then, one can show that

$$r_i - r = (1 - R_i)\lambda_A. \tag{8}$$

We show (8) formally. Begin by integrating the P_z-dynamics,

$$P_z(t) - P_z(0) = \int_0^t (r + s_A)P_z^-(s) \; ds - \int_0^t P_z^-(s) \; dJ_A(s)$$

Now, by the heuristic $\mathbb{E}[dN_A(s) \mid \mathcal{F}_{s^-}] = \lambda_A ds$ from [11] page 34, we have that

$$\begin{aligned}
\mathbb{E}[dJ_A(s) \mid \mathcal{F}_{s^-}] &= \mathbb{E}[dJ_A(s) \mid \mathcal{F}_{s^-}, J_A(s^-) \leq 1] \underbrace{\mathbb{P}\left(J_A(s^-) \leq 1\right)}_{=1} \\
&\quad + \mathbb{E}[dJ_A(s) \mid \mathcal{F}_{s^-}, J_A(s^-) > 2] \underbrace{\mathbb{P}\left(J_A(s^-) > 2\right)}_{=0} \\
&= \mathbb{E}[dJ_A(s) \mid \mathcal{F}_{s^-}, J_A(s^-) \leq 1] \\
&= \mathbb{E}[dN_A(s) \mid \mathcal{F}_{s^-}] \\
&= \lambda_A ds. \qquad (9)
\end{aligned}$$

Using the above equality (9), taking expectations and changing the order of integration, we have

$$\begin{aligned}
\mathbb{E}\left[P_z(t) - P_z(0)\right] &= \mathbb{E}\left[\int_0^t (r + s_A)P_z^-(s)\ ds\right] - \mathbb{E}\left[\int_0^t P_z^-\ dJ_A(s)\right] \\
&= \mathbb{E}\left[\int_0^t (r + s_A)P_z^-(s)\ ds\right] - \mathbb{E}\left[\mathbb{E}\left[\int_0^t P_z^-\ dJ_A(s) \mid \mathcal{F}_{s^-}\right]\right] \\
&= \int_0^t (r + s_A)\mathbb{E}\left[P_z^-(s)\right]\ ds - \int_0^t \mathbb{E}\left[P_z^-(s)\right]\lambda_A\ ds \\
&= \int_0^t (r + s_A - \lambda_A)\mathbb{E}\left[P_z^-(s)\right]\ ds
\end{aligned}$$

and thus,

$$\frac{d}{dt}\mathbb{E}\left[P_z(t)\right] = (r + s_A - \lambda_A)\mathbb{E}\left[P_z^-(t)\right],$$

we find

$$\mathbb{E}\left[P_z(t)\right] = P_z(0)e^{(r+s_A-\lambda_A)t}.$$

Furthermore, since we assume

$$dB(t) = rB(t)\ dt,$$

then

$$B(t) = B(0)e^{rt}.$$

Since B is deterministic, the expectation of $B(t)$ is just $B(t)$. Furthermore, because of the zero bond basis assumption, if we start with the same amount of money invested in B and P_z, they should have he same expected value for all times t Therefore, for all t, we have

$$\mathbb{E}\left[B(t)\right] = \mathbb{E}\left[P_z(t)\right] \implies e^{rt} = e^{(r+s_A-\lambda_A)t} \implies \lambda_A = s_A.$$

So we can conclude that the intensity of J_A is s_A. Furthermore, by a similar argument we can show that

$$\mathbb{E}\left[P_i(t)\right] = P_i(0)e^{(r_i-(1-R_i)\lambda_A)t}$$

because of the zero basis assumption, should be equal to $\mathbb{E}\left[B_t\right]$ if they have the same initial value. Thus,

$$e^{(r_i-(1-R_i)\lambda_A)t} = e^{rt} \implies r_i - (1 - R_i)\lambda_A = r,$$

which shows (8).

Because of the assumed tradable assets, one can write the derivative value as $\hat{\mathcal{V}} = \hat{\mathcal{V}}(t, S, J_A, J_B)$. Moreover, in the case of a default by either A or B, the derivative value might drastically change, and this is modeled by the boundary values (see subsection 2.4)

$$\hat{\mathcal{V}}(t, S, 1, 0) = g_A(\mathcal{V}, X), \tag{10}$$

$$\hat{\mathcal{V}}(t, S, 0, 1) = g_B(\mathcal{V}, X), \tag{11}$$

where X is the collateral, for simplicity assumed to be in cash, and $\mathcal{V}$ the Black-Scholes price solving (28). Recall that g_A and g_B are functions that represent the value that can be recover after A's and B's default, respectively.

3.2 Semi-replicating portfolio

In this subsection, we will define a hedging self-financing portfolio. This portfolio will be used in subsection 3.3 to formulate a pricing partial differential equation (PDE) for the derivative value $\hat{\mathcal{V}}$.

Let Π be a portfolio, and assume it is self-financing.

$$\Pi = \delta S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_B P_B + \beta_S + \beta_B - X, \tag{12}$$

where $\delta, \alpha_1, \alpha_2$ and α_B are units of each traded asset respectively, β_S and β_B are cash accounts and X is a cash account called collateral account. The self-financing assumption of the portfolio Π means that the prior-to-rebalancing change $d\bar{\Pi}$ fulfills

$$d\bar{\Pi} = \delta dS + \alpha_1 dP_1 + \alpha_2 dP_2 + \alpha_B dP_B + d\bar{\beta}_S + d\bar{\beta}_B - d\bar{X}, \tag{13}$$

where $d\bar{\beta}_S$, $d\bar{\beta}_B$, and $d\bar{X}$ are prior-to-rebalancing (i.e., before changes in the portfolio) differentials of β_S, β_B, and X. The use of prior-to-rebalancing differentials is a technical detail that Bugard and Kjaer employ to address a previous issue with the portfolio self-financing assumption in [5], as pointed out by Brigo et al. [12].

The cash accounts β_S and β_B exist to fund the positions in S and P_B, respectively. This means that at time t, money from the cash accounts β_S and β_B is borrowed to purchase the amount $\delta(t)$ of asset $S(t)$ and the amount $\alpha_B(t)$ of bond $P_B(t)$. This can be formulated as

$$\alpha_B(t)P_B(t) + \beta_B(t) = 0 \quad \text{and} \quad \delta(t)S(t) + \beta_S(t) = 0.$$

Solving for the cash accounts, i.e., determining how much money needs to be borrowed by A, gives

$$\beta_B(t) = -\alpha_B(t)P_B(t) \quad \text{and} \quad \beta_S(t) = -\delta(t)S(t). \tag{14}$$

Inserting (14) into the cash account dynamics (7) and using prior-to-rebalancing differentials yields

$$d\bar{\beta}_B = -q_B \alpha_B P_B \, dt, \tag{15}$$
$$d\bar{\beta}_S = -q_S \delta S \, dt. \tag{16}$$

The collateral X is assumed to be in cash and is also assumed to be rehypothecable (see subsection 2.2), meaning that it can be reused and pays the rate r_X. That is, the dynamics are

$$d\bar{X} = -r_X X \, dt. \tag{17}$$

Furthermore, notice that the amount not covered by collateral, $\hat{\mathcal{V}} - X$, needs to be funded by A. This can be achieved by choosing to issue an amount α_1 of bond P_1 and an amount α_2 of bond P_2 such that

$$\hat{\mathcal{V}} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0. \tag{18}$$

We will call the above equality (18) the funding constraint.

3.3 Pricing PDE

The self-financing portfolio Π in (12) will now be used to formulate a pricing PDE. Using the self-financing assumption and dynamics (3), (4) and (5), and (15), (16) and (17) yields

$$\begin{aligned} d\bar{\Pi} =& \delta dS + \alpha_1 dP_1 + \alpha_2 dP_2 + \alpha_B dP_B + d\bar{\beta}_S + d\bar{\beta}_B - dX, \\ =& \big(r_1\alpha_1 P_1^- + r_2\alpha_2 P_2^- + (r_B - q_B)\alpha_B P_B^- - q_S\delta S - r_X X\big)\, dt, \\ &+ (\alpha_1 R_1 P_1^- + \alpha_2 R_2 P_2^- - \alpha_1 P_1^- - \alpha_2 P_2^-) dJ_A - \alpha_B P_B^- dJ_B + \delta dS. \end{aligned} \tag{19}$$

Letting

$$\begin{aligned} P_D &= \alpha_1 R_1 P_1 + \alpha_2 R_2 P_2, \\ P &= \alpha_1 P_1 + \alpha_2 P_2, \\ \lambda_B &= r_B - q_B, \end{aligned}$$

we find

$$\begin{aligned} d\bar{\Pi} =& \big(r_1\alpha_1 P_1^- + r_2\alpha_2 P_2^- + \lambda_B \alpha_B P_B^- - q_S\delta S - r_X X\big)\, dt \\ &+ (P_D^- - P^-) dJ_A - \alpha_B P_B^- dJ_B + \delta dS. \end{aligned} \tag{20}$$

Using Itô's lemma for jump-diffusions [13] gives

$$d\hat{\mathcal{V}} = \left(\frac{\partial \hat{\mathcal{V}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\mathcal{V}}}{\partial S^2}\right) dt + \frac{\partial \hat{\mathcal{V}}}{\partial S} dS + \Delta\hat{\mathcal{V}}_A\, dJ_A + \Delta\hat{\mathcal{V}}_B\, dJ_B, \tag{21}$$

where the size of the drawdown in the value $\hat{\mathcal{V}}$ in the event of A's or B's default are given by

$$\begin{aligned} \Delta\hat{\mathcal{V}}_A &= g_A(\mathcal{V}, X) - \hat{\mathcal{V}}(t, S, 0, 0), \\ \Delta\hat{\mathcal{V}}_B &= g_B(\mathcal{V}, X) - \hat{\mathcal{V}}(t, S, 0, 0), \end{aligned}$$

respectively. Combining (20) and (21) yields

$$\begin{aligned} d\hat{\mathcal{V}} + d\bar{\Pi} =& \left(\frac{\partial \hat{\mathcal{V}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\mathcal{V}}}{\partial S^2} + r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_B\alpha_B P_B - q_S\delta S - r_X X\right) dt \\ &+ \left(\delta + \frac{\partial \hat{\mathcal{V}}}{\partial S}\right) dS + (\Delta\hat{\mathcal{V}}_B - \alpha_B P_B^-)\, dJ_B + (\Delta\hat{\mathcal{V}}_A + P_D^- - P^-)\, dJ_A. \end{aligned} \tag{22}$$

The issuer can eliminate the market risk and counterparty risk to B by choosing

$$\delta = -\frac{\partial \hat{\mathcal{V}}}{\partial S} \quad \text{and} \quad \alpha_B = \frac{\Delta\hat{\mathcal{V}}_B}{P_B^-},$$

which yields

$$\begin{aligned} d\hat{\mathcal{V}} + d\bar{\Pi} =& \left(\frac{\partial \hat{\mathcal{V}}}{\partial t} + q_S\frac{\partial \hat{\mathcal{V}}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\mathcal{V}}}{\partial S^2} + r_1\alpha_1 P_1^- + r_2\alpha_2 P_2^- + \lambda_B \Delta\hat{\mathcal{V}}_B - r_X X\right) dt \\ &+ (g_A - \hat{\mathcal{V}} + P_D^- - P^-)\, dJ_A. \end{aligned} \tag{23}$$

But by the funding constraint (18) $\hat{\mathcal{V}} + P = X$ and also defining the operator $\mathcal{A}_t = q_S S \frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}$ one obtains

$$\begin{aligned} d\hat{\mathcal{V}} + d\bar{\Pi} = &\left(\frac{\partial \hat{\mathcal{V}}}{\partial t} + \mathcal{A}_t \hat{\mathcal{V}} + r_1 \alpha_1 P_1^- + r_2 \alpha_2 P_2^- + \lambda_B \Delta \hat{\mathcal{V}}_B - r_X X \right) dt \\ &+ (g_A + P_D^- - X)\, dJ_A. \end{aligned} \tag{24}$$

Now we want to formulate the drift in terms of $\hat{\mathcal{V}}$ instead of the portfolio assets. Using (8) yields

$$\begin{aligned} r_1\alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_B \Delta \hat{\mathcal{V}}_C - r_X X =& \big(r + (1-R_1)\lambda_A\big)\alpha_1 P_1 + \big(r + (1-R_2)\lambda_A\big)\alpha_2 P_2 \\ &+ \lambda_B g_B - \lambda_B \hat{\mathcal{V}} - r_X X \\ =& \, r(\alpha_1 P_1 + \alpha_2 P_2) + \lambda_A(\alpha_1 P_1 + \alpha_2 P_2) \\ &- \lambda_A(\alpha_1 R_1 P_1 + \alpha_2 R_2 P_2) \\ &+ \lambda_B g_B - \lambda_B \hat{\mathcal{V}} - r_X X \\ =& \, rP + \lambda_A P - \lambda_A P_D + \lambda_B g_B - \lambda_B \hat{\mathcal{V}} - r_X X. \end{aligned}$$

Now using the funding constraint (18)

$$\begin{aligned} r_1\alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_B \Delta \hat{\mathcal{V}}_B - r_X X &= r(X - \hat{\mathcal{V}}) + \lambda_A (X - \hat{\mathcal{V}}) - \lambda_A P_D + \lambda_B g_B - \lambda_B \hat{\mathcal{V}} - r_X X \\ &= -(r + \lambda_A + \lambda_B)\hat{\mathcal{V}} - (r_X - r)X - \lambda_A(P_D - X) + \lambda_B g_B \\ &= -(r + \lambda_A + \lambda_B)\hat{\mathcal{V}} - (r_X - r)X + \lambda_A g_A - \lambda_A(g_A + P_D - X) + \lambda_B g_B. \end{aligned}$$

Letting $s_X = r_X - r$ and $\epsilon_h = g_A + P_D - X$, called the hedging error, one gets

$$r_1\alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_B \Delta \hat{\mathcal{V}}_B - r_X X = -(r + \lambda_A + \lambda_B)\hat{\mathcal{V}} - s_X X + \lambda_B g_B + \lambda_A g_A - \lambda_A \epsilon_h. \tag{25}$$

Inserting (25) into (24) gives

$$\begin{aligned} d\hat{\mathcal{V}} + d\bar{\Pi} = &\left(\frac{\partial \hat{\mathcal{V}}}{\partial t} + \mathcal{A}_t \hat{\mathcal{V}} - (r + \lambda_A + \lambda_B)\hat{\mathcal{V}} - s_X X + \lambda_B g_B + \lambda_A g_A - \lambda_A \epsilon_h \right) dt \\ &+ \epsilon_h dJ_A. \end{aligned} \tag{26}$$

While the issuer is alive, i.e $J_A = 0$ and thus $dJ_A = 0$, the issuer wants the portfolio to be a hedging portfolio $d\hat{\mathcal{V}} + d\bar{\Pi} = 0$. The above differential (26) gives the following PDE of the derivative value

$$\begin{cases} \frac{\partial \hat{\mathcal{V}}}{\partial t} + \mathcal{A}_t \hat{\mathcal{V}} - (r + \lambda_A + \lambda_B)\hat{\mathcal{V}} = s_X X - \lambda_B g_B - \lambda_A g_A + \lambda_A \epsilon_h, \\ \hat{\mathcal{V}}(T, S) = H(S). \end{cases} \tag{27}$$

3.4 The value adjustments

Using the pricing PDE just derived, one can now find an additive adjustment, which we will define as the sum of the credit value adjustment (CVA), debit value adjustment (DVA), funding cost adjustment (FCA), and collateral value adjustment (COLLVA). Specifically, let

$$U = \text{CVA} + \text{DVA} + \text{FCA} + \text{COLLVA},$$

such that $\hat{\mathcal{V}} = \mathcal{V} + U$, where $\mathcal{V}$ is the derivative price that does not incorporate funding costs and counterparty risk, and that satisfies the Black-Scholes equation

$$\begin{cases} \frac{\partial \mathcal{V}}{\partial t} + \mathcal{A}_t \mathcal{V} - r\mathcal{V} = 0, \\ \mathcal{V}(T, S) = H(S). \end{cases} \tag{28}$$

Thus, the adjustment addend U solves

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_A + \lambda_B)U = s_X X - \lambda_B(g_B - \mathcal{V}) - \lambda_A(g_A - \mathcal{V}) + \lambda_A \epsilon_h, \\ U(T, S) = 0. \end{cases}$$

Applying the Feynman-Kac formula [14] to the above PDE yields the value adjustments. Let $\mathcal{V}(\cdot) = \mathcal{V}(\cdot, S(\cdot))$ be the derivative price (28) along the market asset path $S(\cdot)$. We get

$$\begin{aligned} U = &- \int_t^T \lambda_B D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[\mathcal{V}(u) - g_B(\mathcal{V}(u), X(u)) \right] du \\ &- \int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[V(u) - g_A(\mathcal{V}(u), X(u)) \right] du \\ &- \int_t^T s_X D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[X(u) \right] du - \int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[\epsilon_h(u) \right] du, \end{aligned} \tag{29}$$

where $\mathbb{E}_t^{\mathbb{Q}}[\,\cdot\,] = \mathbb{E}^{\mathbb{Q}}[\,\cdot \mid \mathcal{F}_t]$ and $D_z(t,u) = \exp(-z(u-t))$ is called the discount factor in the interval $[t,u]$ using rate z. Since we assume $U = \text{CVA} + \text{DVA} + \text{FCA} + \text{COLLVA}$, we let the above integrals in (29) be

$$\begin{aligned} \text{CVA} &= - \int_t^T \lambda_B D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[\mathcal{V}(u) - g_B(\mathcal{V}(u), X(u)) \right] du, \\ \text{DVA} &= - \int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[\mathcal{V}(u) - g_A(\mathcal{V}(u), X(u)) \right] du, \\ \text{COLLVA} &= - \int_t^T s_X D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[X(u) \right] du, \\ \text{FCA} &= - \int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u) \mathbb{E}_t^{\mathbb{Q}} \left[\epsilon_h(u) \right] du. \end{aligned}$$

The pricing measure $\mathbb{Q}$ is chosen such that the $\mathbb{Q}$-dynamics of S are given by

$$dS = q_S S\, dt + \sigma S\, dW^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Wiener process. In other words, $\mathbb{Q}$ is chosen such that S_t/β_S is a martingale under $\mathbb{Q}$. The volatility σ can be specified using implied volatility, as described in [10].

From these value adjustment formulas we see that they play an important role in the price. The DVA and FCA terms will be used in subsections 3.5 and 3.6.

3.5 Symmetric FVA for uncollateralized derivatives

In this subsection, we follow the argument in (3.2) of Bugard and Kjaer [1] to define what we will call the symmetric funding value adjustment (SFVA) charge, $\delta\,\text{SFVA}$. This charge represents the charge that the issuer needs to pay to, or receive from, the counterparty for the additional funding costs or benefits associated with entering a new uncollateralized derivative trade with the counterparty.

For simplicity, assume that there is no collateral, i.e. $X = 0$. Also, assume that issuer recovery rates from subsection 3.1 are $R_1 = 0$ and $R_2 = R_A$. Let the P_2-bond rate $r_2 = r_F$, where r_F will be called a funding rate.

As discussed in subsection 3.2, we need to ensure that the funding constraint (18) is satisfied:

$$\hat{\mathcal{V}} + \alpha_1 P_1 + \alpha_2 P_2 = 0.$$

This can be done by letting the amounts α_1 and α_2 of bonds purchased satisfy

$$\alpha_1 P_1 = -(\hat{\mathcal{V}} - \mathcal{V}) = -U, \tag{30}$$

$$\alpha_2 P_2 = -\alpha_1 P_1 - \hat{\mathcal{V}} = -\mathcal{V}, \tag{31}$$

which gives

$$\hat{\mathcal{V}} + \alpha_1 P_1 + \alpha_2 P_2 = \hat{\mathcal{V}} + \alpha_1 P_1 - \alpha_1 P_1 - \hat{\mathcal{V}} = 0.$$

As described by Bugard and Kjaer, this funding strategy (choice of α_1 and α_2) corresponds to the bond positions

1. Buying or selling the amount α_1 of bonds P_1 to cover the difference $\hat{\mathcal{V}} - \mathcal{V}$, which is the value adjustment addend U.
2. Buying or selling the amount α_2 of bonds P_2 to cover the remaining value $-\alpha_1 P_1 - \hat{\mathcal{V}}$, which is the standard Black-Scholes value $\mathcal{V}$.

Using this funding strategy, the hedging error ϵ_h, becomes

$$\epsilon_h = g_A + P_D = (1 - R_A)\mathcal{V}^+.$$

Bugard and Kjaer justify the above choices of α_1 and α_2 in (30) and (31) by arguing that they are feasible in practice from a regulatory perspective, since only zero-recovery bonds (P_1 bonds) are used to fund (or invest) the value adjustment addend U, which they claim "falls away upon default" of the issuer. Furthermore, bonds with recovery are only used to cover the value that does not include counterparty risk $\mathcal{V}$.

Since we assume $X = 0$, the recovered value of the derivative after default of A or B (see 1 and 2) are

$$g_A = \mathcal{V}^+ + R_A \mathcal{V}^-,$$

$$g_B = R_B \mathcal{V}^+ + \mathcal{V}^-,$$

respectively, where $\mathcal{V}$ is still is the Black-Scholes price in (28) and R_A the recovery rate of the issuer and R_B the recovery rate of the counterparty. Using this, Bugard and Kjaer show that (see subsection 3.2 in [1]) the funding cost adjustment (FCA) and debit value adjustment (DVA) satisfy

$$\mathrm{FCA}_{\mathcal{V}} = -(1-R_A)\int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^+\right] du,$$

$$\mathrm{DVA}_{\mathcal{V}} = -(1-R_A)\int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^-\right] du.$$

The adjustments are indexed by $\mathcal{V}$ to indicate their dependence on the portfolio value. Bugard and Kjaer [1] combine these two adjustments in their equation (27) to form a symmetric funding valuation adjustment for the portfolio

$$\begin{aligned}\mathrm{SFVA}_{\mathcal{V}} &:= \mathrm{FCA}_{\mathcal{V}} + \mathrm{DVA}_{\mathcal{V}} \\ &= -(1-R_A)\int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}\right] du.\end{aligned}$$

This symmetric funding valuation adjustment represents the total cost or benefit for the issuer in funding its uncollateralized derivatives. If the issuer enters into a new deal with a counterparty, resulting in a new uncollateralized derivative with value $\mathcal{E}$ being added to the portfolio, what amount should the issuer charge for the additional FVA? This charge, which we will call the symmetric funding charge, is denoted by $\delta\,\mathrm{SFVA}$. The $\delta\,\mathrm{SFVA}$ can be calculated as follows:

$$\delta\,\mathrm{SFVA} := \mathrm{SFVA}_{\mathcal{V}+\mathcal{E}} - \mathrm{SFVA}_{\mathcal{V}} = -(1-R_A)\int_t^T \lambda_A D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{E}\right] du.$$

Instead of specifying the above formula in terms of the recovery rate R_B, one might prefer to express it in terms of the funding rate r_F. According to (8) and noting that we denote $r_2 = r_F$, we have $r_F - r = (1-R_A)\lambda_A$. Thus, the symmetric funding charge is given by

$$\delta\,\mathrm{SFVA} = -(r_F - r)\int_t^T D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{E}\right] du. \tag{32}$$

From a computational viewpoint, the SFVA-charge formula has the advantage that it is sufficient to sample only the value process $\mathcal{E}$, rather than the total portfolio value $\mathcal{V}$, if we estimate $\delta\,\mathrm{SFVA}$ by Monte Carlo method. One is also usually able to compute the expectation $\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{E}\right]$ analytically.

3.6 Asymmetric FVA for uncollateralized derivatives

In practice, an issuer might not have the same funding rate r_F for borrowing and lending. This means that the issuer can borrow at the rate r_I and lend at a different rate r_{II}. This asymmetric case is studied by Gregory [3] in a discrete form (see p.343).

The asymmetric funding cost adjustment (AFCA) and asymmetric funding benefit adjustment (AFBA) at time t for an uncollateralized derivative portfolio are defined as

$$\mathrm{AFCA}_{\mathcal{V}} := -(r_I - r)\int_t^T D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^+\right] du, \tag{33}$$

$$\mathrm{AFBA}_{\mathcal{V}} := -(r_{II} - r)\int_t^T D_{r+\lambda_A+\lambda_B}(t,u)\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^-\right] du. \tag{34}$$

Note that the $\mathrm{AFBA}_{\mathcal{V}}$ replaces the DVA from the previous subsection. Combining the AFCA and AFBA gives the asymmetric funding value adjustment for the portfolio

$$\mathrm{AFVA}_{\mathcal{V}} := \mathrm{AFCA}_{\mathcal{V}} + \mathrm{AFBA}_{\mathcal{V}}\,. \tag{35}$$

The charge $\delta\,\mathrm{AFVA}$ for adding a new uncollateralized derivative is therefore

$$\delta\,\mathrm{AFVA} := \mathrm{AFVA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFVA}_{\mathcal{V}}, \tag{36}$$

which we will refer to as the asymmetric funding charge. Although this charge is based on the more realistic assumption that the issuer has different rates for borrowing and lending, it could be computationally expensive. This is because the entire portfolio value $\mathcal{V}$ needs to be sampled, as estimates are required for $\mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^{+}\right], \mathbb{E}_t^{\mathbb{Q}}\left[\mathcal{V}^{-}\right]$, $\mathbb{E}_t^{\mathbb{Q}}\left[(\mathcal{V}+\mathcal{E})^{+}\right]$ and $\mathbb{E}_t^{\mathbb{Q}}\left[(\mathcal{V}+\mathcal{E})^{-}\right]$.

In the following two sections, we will study and compare these two types of funding charges, i.e, $\delta\,\mathrm{SFVA}$ as in (32) and $\delta\,\mathrm{AFVA}$ as in (36).

4 A simplified FVA model

As discussed in subsection 3.6, computing the asymmetric funding charge $\delta\,\mathrm{AFVA}$ using Monte Carlo methods can be computationally expensive for a portfolio with many derivatives, as it requires sampling the entire portfolio, which leads to a high number of random samples. Therefore, in this section, we will consider a simplified model in subsection 4.1 and show through an asymptotic argument, in subsection 4.2, that the asymmetric funding charge $\delta\,\mathrm{AFVA}$ converges to a symmetric funding charge $\delta\,\mathrm{SFVA}$ (see subsection 3.5). Under the same model, we will then derive an approximation formula for $\delta\,\mathrm{AFVA}$, which we will denote as $\delta\overline{\mathrm{AFVA}}$, in subsection 4.3.

4.1 A Gaussian model

Everything in this model is formulated under the pricing measure $\mathbb{Q}$, and for notational simplicity, the notation on $\mathbb{Q}$ and the filtration up to time t in the expectations is dropped. Assume that the bank's total uncollateralized derivatives portfolio value in domestic currency, denoted $\mathcal{V}$, follows a Gaussian distribution with mean $\mu(t)$ and variance $\sigma_{\mathcal{V}}^2(t)$, both bounded functions of t. Suppose that a new derivative is added to the portfolio, with its value $\mathcal{E}$ also distributed according to a Gaussian distribution with mean $\alpha(t)$ and variance $\sigma_{\mathcal{E}}^2(t)$, again, both bounded functions of t. Additionally, assume that the correlation between the portfolio value and the derivative value is $\rho(t)$. Furthermore, it is assumed that the derivative is fully uncollateralized. By properties of the Gaussian distribution, we know that

$$\mathcal{V}+\mathcal{E} \sim \mathcal{N}\left(\mu(t)+\alpha(t), \sigma_{\mathcal{V}}^2(t)+\sigma_{\mathcal{E}}^2(t)+2\rho(t)\sigma_{\mathcal{V}}(t)\sigma_{\mathcal{E}}(t)\right).$$

The choice of Gaussian distributions makes it possible to derive an analytical expression for $\delta\,\mathrm{AFVA}$, in the following section 4.2. Except for when we want to emphasize the time dependence of $\mu, \alpha, \sigma_{\mathcal{V}}, \sigma_{\mathcal{E}}$ and ρ, we will not write it out.

4.2 An asymptotic argument

We will now perform an asymptotic analysis to understand what happens to the asymmetric funding charge $\delta\,\mathrm{AFVA}$ as the portfolio expected value μ tends to $\pm\infty$. As discussed in subsection 3.6 , the funding charge $\delta\,\mathrm{AFVA}$ from adding a new derivative with value $\mathcal{E}$ is given by (36), that is,

$$\begin{aligned}
\delta\,\mathrm{AFVA} &= \mathrm{AFVA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFVA}_{\mathcal{V}} && (37)\\
&= (\mathrm{AFCA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFCA}_{\mathcal{V}}) + (\mathrm{AFBA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFBA}_{\mathcal{V}}) && (38)\\
&= -(r_I - r)\int_t^T \left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right]\right) D_{r+\lambda_A+\lambda_B}(t,u)\,du && (39)\\
&\quad - (r_{II} - r)\int_t^T \left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right] - \mathbb{E}\left[\mathcal{V}^-\right]\right) D_{r+\lambda_A+\lambda_B}(t,u)\,du. && (40)
\end{aligned}$$

More specifically, we will look at

$$\lim_{\mu\to\infty} \delta\,\mathrm{AFVA} \quad \text{and} \quad \lim_{\mu\to-\infty} \delta\,\mathrm{AFVA}\,.$$

Theorem 4.1. *Assume that the portfolio and derivative price process, $\mathcal{V}$ and $\mathcal{E}$, respectively, follow the distributions of the Gaussian model in subsection 4.1. Also assume that μ and $\sigma_{\mathcal{V}}^2$ are real constants. Then,*

$$\lim_{\mu\to\infty} \delta \, \mathrm{AFVA} = -(r_I - r) \int_t^T \mathbb{E}\left[\mathcal{E}\right] D_{r+\lambda_A+\lambda_B}(t,u) \, du, \tag{41}$$

$$\lim_{\mu\to-\infty} \delta \, \mathrm{AFVA} = -(r_{II} - r) \int_t^T \mathbb{E}\left[\mathcal{E}\right] D_{r+\lambda_A+\lambda_B}(t,u) \, du. \tag{42}$$

Proof. Notice that the only μ dependence in (39) and (40) comes from,

$$\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right] \quad \text{and} \quad \mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right] - \mathbb{E}\left[\mathcal{V}^-\right].$$

So we will look at the limits

$$\lim_{\mu\to\pm\infty} (\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right]) \quad \text{and} \quad \lim_{\mu\to\pm\infty} (\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right] - \mathbb{E}\left[\mathcal{V}^-\right]).$$

First, note that by claim B.1, in the appendix, with $s^2 = \sigma_{\mathcal{V}}^2 + \sigma_{\mathcal{E}}^2 + 2\rho\sigma_{\mathcal{V}}\sigma_{\mathcal{E}}$, we have

$$\begin{aligned}
\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right] = &\frac{1}{2}\mu\left(\mathrm{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right) - \mathrm{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right) \\
&+ \frac{1}{2}\left(\sqrt{\frac{2s^2}{\pi}}\exp\left(\frac{-(\mu+\alpha)^2}{2s^2}\right) - \sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}}\exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)\right) \\
&+ \frac{\alpha}{2}\,\mathrm{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right) + \frac{\alpha}{2},
\end{aligned} \tag{43}$$

where $\mathrm{erf} : \mathbb{R} \to [-1, 1]$ is given by

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

Furthermore, we have

$$\lim_{x\to\pm\infty} \mathrm{erf}\,(x) = \pm 1.$$

Then, taking the limit yields

$$
\begin{aligned}
\lim_{\mu\to\infty}\left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right]-\mathbb{E}\left[\mathcal{V}^+\right]\right) &= \lim_{\mu\to\infty}\frac{1}{2}\mu\left(\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)-\operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right)+\lim_{\mu\to\infty}\frac{\alpha}{2}\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)+\frac{1}{2}\alpha \\
&+\lim_{\mu\to\infty}\frac{1}{2}\left(\sqrt{\frac{2s^2}{\pi}}\exp\left(\frac{-(\mu+\alpha)^2}{2s^2}\right)-\sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}}\exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)\right) \\
&= \lim_{\mu\to\infty}\frac{1}{2}\mu\left(\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)-\operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right)+\frac{\alpha}{2}+\frac{\alpha}{2} \\
&= \lim_{\mu\to\infty}\frac{1}{2}\frac{\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)-\operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)}{\frac{1}{\mu}}+\alpha \\
&= \lim_{\mu\to\infty}\frac{1}{2}\frac{\frac{\sqrt{2}}{\sqrt{\pi s^2}}\exp\left(\frac{-(\mu+\alpha)^2}{2s^2}\right)-\frac{\sqrt{2}}{\sqrt{\pi\sigma_{\mathcal{V}}^2}}\exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{-\frac{1}{\mu^2}}+\alpha \qquad \text{(L'Hôpital's rule)} \\
&= \alpha-\frac{1}{\sqrt{2\pi s^2}}\lim_{\mu\to\infty}\mu^2\exp\left(\frac{-(\mu+\alpha)^2}{2s^2}\right)+\frac{1}{\sqrt{2\pi\sigma_{\mathcal{V}}^2}}\lim_{\mu\to\infty}\mu^2\exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right) \\
&= \alpha, \qquad (44)
\end{aligned}
$$

where the last equality holds due to that an exponential function decay faster than a polynomial function. A similar calculation as (44) gives

$$
\lim_{\mu\to-\infty}\left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right]-\mathbb{E}\left[\mathcal{V}^+\right]\right)=0. \qquad (45)
$$

Furthermore, using claim B.2, in the appendix, we have that

$$
\begin{aligned}
\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right]-\mathbb{E}\left[\mathcal{V}^-\right] &= -\frac{1}{2}\mu\left(\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)-\operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right) \\
&-\frac{1}{2}\left(\sqrt{\frac{2s^2}{\pi}}\exp\left(\frac{-(\mu+\alpha)^2}{2s^2}\right)-\sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}}\exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)\right) \qquad (46) \\
&-\frac{\alpha}{2}\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2s^2}}\right)+\frac{\alpha}{2}.
\end{aligned}
$$

Similar calculations as for (44) gives

$$
\lim_{\mu\to\infty}\left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right]-\mathbb{E}\left[\mathcal{V}^-\right]\right)=0 \quad \text{and} \quad \lim_{\mu\to-\infty}\left(\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right]-\mathbb{E}\left[\mathcal{V}^-\right]\right)=\alpha. \qquad (47)
$$

Therefore, using integrals (39) and (40) and limits (44), (45) and (47), and $\alpha = \mathbb{E}[\mathcal{E}]$, and interchanging limit and integral sign, yields

$$
\lim_{\mu\to\infty}\delta\,\mathrm{AFVA} = -(r_I - r)\int_t^T \mathbb{E}[\mathcal{E}]\,D_{r+\lambda_A+\lambda_B}(t,u)\,du, \qquad (48)
$$

$$
\lim_{\mu\to-\infty}\delta\,\mathrm{AFVA} = -(r_{II} - r)\int_t^T \mathbb{E}[\mathcal{E}]\,D_{r+\lambda_A+\lambda_B}(t,u)\,du. \qquad (49)
$$

□

We can interchange limit and integral sign because, by definition, $\mathbb{E}[\mathcal{E}]$ is bounded and

$$|D_{r+\lambda_A+\lambda_B}(t,u)| = \left|e^{-(r+\lambda_A+\lambda_B)(u-t)}\right| \leq 1 \quad \text{for all } t \leq u \leq T,$$

also shows that $D_{r+\lambda_A+\lambda_B}(t,u)$ is bounded. We can therefore apply the Lebesgue dominated convergence theorem.

Notice that, the above limits (48) and (49) are very similar to the symmetric funding charge formula δ SFVA in (32) (subsection 3.5), (48) and (49) becomes (32) if we replace r_F by r_I and r_{II}, respectively.

It is important to note that in the differences (43) and (46), the expected portfolio value μ has to increase faster relative to $\sigma_{\mathcal{V}}$. Suppose $\sigma_{\mathcal{V}} = \sigma_{\mathcal{V}}(\mu)$, such that $\sigma_{\mathcal{V}}$ increases at the same rate as μ, i.e., $\mu/\sigma_{\mathcal{V}}$ is a constant, then the results (48) and (49) would not hold. This observation will be important in section 5 where we will specify different portfolios with increasing expected value, to see if the results (48) and (49) hold in another model.

An important insight gained from the asymptotic result is that one should use δ SFVA instead of δ AFVA if the expected value of the portfolio is large in magnitude.

4.3 Approximate AFVA charge

The asymptotic argument in the above subsection 4.2 gives us information that the asymmetric funding charge δ AFVA converges to a symmetric funding charge δ SFVA when the expected portfolio value is large (tends to infinity) in the Gaussian model. But we are also interested in the case when the expected portfolio value is close to zero. Therefore, we derive an approximation formula for δ AFVA.

To begin with, we first estimate how much $\mathbb{E}[\mathcal{V}^+]$ and $\mathbb{E}[\mathcal{V}^-]$ change after adding the derivative with value $\mathcal{E}$. We use the notation

$$E_+(\alpha,\sigma_{\mathcal{E}}) := \mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] \quad \text{and} \quad E_-(\alpha,\sigma_{\mathcal{E}}) := \mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right], \tag{50}$$

which are the expected positive exposure and expected negative exposure after adding a new derivative to the portfolio, respectively. Then the marginal changes in the expected positive exposure $\mathbb{E}[\mathcal{V}^+]$ and the expected negative exposure $\mathbb{E}[\mathcal{V}^-]$ after adding $\mathcal{E}$ are defined by

$$\begin{aligned}
\epsilon_+ &:= \lim_{\sigma_{\mathcal{E}}\to 0} \frac{E_+(0,\sigma_{\mathcal{E}}) - \mathbb{E}[\mathcal{V}^+]}{\sigma_{\mathcal{E}}}, \\
\delta_+ &:= \lim_{\alpha\to 0} \frac{E_+(\alpha,0) - \mathbb{E}[\mathcal{V}^+]}{\alpha}, \\
\epsilon_- &:= \lim_{\sigma_{\mathcal{E}}\to 0} \frac{E_-(0,\sigma_{\mathcal{E}}) - \mathbb{E}[\mathcal{V}^-]}{\sigma_{\mathcal{E}}}, \\
\delta_- &:= \lim_{\alpha\to 0} \frac{E_-(\alpha,0) - \mathbb{E}[\mathcal{V}^-]}{\alpha}.
\end{aligned}$$

The values ϵ_+ and ϵ_- can be seen as incremental changes, per unit of expected derivative value, in the expected positive exposure $\mathbb{E}[\mathcal{V}^+]$ and expected negative exposure $\mathbb{E}[\mathcal{V}^-]$ after adding $\mathcal{E}$. Likewise, δ_+ and δ_- can be seen as the incremental changes, per unit of standard deviation in the derivative value.

Theorem 4.2. *The marginal changes satisfy*

$$\epsilon_+ = \frac{\rho}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2\sigma_\mathcal{V}^2}} \quad \textit{and} \quad \delta_+ = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_\mathcal{V}^2}}\right)\right),$$

$$\epsilon_- = -\frac{\rho}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2\sigma_\mathcal{V}^2}} \quad \textit{and} \quad \delta_- = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_\mathcal{V}^2}}\right)\right).$$

The proof of theorem 4.2 is split up into two lemmas. The claims B.1 and B.2 in appendix B are computational foundations which are used to show the two lemmas.

Lemma 4.1. *If $\alpha = 0$, then*

$$\epsilon_+ = \lim_{\sigma_\mathcal{E} \to 0} \frac{\mathbb{E}[(\mathcal{V}+\mathcal{E})^+] - \mathbb{E}[\mathcal{V}^+]}{\sigma_\mathcal{E}} = \frac{\rho}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2\sigma_\mathcal{V}^2}} \tag{51}$$

and

$$\epsilon_- = \lim_{\sigma_\mathcal{E} \to 0} \frac{\mathbb{E}[(\mathcal{V}+\mathcal{E})^-] - \mathbb{E}[\mathcal{V}^-]}{\sigma_\mathcal{E}} = -\frac{\rho}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2\sigma_\mathcal{V}^2}}. \tag{52}$$

Proof. We begin by showing (51). Using claim B.1, then with $s^2 = \sigma_\mathcal{V}^2 + \sigma_\mathcal{E}^2 + 2\rho\sigma_\mathcal{V}\sigma_\mathcal{E}$, we have

$$\begin{aligned} &\frac{\mathbb{E}[(\mathcal{V}+\mathcal{E})^+] - \mathbb{E}[\mathcal{V}^+]}{\sigma_\mathcal{E}} \\ &= \frac{1}{2}\mu \frac{\operatorname{erf}\left(\frac{\mu}{\sqrt{2s^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_\mathcal{V}^2}}\right)}{\sigma_\mathcal{E}} + \frac{1}{2}\frac{\sqrt{\frac{2s^2}{\pi}}\exp\left(\frac{-\mu^2}{2s^2}\right) - \sqrt{\frac{2\sigma_\mathcal{V}^2}{\pi}}\exp\left(\frac{-\mu^2}{2\sigma_\mathcal{V}^2}\right)}{\sigma_\mathcal{E}}. \end{aligned} \tag{53}$$

Furthermore, we have that the derivative

$$\frac{ds}{d\sigma_\mathcal{E}} = \frac{2\sigma_\mathcal{E} + 2\rho\sigma_\mathcal{V}}{2\sqrt{\sigma_\mathcal{V}^2 + \sigma_\mathcal{E}^2 + 2\rho\sigma_\mathcal{V}\sigma_\mathcal{E}}} = \frac{\sigma_\mathcal{E} + \rho\sigma_\mathcal{V}}{s}. \tag{54}$$

Thus, using the above equation (54) yields

$$\frac{d}{d\sigma_\mathcal{E}}\left(\frac{\mu}{\sqrt{2}s}\right) = \frac{-\mu}{\sqrt{2}s^2}\frac{ds}{d\sigma_\mathcal{E}} = \frac{-\mu(\sigma_\mathcal{E} + \rho\sigma_\mathcal{V})}{\sqrt{2}s^3}. \tag{55}$$

Using L'Hôpital's rule, the derivative of the erf-function and equation (55), we find

$$\begin{aligned} \lim_{\sigma_\mathcal{E} \to 0} \frac{\operatorname{erf}\left(\frac{\mu}{\sqrt{2s^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_\mathcal{V}^2}}\right)}{\sigma_\mathcal{E}} &= \lim_{\sigma_\mathcal{E} \to 0} \frac{\frac{2}{\sqrt{\pi}}\exp\left(\frac{-\mu^2}{2s^2}\right) \cdot \frac{-\mu(\sigma_\mathcal{E}+\rho\sigma_\mathcal{V})}{\sqrt{2}s^3}}{1} \\ &= -\sqrt{\frac{2}{\pi}}\frac{\mu\rho}{\sigma_\mathcal{V}^2}\exp\left(\frac{-\mu^2}{2\sigma_\mathcal{V}^2}\right), \end{aligned} \tag{56}$$

where the last equation holds due to $s \to \sigma_{\mathcal{V}}$ as $\sigma_{\mathcal{E}} \to 0$. Furthermore, using L'Hôpital's rule again,

$$\lim_{\sigma_{\mathcal{E}} \to 0} \frac{\sqrt{\frac{2s^2}{\pi}} \exp\left(\frac{-\mu^2}{2s^2}\right) - \sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}} \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{\sigma_{\mathcal{E}}}$$
$$= \lim_{\sigma_{\mathcal{E}} \to 0} \left(\sqrt{\frac{2}{\pi}} \frac{\sigma_{\mathcal{E}} + \rho\sigma_{\mathcal{V}}}{s} \exp\left(\frac{-\mu^2}{2s^2}\right) + \sqrt{\frac{2s^2}{\pi}} \frac{\mu^2(\sigma_{\mathcal{E}} + \rho\sigma_{\mathcal{V}})}{s^4} \exp\left(\frac{-\mu^2}{2s^2}\right) \right)$$
$$= \sqrt{\frac{2}{\pi}} \left(\rho + \frac{\mu^2\rho}{\sigma_{\mathcal{V}}^2}\right) \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right). \tag{57}$$

Combining the two limits (56) and (57) gives

$$\epsilon_+ = \frac{\rho}{\sqrt{2\pi}} \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right). \tag{58}$$

This gives the first part (51) of the lemma. We now turn to equation (52). With $\alpha = 0$, using claim B.2 we get

$$\frac{\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right] - \mathbb{E}\left[\mathcal{V}^-\right]}{\sigma_{\mathcal{E}}} = -\frac{1}{2}\mu \frac{\operatorname{erf}\left(\frac{\mu}{\sqrt{2s^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)}{\sigma_{\mathcal{E}}}$$
$$- \frac{1}{2} \frac{\sqrt{\frac{2s^2}{\pi}} \exp\left(\frac{-\mu^2}{2s^2}\right) - \sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}} \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{\sigma_{\mathcal{E}}}.$$

Using the same argument as we did in showing (51) yields

$$\epsilon_- = -\frac{\rho}{\sqrt{2\pi}} \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right).$$

This completes the proof. □

Lemma 4.2. *If* $\sigma_{\mathcal{E}} = 0$*, then*

$$\delta_+ = \lim_{\alpha \to 0} \frac{\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right]}{\alpha} = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right) \tag{59}$$

and

$$\delta_- = \lim_{\alpha \to 0} \frac{\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^-\right] - \mathbb{E}\left[\mathcal{V}^-\right]}{\alpha} = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right). \tag{60}$$

Proof. We begin by showing (59). Again, using claim B.1 with $\sigma_{\mathcal{E}} = 0$ gives

$$\frac{\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right]}{\alpha}$$
$$= \frac{1}{2}\mu \frac{\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)}{\alpha} + \frac{1}{2}\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)$$
$$+ \frac{1}{2}\sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}} \frac{\exp\left(\frac{-(\mu+\alpha)^2}{2\sigma_{\mathcal{V}}^2}\right) - \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{\alpha} + \frac{1}{2}.$$

Then, using L'Hôpital's rule and the derivative of the erf-function,

$$\lim_{\alpha \to 0} \frac{\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)}{\alpha} = \lim_{\alpha \to 0} \frac{2}{\sqrt{2\pi\sigma_{\mathcal{V}}^2}} \exp\left(-\frac{(\mu+\alpha)^2}{2\sigma_{\mathcal{V}}^2}\right)$$
$$= \frac{2}{\sqrt{2\pi\sigma_{\mathcal{V}}^2}} \exp\left(-\frac{\mu^2}{2\sigma_{\mathcal{V}}^2}\right). \tag{61}$$

Additionally, using L'Hôpital's rule gives

$$\lim_{\alpha \to 0} \frac{\exp\left(\frac{-(\mu+\alpha)^2}{2\sigma_{\mathcal{V}}^2}\right) - \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{\alpha} = \lim_{\alpha \to 0} \frac{-2(\mu+\alpha)}{2\sigma_{\mathcal{V}}^2} \exp\left(\frac{-(\mu+\alpha)^2}{2\sigma_{\mathcal{V}}^2}\right)$$
$$= -\frac{\mu}{\sigma_{\mathcal{V}}^2} \exp\left(-\frac{\mu^2}{2\sigma_{\mathcal{V}}^2}\right). \tag{62}$$

Clearly, the continuity of the erf-function gives

$$\lim_{\alpha \to 0} \frac{1}{2} \operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) = \frac{1}{2} \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right). \tag{63}$$

Combining the limits (61), (62) and (63) gives

$$\begin{aligned}\delta_+ &= \frac{1}{2}\mu \frac{2}{\sqrt{2\pi\sigma_{\mathcal{V}}^2}} \exp\left(-\frac{\mu^2}{2\sigma_{\mathcal{V}}^2}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) \\ &- \frac{1}{2}\sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}} \frac{\mu}{\sigma_{\mathcal{V}}^2} \exp\left(-\frac{\mu^2}{2\sigma_{\mathcal{V}}^2}\right) + \frac{1}{2} \\ &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right),\end{aligned}$$

which shows (59). We now turn to showing (60), using claim B.2 and $\sigma_{\mathcal{E}} = 0$

$$\begin{aligned}&\frac{\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] - \mathbb{E}\left[\mathcal{V}^+\right]}{\alpha} \\ &= -\frac{1}{2}\mu \frac{\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)}{\alpha} - \frac{1}{2}\operatorname{erf}\left(\frac{\mu+\alpha}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right) \\ &- \frac{1}{2}\sqrt{\frac{2\sigma_{\mathcal{V}}^2}{\pi}} \frac{\exp\left(\frac{-(\mu+\alpha)^2}{2\sigma_{\mathcal{V}}^2}\right) - \exp\left(\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}\right)}{\alpha} + \frac{1}{2}.\end{aligned}$$

Using the same argument as we did in (59), one gets

$$\delta_- = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right),$$

which concludes the proof. □

Now we will derive an approximation formula for $\delta\,\text{AFVA}$ using the marginal changes in theorem 4.2. The approximation formula will be denoted $\delta\overline{\text{AFVA}}$.

Theorem 4.3. *Assume that the portfolio and derivative price process, $\mathcal{V}$ and $\mathcal{E}$, respectively, follow the distributions of the Gaussian model in subsection 4.1. Also assume that $r_I = f + s$ and $r_{II} = f - s$, for some mid-rate f and funding spread s. Then,*

$$\delta\overline{\text{AFVA}}(\alpha, \sigma_{\mathcal{E}}) = -\int_t^T \left(\left(f - r + s \cdot \text{erf}\left(\frac{\mu(u)}{\sqrt{2\sigma_{\mathcal{V}}^2(u)}} \right) \right) \cdot \alpha(u) + \frac{2\rho s}{\sqrt{2\pi}} e^{\frac{-\mu^2(u)}{2\sigma_{\mathcal{V}}^2(u)}} \cdot \sigma_{\mathcal{E}}(u) \right) D_{r+\lambda_B+\lambda_C}(t,u)\, du$$

Proof. With the first-order Taylor linearizations we get

$$E_+(\alpha, \sigma_{\mathcal{E}}) \approx \mathbb{E}\left[\mathcal{V}^+\right] + \delta_+\alpha + \epsilon_+\sigma_{\mathcal{E}}, \tag{64}$$

$$E_-(\alpha, \sigma_{\mathcal{E}}) \approx \mathbb{E}\left[\mathcal{V}^-\right] + \delta_-\alpha + \epsilon_-\sigma_{\mathcal{E}}. \tag{65}$$

So the change in $\mathbb{E}\left[\mathcal{V}^+\right]$ and $\mathbb{E}\left[\mathcal{V}^-\right]$ from adding $\mathcal{E}$ is estimated by

$$e_+(\alpha, \sigma_{\mathcal{E}}) := \delta_+\alpha + \epsilon_+\sigma_{\mathcal{E}} \quad \text{and} \quad e_-(\alpha, \sigma_{\mathcal{E}}) := \delta_-\alpha + \epsilon_-\sigma_{\mathcal{E}},$$

respectively. Using the formulas (33) and (34) from subsection 3.6, the notation in (50) and linearizations (64) and (65), we have the approximate change funding cost adjustment

$$\begin{aligned}
&\text{AFCA}_{\mathcal{V}+\mathcal{E}} - \text{AFCA}_{\mathcal{V}} \\
&= -\int_t^T (r_I - r)(E_+(\alpha, \sigma_{\mathcal{E}}))D_{r+\lambda_B+\lambda_C}(t,u)\, du + \int_t^T (r_I - r)\mathbb{E}\left[\mathcal{V}^+\right] D_{r+\lambda_B+\lambda_C}(t,u)\, du \\
&\approx -\int_t^T (r_I - r)(\mathbb{E}\left[\mathcal{V}^+\right] + \delta_+\alpha + \epsilon_+\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\, du \\
&+ \int_t^T (r_I - r)\mathbb{E}\left[\mathcal{V}^+\right] D_{r+\lambda_B+\lambda_C}(t,u)\, du \\
&= -\int_t^T (r_I - r)(\delta_+\alpha + \epsilon_+\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\, du \\
&= -\int_t^T (r_I - r)e_+(\alpha, \sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\, du.
\end{aligned}$$

We denote the last term as $\delta\overline{\text{AFCA}}(\alpha, \sigma_{\mathcal{E}})$, that is,

$$\delta\overline{\text{AFCA}}(\alpha, \sigma_{\mathcal{E}}) := -\int_t^T (r_I - r)e_+(\alpha, \sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\, du. \tag{66}$$

Furthermore, we have the approximate change funding benefit adjustment

$$\begin{aligned}
&\mathrm{AFBA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFBA}_{\mathcal{V}} \\
&= -\int_t^T (r_{II}-r)(E_-(\alpha,\sigma_{\mathcal{E}}))D_{r+\lambda_B+\lambda_C}(t,u)\,du + \int_t^T (r_{II}-r)\mathbb{E}\left[\mathcal{V}^-\right] D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&\approx -\int_t^T (r_{II}-r)(\mathbb{E}\left[\mathcal{V}^-\right] + \delta_-\alpha + \epsilon_-\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&+ \int_t^T (r_{II}-r)\mathbb{E}\left[\mathcal{V}^-\right] D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&= -\int_t^T (r_{II}-r)(\delta_-\alpha + \epsilon_-\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&= -\int_t^T (r_{II}-r)e_-(\alpha,\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\,du.
\end{aligned}$$

We denote the last term as $\delta\overline{\mathrm{AFBA}}(\alpha,\sigma_{\mathcal{E}})$, that is,

$$\delta\overline{\mathrm{AFBA}}(\alpha,\sigma_{\mathcal{E}}) := -\int_t^T (r_{II}-r)e_-(\alpha,\sigma_{\mathcal{E}})D_{r+\lambda_B+\lambda_C}(t,u)\,du. \tag{67}$$

To find an approximation for the asymmetric funding charge $\delta\,\mathrm{AFVA}$, we use formulas (35) and (36) from subsection 3.6 and the approximations (66) and (67) to show:

$$\begin{aligned}
\delta\,\mathrm{AFVA} &= \mathrm{AFVA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFVA}_{\mathcal{V}} \\
&= (\mathrm{AFCA}_{\mathcal{V}+\mathcal{E}} + \mathrm{AFBA}_{\mathcal{V}+\mathcal{E}}) - (\mathrm{AFCA}_{\mathcal{V}} + \mathrm{AFBA}_{\mathcal{V}}) \\
&= (\mathrm{AFCA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFCA}_{\mathcal{V}}) + (\mathrm{AFBA}_{\mathcal{V}+\mathcal{E}} - \mathrm{AFBA}_{\mathcal{V}}) \\
&\approx \delta\overline{\mathrm{AFCA}}(\alpha,\sigma_{\mathcal{E}}) + \delta\overline{\mathrm{AFBA}}(\alpha,\sigma_{\mathcal{E}}).
\end{aligned}$$

Therefore, the approximate change in AFVA is given by:

$$\delta\overline{\mathrm{AFVA}}(\alpha,\sigma_{\mathcal{E}}) := \delta\overline{\mathrm{AFCA}}(\alpha,\sigma_{\mathcal{E}}) + \delta\overline{\mathrm{AFBA}}(\alpha,\sigma_{\mathcal{E}}).$$

We assume that $r_I = f+s$ and $r_{II} = f-s$ for some mid-rate f and spread $s>0$, i.e., it is more expensive for the issuer to borrow money than to lend money. Then the cost that the issuer wants to charge for the new uncollateralized derivative with value $\mathcal{E}$ is approximated by:

$$\begin{aligned}
\delta\overline{\mathrm{AFVA}}(\alpha,\sigma_{\mathcal{E}}) &= -\int_t^T \left(\left(f - r + s\cdot\mathrm{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\right)\cdot\alpha + \frac{2\rho s}{\sqrt{2\pi}}e^{\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}}\cdot\sigma_{\mathcal{E}}\right) D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&= -\int_t^T (f-r)\cdot\alpha\cdot D_{r+\lambda_B+\lambda_C}(t,u)\,du \\
&- \int_t^T \left(s\cdot\mathrm{erf}\left(\frac{\mu}{\sqrt{2\sigma_{\mathcal{V}}^2}}\right)\cdot\alpha + \frac{2\rho s}{\sqrt{2\pi}}e^{\frac{-\mu^2}{2\sigma_{\mathcal{V}}^2}}\cdot\sigma_{\mathcal{E}}\right) D_{r+\lambda_B+\lambda_C}(t,u)\,du.
\end{aligned}$$

□

Observe that the approximation formula is a sum of a term similar to SFVA,

$$-\int_t^T (f-r)\cdot\alpha(u)\cdot D_{r+\lambda_B+\lambda_C}(t,u)du$$

and an additional addend, which we call the asymmetric correction.

$$-\int_t^T \left(s\cdot\mathrm{erf}\left(\frac{\mu(u)}{\sqrt{2\sigma_{\mathcal{V}}^2(u)}}\right)\cdot\alpha(u)+\frac{2\rho s}{\sqrt{2\pi}}e^{\frac{-\mu^2(u)}{2\sigma_{\mathcal{V}}^2(u)}}\cdot\sigma_{\mathcal{E}}(u)\right)D_{r+\lambda_B+\lambda_C}(t,u)\,du. \tag{68}$$

This approximation formula in theorem 4.3 could, for instance, be used as a rule of thumb for estimating the asymmetric funding charge in cases where an issuer is unable to compute the exact asymmetric funding charge.

A brief analysis of the approximation formula (theorem 4.3) is conducted in appendix C. This analysis aims to investigate the the magnitude of the $\delta\overline{\text{AFVA}}$ charge and the asymmetric correction addend.

5 Foreign Exchange Simulations

In this chapter, inspired by the asymptotic argument from subsection4.2, we will examine both how the asymmetric charge $\delta\,\mathrm{AFVA}$ and the symmetric charge $\delta\,\mathrm{SFVA}$ compare in size to each other, as well as whether the difference between them disappears as the value of the portfolio increases. We will conduct the analysis numerically using Monte Carlo estimates in a diffusion model.

We chose to conduct the analysis using a foreign exchange (FX) model for several reasons. Firstly, this model closely aligns with the approaches used in practice at SEB, making the results more applicable to their existing framework. Additionally, the choice enhances explainability, as it is easier to illustrate and communicate the model's structure when its components have recognizable, real-world names. Furthermore, many derivatives pricing models rely on geometric Brownian motions, which we also employ extensively, and so this model is not only relevant for our specific context but also generalizable to other similar interest rate or equity models.

In subsection 5.1, the FX model used is presented. Then, in subsection 5.2, we describe how we estimate the funding charges using Monte Carlo in the model. Furthermore, in subsection 5.3, we specify what parameters that we will use. Lastly, the resulting funding charges are presented in subsection 5.4.

5.1 The model

We begin by explaining the model we will use to calculate the funding charges. Every stochastic process is constructed under the pricing measure $\mathbb{Q}$.

Assume that there is a bank which uses USD as its main currency (this is the only bank that we consider throughout this subsection). The bank has a counterparty that uses several currencies: EUR, GBP, SEK, JPY, NOK, and USD. The currencies EUR, GBP, SEK, JPY, and NOK have exchange rates (FX rates) with respect to USD, denoted by S_t^i for $i = 1, 2, \ldots, 5$. These FX rates are assumed to be random processes to reflect the seemingly random behavior of FX rates in the real world. Here, $i = 1$ corresponds to the FX rate EUR to USD, and $i = 5$ the FX rate NOK to USD. To clarify the FX rates further, if the bank wants to buy an amount Z EUR at time t, the bank has to pay $S_t^1 \cdot Z$ USD. For instance, if $S_t^1 = 1.2$, then buying 1 EUR costs 1.2 USD.

Furthermore, since all the FX rates are assumed to be priced in USD, it is reasonable to assume that there is some correlation among them. For example, policy changes by the Federal Reserve are likely to impact all FX rates in USD, resulting in correlated movements among the FX rates.

We now specify the model. Let W^i for $i = 1, \ldots, 5$ be correlated Wiener processes, such that their quadratic variation is $\langle W^i, W^j \rangle_t = \rho_{ij} t$, with ρ_{ij}, constant for all i, j, representing the correlation between the FX rates. Let $\sigma_i > 0$ be constant for all i. Assume that the described FX rates follow the dynamics

$$\begin{aligned} dS_t^i &= \sigma_i S_t^i \, dW_t^i, \qquad i = 1, \ldots, 5, \\ S_0^i &= s_0^i. \end{aligned} \tag{69}$$

The process which satisfies (69) is known as geometric Brownian motion, which is commonly used as FX rate models as well as stock/equity models (see, for example, [10], Chapter 18). However, we simplify the model further by choosing the drift terms to be zero, as this results in a convenient form for the fair value (price) of a forward contract on an FX rate (referred to as an FX forward). More specifically, it can be shown that the fair value of a forward contract on the FX rate S^i with contract time 0, delivery time T and forward amount K (see appendix A.9), denoted as $\Pi_t^i(K,T)$, is given by

$$\Pi_t^i(K,T) = S_t^i - K, \quad \text{for all } t \in [0,T]. \tag{70}$$

Furthermore, assume that the bank holds a constant portfolio $\beta \in \mathbb{R}^5$ of FX forwards on all the FX rates S^i, with a maturity time $\mathcal{T}$ years (very far into the future), and with a forward amount $K = 0$. Also assume that the portfolio has some initial amount $\beta_0 \in \mathbb{R}$ in USD, which can be used to translate the portfolio's expected value for analysis purposes.

Assuming that the portfolio β is constant might seem to be a simplification of reality. In a "real portfolio", some of the FX forwards would mature (i.e. reach their maturity date) at different times. However, when an FX forward matures, it is usually replaced by a new FX forward from a trade with another counterparty. Therefore, we may just consider a constant portfolio. Moreover, because banks continuously enter into new derivative trades, we can model this as if the longest maturity time of the FX forwards is very far into the future, hence we use a common maturity time $\mathcal{T}$ years for the portfolio β.

Choosing $K = 0$ for all FX forwards in the portfolio is also for simplicity. Using (70), the value process $\mathcal{V}_t$ of the portfolio β is given by

$$\mathcal{V}_t = \beta_0 + \sum_{i=1}^{5} \beta_i \Pi_t^i(0,\mathcal{T}) = \beta_0 + \sum_{i=1}^{5} \beta_i S_t^i \quad \text{for all } t \in [0,\mathcal{T}].$$

Assume that the counterparty to the bank wants to enter into a new uncollateralized trade with the bank. The counterparty wishes to trade constant amounts $\gamma_1, \gamma_2, \ldots, \gamma_5$ of forwards for each FX rate S^i, with each forward having the same maturity time T and forward amounts K_i. We assume that the maturity time T years is less than the portfolio maturity time $\mathcal{T}$ ($T < \mathcal{T}$). We have

$$\mathcal{E}_t = \sum_{i=1}^{5} \gamma_i \Pi_t^i(K_i,T) \quad \text{for all } t \in [0,T].$$

Because this derivative trade is uncollateralized, the bank wants to compute (and charge) an FVA charge for this trade, as discussed in subsection 2.3. The bank can use the symmetric FVA charge (32) or the asymmetric FVA charge (36), as discussed in subsections 3.5 and 3.6. The simplicity of the model, with only five FX rates considered, makes both the SFVA charge and the AFVA charge computable using Monte Carlo estimation, which allows for a comparison between them. In our numerical experiments, we will focus on plain FX forwards between USD and the other given currencies, as well as FX forwards between two of the given currencies (EUR, GBP, SEK, JPY, NOK). If the amount γ_i is bought of an FX forward between USD and currency i, i.e. γ_i FX forward contracts are transacted, the value process is

$$\mathcal{E}_t = \gamma_i \Pi_t^i(K_i,T).$$

For an FX forward between currency i and j, the value process

$$\mathcal{E}_t = s_0^j S_t^i - s_0^i S_t^j, \tag{71}$$

where specific form of the forward is chosen such that it has zero expectation. To be more clear,

$$\mathbb{E}\left[\mathcal{E}_t\right] = s_0^j \mathbb{E}\left[S_t^i\right] - s_0^i \mathbb{E}\left[S_t^j\right] = s_0^j s_0^i - s_0^i s_0^j = 0,$$

where the second equality holds due to the fact that S_t^i are martingales. A derivative traded with zero expected value is called an at-the-money (ATM) derivative trade and is the usual case in practice.

FX forwards are often quoted (priced) in terms of the forward amount divided by the initial FX rate times the maturity time T [15]. We choose to express the quotes in basis points (bps), where 1% equals 100 bps.

$$\text{FX forward quote}_i(K,T) = 10000 \cdot \frac{K}{s_0^i \cdot T}.$$

However, the funding charge formulas (32) and (36) are defined in terms of money. Therefore, we can define the asymmetric funding charge quote ($\delta\,\mathrm{AFVA}^{\mathrm{Q}}$) and symmetric funding charge quote ($\delta\,\mathrm{SFVA}^{\mathrm{Q}}$) for γ_i amounts of FX forward, of the form (70), as

$$\delta\,\mathrm{AFVA}^{\mathrm{Q}} = 10000 \cdot \frac{\delta\,\mathrm{AFVA}}{\gamma_i s_0^i \cdot T} \quad \text{and} \quad \delta\,\mathrm{SFVA}^{\mathrm{Q}} = 10000 \cdot \frac{\delta\,\mathrm{SFVA}}{\gamma_i s_0^i \cdot T}. \tag{72}$$

Similarly, for a FX forward between currency i and j is of the form (71), the funding charge quotes are

$$\delta\,\mathrm{AFVA}^{\mathrm{Q}} = 10000 \cdot \frac{\delta\,\mathrm{AFVA}}{s_0^i s_0^j \cdot T} \quad \text{and} \quad \delta\,\mathrm{SFVA}^{\mathrm{Q}} = 10000 \cdot \frac{\delta\,\mathrm{SFVA}}{s_0^i s_0^j \cdot T}. \tag{73}$$

The reason for the usage of such quotes is to understand how large the funding charge is as a fraction of the initial value per unit of maturity T, and we will take this quoting system as a definition.

Moreover, we assume that the funding rates in the $\delta\,\mathrm{AFVA}$ are given by $r_I = f+s$ and $r_{II} = f-s$, to reflect the practical view where the funding rates are centered around a so-called mid rate f with a spread of s.

We will use this FX model to compute funding charges for different portfolios β, specified through changes in $(\beta)_{i=1}^5$, as well as different derivative maturities T and spreads s. The computed funding charges will then be analyzed to see how much $\delta\,\mathrm{SFVA}^{\mathrm{Q}}$ differs from $\delta\,\mathrm{AFVA}^{\mathrm{Q}}$.

5.2 Simulation method

In this subsection, we will describe how we produce our funding charge estimates numerically using Monte Carlo methods.

The dynamics (69) depends on the Wiener processes W^i, $i = 1, \ldots, 5$. Therefore, we first need a method for sampling the Wiener processes.

Algorithm 1: Sample correlated k-dimensional Wiener process [16]

Data: Covariance matrix $\Sigma = (\rho_{ij})_{i,j} \in \mathbb{R}^{k\times k}$, Number of samples M, End time $T \in \mathbb{R}^+$
Result: Samples $\{\tilde{W}_{m\Delta t}\}_{m=0}^{M}$
Assign $\Delta t = \frac{T}{M-1}$
Assign $\tilde{W}_0 = 0 \in \mathbb{R}^k$
for *m = 1, ..., M-1* **do**
  Sample $Z_n \sim \mathcal{N}(0, \Sigma)$
  Assign $\tilde{W}_{m\Delta t} = \tilde{W}_{(m-1)\Delta t} + \sqrt{\Delta t} Z_n$
end

Using Algorithm 1, we can sample $\{W_{m\Delta t}\}_{m=0}^{M}$, a Wiener process path on the interval $[0, T]$. Then, using the fact that a GBM satisfying (69) has an explicit solution (see, [14])

$$S_t^i = e^{-\frac{\sigma_i^2}{2}t + \sigma_i W_t^i},$$

and by substituting in Wiener process samples $\{W_{m\Delta t}\}_{m=0}^{M}$, we can obtain samples for FX rates,

$$\tilde{S}_{m\Delta t}^i = e^{-\frac{\sigma_i^2}{2}m\Delta t + \sigma_i \tilde{W}_{m\Delta t}^i} \quad \text{for } m = 1, 2, \dots, M. \tag{74}$$

Recall that, as discussed in subsection 3.6, to compute the δ AFVA, we need estimates of $\mathbb{E}\left[\mathcal{V}_t^+\right], \mathbb{E}\left[\mathcal{V}_t^-\right]$, $\mathbb{E}\left[(\mathcal{V}+\mathcal{E})_t^+\right]$ and $\mathbb{E}\left[(\mathcal{V}+\mathcal{E})_t^-\right]$. We describe how we estimate $\mathbb{E}\left[(\mathcal{V}+\mathcal{E})_t^+\right]$ using Monte Carlo, the same approach can be applied to the other expected values. Using Algorithm 1 and (74), we can obtain N sample paths $\{\{\tilde{S}_{m\Delta t,n}^i\}_{m=0}^{M}\}_{n=1}^{N}$ over $[0, T]$ for $i = 1, 2, \dots, 5$. Using the FX rate samples we can obtain N portfolios with added derivative sample paths

$$(\tilde{\mathcal{V}} + \tilde{\mathcal{E}})_{m\Delta t,n} = \sum_{i=1}^{5} (\beta_i + \gamma_i)\tilde{S}_{m\Delta t,n}^i, \quad \text{for } m = 1, 2, \dots, M,$$

for $n = 1, 2, \dots, N$. Using the antithetic sampling method as mentioned in appendix D, we get estimates

$$\tau_{m\Delta t} \simeq \mathbb{E}\left[(\tilde{\mathcal{V}} + \tilde{\mathcal{E}})_{m\Delta t}^+\right], \quad \text{for } m = 0, 1, \dots, M.$$

The above estimates of the expected positive exposure $\tau_{m\Delta t}$ are then used to calculate the asymmetric funding cost after adding $\mathcal{E}$,

$$\text{AFCA} = -\int_t^T (r_I - r) D_{r+\lambda_A+\lambda_B}(t, u)\mathbb{E}\left[(\mathcal{V}+\mathcal{E})^+\right] du, \tag{75}$$

where $D_z(t, u) = \exp(-z(u-t))$. To approximate the integral in (75), we use the trapezoid method (see [17], p. 255), such that

$$\text{AFCA} \approx -(r_I - r) \sum_{m=1}^{M} \frac{D_{r+\lambda_A+\lambda_B}(t, t+(m-1)\Delta t)\tau_{(m-1)\Delta t} + D_{r+\lambda_A+\lambda_B}(t, t+m\Delta t)\tau_{m\Delta t}}{2} \Delta t.$$

5.3 Model specification

In this subsection, we will specify the parameters used in our simulations of the model presented in subsection 5.1. The goal of these simulations is to compute funding charges for different model scenarios (parameter specifications).

For the number of Monte Carlo samples, we use $N = 100000$ (and therefore 200000 samples counting the antithetic samples). Five estimates are made, and their average gives the final estimate, which is equivalent to using 1000000 samples. The reason for using 5 estimates instead of 1 estimate of 1000000 samples is due to RAM limitations.

For the time discretization, we use $M = \lceil\sqrt{2N}\rceil = 448$. The choice of both N and M are empirical. Because the trapezoid method is of convergence order $\mathcal{O}(1/M^2)$ [17] and Monte Carlo is of order $\mathcal{O}(1/\sqrt{N})$, by the Central Limit Theorem (CLT), we expect the statistical error from Monte Carlo to dominate the approximation error of the trapezoid method. Hence, we expect to be able to use significantly fewer steps in the time discretization than samples in the Monte Carlo. This is why we chose $M = \lceil\sqrt{2N}\rceil$.

Choosing M as a function of N is also practical, in this case, since one only has to empirically experiment with one parameter N. Furthermore, to choose N, we increased N until we obtained a Monte Carlo standard deviation of approximately 0.001 bps in the AFVA charge quote. The choice of 0.001 bps was a trade-off between accuracy and compute time. These calculations were done with portfolio 1 (Table 4) and derivative 1 (Table 11) with maturity $T = 20$.

The parameters presented in the two following tables (Table 1 and Table 2) were provided by the external project provider SEB and reflect the approximate current market state (summer 2024).

Table 1: Model volatility and initial FX rate parameters

σ_1	σ_2	σ_3	σ_4	σ_5	s_0^1	s_0^2	s_0^3	s_0^4	s_0^5
0.1	0.1	0.1	0.1	0.1	1.07	1.26	0.094	0.0062	0.094

The first five values of Table 1 are the standard deviations of the FX rates and the last five values are the initial FX rates at time zero.

Table 2: Model correlation parameters

ρ_{ij}					
	1	0.75	0.8	0.41	0.7
	0.75	1	0.7	0.41	0.6
	0.8	0.7	1	0.36	0.3
	0.41	0.41	0.36	1	0.76
	0.7	0.6	0.3	0.76	1

Table 2 shows the correlations between the driving Wiener processes in the model.

Table 3: Model rate parameters

f	r	s	λ_A	λ_B
0.01	0	0.0005	0.005	0.01

Table 3 shows the funding rate parameter f, spread s, spreads λ_A and λ_B. These numbers are all provided by SEB to resemble a realistic scenario. The risk-free rate r is assumed to be zero since all the drifts in the model are zero.

5.3.1 Model portfolios

We will calculate funding charges for seven different portfolios, which we will present and justify here.

The first portfolio we choose to test is the empty one, meaning it does not contain any derivatives. This portfolio is chosen for its simplicity.

Table 4: Model **Portfolio 1**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
0	0	0	0	0	0

For the above portfolio we have expectation $\mu = 0$ and yearly volatility $\sigma = 0$.

Inspired by the asymptotic argument and the discussion at the end of subsection 4.2, we investigate whether $\delta\,\mathrm{AFVA}^Q$ converges to $\delta\,\mathrm{SFVA}^Q$ with funding rate $r_F = f + s$. To do this, we look at portfolios $\mathcal{V}$ that move further away from zero, meaning that we investigate when the portfolio expectation is $\mathbb{E}\left[\mathcal{V}_t\right] = k\sqrt{\mathrm{Var}\left(\mathcal{V}_1\right)}$ for increasing values of k. Since the portfolio is constant,

$$\mathbb{E}\left[\mathcal{V}_t\right] = \beta_0 + \sum_{i=1}^{5} \beta_i \mathbb{E}\left[S_t^i\right] = \beta_0 + \sum_{i=1}^{5} \beta_i s_0^i,$$

where we use that the S^is are martingales. In addition, the covariance between S^i and S^j is given by the following claim 5.1.

Claim 5.1.

$$\mathrm{Cov}(S_t^i, S_t^j) = s_0^i s_0^j (e^{\rho_{ij}\sigma_i\sigma_j t} - 1) \tag{76}$$

Proof. Notice that

$$S_t^i = s_0^i e^{-\frac{\sigma_i^2}{2}t + \sigma_i W_t^i} \quad \text{and} \quad S_t^j = s_0^j e^{-\frac{\sigma_j^2}{2}t + \sigma_i W_t^j}. \tag{77}$$

By theorem 4.5.3 in [18] and (77),

$$
\begin{aligned}
\mathrm{Cov}(S_t^i, S_t^j) &= \mathbb{E}\left[S_t^i S_t^j\right] - \mathbb{E}\left[S_t^i\right]\mathbb{E}\left[S_t^j\right] \\
&= s_0^i s_0^j \left(\mathbb{E}\left[e^{-\frac{\sigma_i^2+\sigma_j^2}{2}t+\sigma_i W_t^i+\sigma_j W_t^j}\right] - 1\right) \\
&= s_0^i s_0^j \left(e^{-\frac{\sigma_i^2+\sigma_j^2}{2}t}\mathbb{E}\left[e^{\sigma_i W_t^i+\sigma_j W_t^j}\right] - 1\right) \\
&= s_0^i s_0^j \left(e^{-\frac{\sigma_i^2+\sigma_j^2}{2}t}\mathbb{E}\left[e^{\sigma_i W_t^i+\sigma_j W_t^j}\right] - 1\right) \\
&= s_0^i s_0^j \left(e^{-\frac{\sigma_i^2+\sigma_j^2}{2}t}\mathbb{E}\left[e^{\sqrt{\sigma_i^2+\sigma_j^2+2\rho_{ij}\sigma_i\sigma_j}\frac{\sigma_i W_t^i+\sigma_j W_t^j}{\sqrt{\sigma_i^2+\sigma_j^2+2\rho_{ij}\sigma_i\sigma_j}}}\right] - 1\right).
\end{aligned} \tag{78}
$$

Using that

$$
\frac{\sigma_i W_t^i + \sigma_j W_t^j}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}} \sim \mathcal{N}(0, t)
$$

and the moment generating function of a Gaussian (see [18] p. 625) and (78), we get

$$
\begin{aligned}
\mathrm{Cov}(S_t^i, S_t^j) &= s_0^i s_0^j \left(e^{-\frac{\sigma_i^2+\sigma_j^2}{2}t} e^{\sigma_i^2+\sigma_j^2+2\rho_{ij}\sigma_i\sigma_j \frac{t}{2}} - 1\right) \\
&= s_0^i s_0^j (e^{\rho_{ij}\sigma_i\sigma_j t} - 1).
\end{aligned}
$$

□

Furthermore, the claim 5.1 yields

$$
\mathrm{Var}(\mathcal{V}_t) = \sum_{i=1}^{5}\sum_{j=1}^{5} \beta_i\beta_j \,\mathrm{Cov}(S_t^i, S_t^j) = \sum_{i=1}^{5}\sum_{j=1}^{5} \beta_i\beta_j s_0^i s_0^j (e^{\rho_{ij}\sigma_i\sigma_j t} - 1). \tag{79}
$$

However, since the expected portfolio value is constant and the variance is time-dependent, we cannot have $\mathbb{E}\left[\mathcal{V}_t\right] = k\sqrt{\mathrm{Var}\left(\mathcal{V}_t\right)}$ for all times t. Instead, we will use $\mathbb{E}\left[\mathcal{V}_t\right] = k\sqrt{\mathrm{Var}\left(\mathcal{V}_1\right)}$ for increasing k, since $\mathrm{Var}\left(\mathcal{V}_1\right)$ gives a variance per unit of time value, or "yearly variance". This can be seen by linearizing (79), using that the first order approximation of $e^z \approx 1 + z$, yields

$$
\begin{aligned}
\mathrm{Var}(\mathcal{V}_t) &= \sum_{i=1}^{5}\sum_{j=1}^{5} \beta_i\beta_j s_0^i s_0^j (e^{\rho_{ij}\sigma_i\sigma_j t} - 1) \\
&\approx \sum_{i=1}^{5}\sum_{j=1}^{5} \beta_i\beta_j s_0^i s_0^j (1 + \rho_{ij}\sigma_i\sigma_j t - 1) \\
&= t\sum_{i=1}^{5}\sum_{j=1}^{5} \beta_i\beta_j s_0^i s_0^j \rho_{ij}\sigma_i\sigma_j,
\end{aligned}
$$

where setting $t = 1$ gives the coefficient of increase. We will use the notation $\sigma = \sqrt{\mathrm{Var}\left(\mathcal{V}_1\right)}$ and call it yearly volatility of the portfolio, or just yearly volatility.

The following portfolios in Tables 5, 6 and 7 are chosen such that Portfolio 2 has $\mathbb{E}[\mathcal{V}_t] = \sigma$, Portfolio 3 has $\mathbb{E}[\mathcal{V}_t] = 2\sigma$ and Portfolio 4 has $\mathbb{E}[\mathcal{V}_t] = 3\sigma$ by keeping the coefficients β_i constant for $i = 1, 2, \ldots, 5$ and increasing β_0. This approach keeps σ constant.

Table 5: Model **Portfolio 2**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
814.55	10000	-10000	10000	10000	10000

For the above portfolio 2 we have that the portfolio expected vale $\mu = 1\sigma$, with $\sigma \approx 853$.

Table 6: Model **Portfolio 3**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
1671.11	10000	-10000	10000	10000	10000

For the above portfolio 3 we have that the portfolio expected vale $\mu = 2\sigma$, with $\sigma \approx 853$.

Table 7: Model **Portfolio 4**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
2527.66	10000	-10000	10000	10000	10000

For the above portfolio 4 we have that the portfolio expected vale $\mu = 3\sigma$, with $\sigma \approx 853$.

Furthermore, portfolios 1-4 will be used to investigate the funding charges when the expected value increases. For a more comprehensive analysis, we will also calculate the funding charges for portfolios where the expected portfolio value is kept constant while the yearly volatility of the portfolio, σ, is increased. The coefficients β_i for $i = 1, 2 \ldots, 5$ are chosen such that $\sigma = 1$ for portfolio 5, $\sigma = 10$ for portfolio 6, and $\sigma = 100$ for portfolio 7. The parameter β_0 is chosen such that $\mathbb{E}[\mathcal{V}_t] = 0$, since we are investigating the effect of increasing σ on the funding charges. The exact parameters that we used for portfolios 5-7 are presented in Tables 8, 9 and 10 bellow.

Table 8: Model **Portfolio 5**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
-12.18	-1.47	-3.42	93.73	99.46	91.93

For the above portfolio 5 the expected value $\mu = 0$ and yearly volatility $\sigma = 1$

Table 9: Model **Portfolio 6**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
-100.51	90.24	-12.18	99.41	99.97	99.42

For the above portfolio 6 the expected value $\mu = 0$ and yearly volatility $\sigma = 10$

Table 10: Model **Portfolio 7**

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	$\boldsymbol{\beta_5}$
-1070.17	557.79	355.75	131.80	101.17	128.53

For the above portfolio 7 the expected value $\mu = 0$ and yearly volatility $\sigma = 100$

5.3.2 Derivatives added

For the seven different portfolios, we will calculate funding charges for three different derivatives added to each portfolio. The derivatives are selected to provide a comprehensive overview of potential trades.

The first derivative to be used is 10, at-the-money, (zero expected value) FX forwards on the GBP/USD FX rate S^1, with a forward amount of 1.07 USD. That is, the price process of one FX forward is

$$S_t^1 - 1.07.$$

Therefore, for all 10 FX forwards, we have the price process of the first derivative

$$\mathcal{E}_t^1 = 10(S_t^1 - 1.07) = 10S_0^1 - 10.7.$$

Then, using values from Table 1,

$$\mathbb{E}\left[\mathcal{E}_t^1\right] = 10S_0^1 - 10.7 = 10 \cdot 1.07 - 10.7 = 0,$$

which shows that the forward is ATM.

Table 11: Model **Derivative 1**

$\boldsymbol{\gamma_1}$	$\boldsymbol{\gamma_2}$	$\boldsymbol{\gamma_3}$	$\boldsymbol{\gamma_4}$	$\boldsymbol{\gamma_5}$	K_1
10	0	0	0	0	1.07

The yearly volatility of derivative 1 is 1.07.

The second derivative to be used is an ATM FX forward between EUR and GBP, as this is a common type of derivative. The value process is

$$\mathcal{E}_t^2 = 1.26S_t^1 - 1.07S_t^2.$$

Table 12: Model **Derivative 2**

γ_1	γ_2	γ_3	γ_4	γ_5	K_i
1.26	-1.07	0	0	0	0

The yearly volatility of derivative 2 is 0.095.

The third, and last derivative, used is 100, in-the-money (ITM), FX forwards on the SEK to USD FX rate S^3, with forward amount 0.084. In-the-money means that the forward has positive expectation. The price process of one of these FX forwards is

$$S_t^3 - 0.084.$$

Therefore, the combined price process of the 100 FX forwards is given by

$$\mathcal{E}_t^3 = 100(S_t^3 - 0.084) = 100S_t^3 - 8.4.$$

We see that the expected value

$$\mathbb{E}\left[\mathcal{E}_t^3\right] = 100\mathbb{E}\left[S_t^3\right] - 8.4 = 100s_0^3 - 8.4 = 100 \cdot 0.094 - 8.4 = 1,$$

showing that they are ITM.

Table 13: Model **Derivative 3**

γ_1	γ_2	γ_3	γ_4	γ_5	K_3
0	0	100	0	0	0.084

The yearly volatility of derivative 3 is 0.94.

To provide an overview of possible derivatives, we will consider two cases. The first case is keeping the spread $s = 0.0005$ constant, and varying the maturity $T = 1, 2, 3, 5, 7, 10, 15, 20$ years. For the second case, we keep the maturity $T = 5$ constant and vary the spread $s = 0.0001, 0.0005, 0.001, 0.002, 0.004$. Furthermore, for both cases, funding charges are calculated using the seven portfolios specified in subsubsection 5.3.1.

Note that the $\delta\,$SFVA formula (32) does not depend on the portfolio value $\mathcal{V}$. Therefore, we will calculate $\delta\,$SFVA charges once for $r_F = f + s$ and $r_F = f$, across all T and s values. That is, we will not make a new $\delta\,$SFVA estimate for each portfolio. Furthermore, for derivatives 1 and 2 the symmetric charge $\delta\,$SFVA will be zero, since we have that $\mathbb{E}\left[\mathcal{E}^1\right] = \mathbb{E}\left[\mathcal{E}^2\right] = 0$, but we compute them anyways to make sure that the Monte Carlo method seems to be correctly implemented. We also know that $\mathbb{E}\left[\mathcal{E}^3\right] = 1$, and could have computed the symmetric charge analytically as well.

5.4 Results and analysis

The resulting funding charges will be presented in this subsection. As a reminder, funding charges have been estimated for seven different portfolios and three different derivatives, across various maturities T and spreads s. The funding charges will be displayed in the form given by (72) or (73), depending on whether the derivative is an FX forward between currency i and USD or an FX forward between currencies i and j.

First, in subsubsection 5.4.1 the compute times for two funding charges are shown. In subsubsection 5.4.2 the results for derivative 1 (Table 11) are shown. The results for derivative 2 (Table 12) are displayed in 5.4.3. Lastly, 5.4.4 shows the results for derivative 3 (Table 13). The results are both presented in tables and figures.

Note that $\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{(i-1)\sigma}$ will denote the funding charge quote ((72) or (73)) for portfolio i for $i = 1, 2, 3, 4$, such that each quote is in basis points (bps), where 1 bps = 0.01% = 0.0001. See Table 14 bellow.

Table 14: Notation used for AFVA quotes for **Portfolio 1-4**

Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**
$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{0\sigma}$	$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{1\sigma}$	$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{2\sigma}$	$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{3\sigma}$

Furthermore, see Table 15 for how we will denote the asymmetric funding charge quote for portfolios 5 to 7.

Table 15: Notation used for AFVA quotes for **Portfolio 5-7**

Portfolio 5	**Portfolio 6**	**Portfolio 7**
$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{\sigma=1}$	$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{\sigma=10}$	$\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{\sigma=100}$

Also note that a negative funding charge value means that the issuer should charge, while a positive value means that the issuer should compensate. Therefore, when the funding charge value becomes more negative we will say that the funding charge increase.

5.4.1 Computation time

The computation time for a funding charge is expected to primarily depend on the number of samples used. Therefore, we chose to investigate the compute time for Derivative 1 and Derivative 2. Since both Derivative 1 and Derivative 3 depend on a single geometric Brownian motion, their compute times are expected to be the same, and thus we have excluded Derivative 3 from this analysis. Additionally, we focus on an asymmetric charge using only Portfolio 2, as this portfolio depends on all five geometric Brownian motions. The maturity time is set to $T = 1$. The funding charges are computed five times, and the average computation time is presented in the table below.

Table 16: Computation times.

	δ **AFVA**$_{1\sigma}$	δ **SFVA**$_{(r_F=f)}$
Derivative 1	87.54	10.61
Derivative 2	85.72	25.43

The values in the above Table 16 are given in seconds. From the table, we observe that the computation time for the symmetric charges is shorter than for the corresponding asymmetric charges. This outcome is expected since Derivatives 1 and 2 rely on only one and two geometric Brownian motions, respectively, while the asymmetric charge depends on all five geometric Brownian motions. Additionally, the observed increase in computation time for the symmetric charge from Derivative 1 to Derivative 2 is also anticipated, as the symmetric charge for Derivative 2 requires sampling from more geometric Brownian motions compared to Derivative 1. The result indicates that one seems to be able to save computation time using a symmetric funding charge.

5.4.2 Derivative 1

The resulting funding charges for the first derivative are now presented.

Table 17: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 1-4** and **Derivative 1**

Maturity	δ **AFVA**$^{\mathbb{Q}}_{0\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{1\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{2\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{3\sigma}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
T = 1	-0.264713	-0.023207	-0.001013	-0.000885	-0.000881	-0.000839
T = 2	-0.259969	-0.022691	-0.002366	0.000047	0.000771	0.000734
T = 3	-0.25851	-0.023604	-0.001562	-0.000181	0.000158	0.000151
T = 5	-0.253128	-0.023236	-0.001725	0.000259	0.000853	0.000813
T = 7	-0.250734	-0.022021	-0.001645	-0.000169	-0.000625	-0.000595
T = 10	-0.242978	-0.021169	-0.002661	0.000254	0.000193	0.000183
T = 15	-0.2325	-0.019916	-0.00177	-0.000722	0.000196	0.000186
T = 20	-0.222561	-0.01941	-0.001865	-0.000068	0.000147	0.00014

Table 18: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 1-4** and **Derivative 1**

Spread	$\delta\,\mathrm{AFVA}^{\mathrm{Q}}_{0\sigma}$	$\delta\,\mathrm{AFVA}^{\mathrm{Q}}_{1\sigma}$	$\delta\,\mathrm{AFVA}^{\mathrm{Q}}_{2\sigma}$	$\delta\,\mathrm{AFVA}^{\mathrm{Q}}_{3\sigma}$	$\delta\,\mathrm{SFVA}^{\mathrm{Q}}_{(r_F=f+s)}$	$\delta\,\mathrm{SFVA}^{\mathrm{Q}}_{(r_F=f)}$
s = 0.0001	-0.051077	-0.004332	-0.000533	-0.000722	0.000161	0.00016
s = 0.0005	-0.254044	-0.022322	-0.001887	0.00007	0.000853	0.000813
s = 0.001	-0.509041	-0.045686	-0.004156	-0.000087	-0.000763	-0.000693
s = 0.002	-1.016848	-0.090936	-0.006751	-0.000343	-0.000556	-0.000463
s = 0.004	-2.033685	-0.178599	-0.015317	-0.000554	0.000143	0.000102

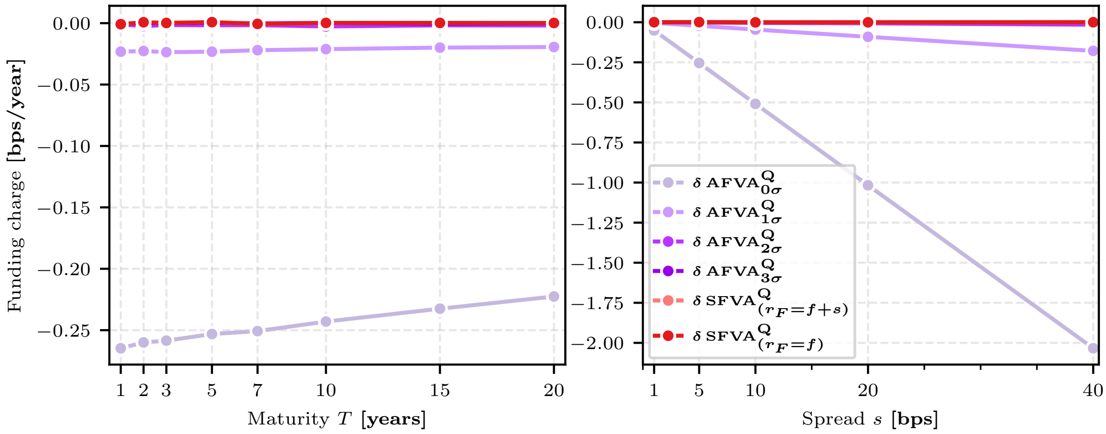

Figure 5: MC-estimated funding charges for **Portfolio 1-4** and **Derivative 1**

Looking at the above Figure 5, and recalling that an increase in portfolio number corresponds to a larger expected value relative to yearly volatility, we see that when the expected portfolio value μ increases relative to the yearly volatility σ, the asymmetric charge $\delta\,\mathrm{AFVA}^{Q}$ converges to the symmetric charge $\delta\,\mathrm{SFVA}^{Q}$. This convergence appears to hold for all the maturities T tested and all spreads s tested. This observation seems to support the asymptotic result in subsection 4.2 even in this model.

If we look at Figure 5, we see that the asymmetric funding quotes for each portfolio seem to slightly decrease (less negative) as the maturity time T increase. This seems to indicate that increasing T does not make the symmetric quotes worse approximators of the asymmetric quotes.

Furthermore, in practice, a difference of 0.01 basis points (bps) is considered negligible, meaning that two funding charges can be treated as the same if they do not differ by more than this amount. When examining Table 18, we observe that for the first four spreads, the asymmetric charge quote is within 0.01 bps of both $\delta\,\mathrm{SFVA}^{\mathrm{Q}}$ when $r_F = f + s$ and $r_F = f$, particularly when portfolio 3 is used ($\mu = 2\sigma$). This suggests that one of the two symmetric charges ($\delta\,\mathrm{SFVA}^{\mathrm{Q}}\,(r_F = f + s)$ or $\delta\,\mathrm{SFVA}^{\mathrm{Q}}\,(r_F = f)$) could be used when the expected portfolio value equals or exceeds two times the yearly volatility, for spreads of $s = 0.0005, 0.001$, or 0.002.

For the spread $s = 0.004$, this conclusion also holds when the expected portfolio value is at least thrice the yearly volatility. Additionally, for $s = 0.0001$, the results suggest that an asymmetric charge could be substituted with a symmetric charge even when $\mu = 0$.

It is important to note that, based on the two Tables 17 and 18, we cannot distinguish whether the funding rate $r_F = f$ or $r_F = f + s$ should be used when substituting for the asymmetric funding charge quote, as the symmetric charges are very close. This is expected, as we previously noted (see end of subsection 5.3), both symmetric funding charges should be zero, since the expected value of derivative 1 is zero.

Table 19: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 5-7** and **Derivative 1**

Maturity	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=1}$	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=10}$	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=100}$	δ $\mathbf{SFVA}^{\mathbf{Q}}_{(r_F=f+s)}$	δ $\mathbf{SFVA}^{\mathbf{Q}}_{(r_F=f)}$
T = 1	-0.24538	-0.261955	-0.251653	-0.000881	-0.000839
T = 2	-0.243098	-0.260097	-0.24909	0.000771	0.000734
T = 3	-0.239689	-0.257675	-0.247341	0.000158	0.000151
T = 5	-0.236881	-0.254028	-0.243125	0.000853	0.000813
T = 7	-0.232154	-0.249073	-0.239207	-0.000625	-0.000595
T = 10	-0.225807	-0.24245	-0.232101	0.000193	0.000183
T = 15	-0.215595	-0.231601	-0.222553	0.000196	0.000186
T = 20	-0.206375	-0.222242	-0.212037	0.000147	0.00014

Table 20: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 5-7** and **Derivative 1**

Spread	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=1}$	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=10}$	δ $\mathbf{AFVA}^{\mathbf{Q}}_{\sigma=100}$	δ $\mathbf{SFVA}^{\mathbf{Q}}_{(r_F=f+s)}$	δ $\mathbf{SFVA}^{\mathbf{Q}}_{(r_F=f)}$
s = 0.0001	-0.047987	-0.05081	-0.049053	0.000161	0.00016
s = 0.0005	-0.236806	-0.252391	-0.242665	0.000853	0.000813
s = 0.001	-0.472895	-0.50689	-0.484448	-0.000763	-0.000693
s = 0.002	-0.944839	-1.012457	-0.968881	-0.000556	-0.000463
s = 0.004	-1.890698	-2.025503	-1.939154	0.000143	0.000102

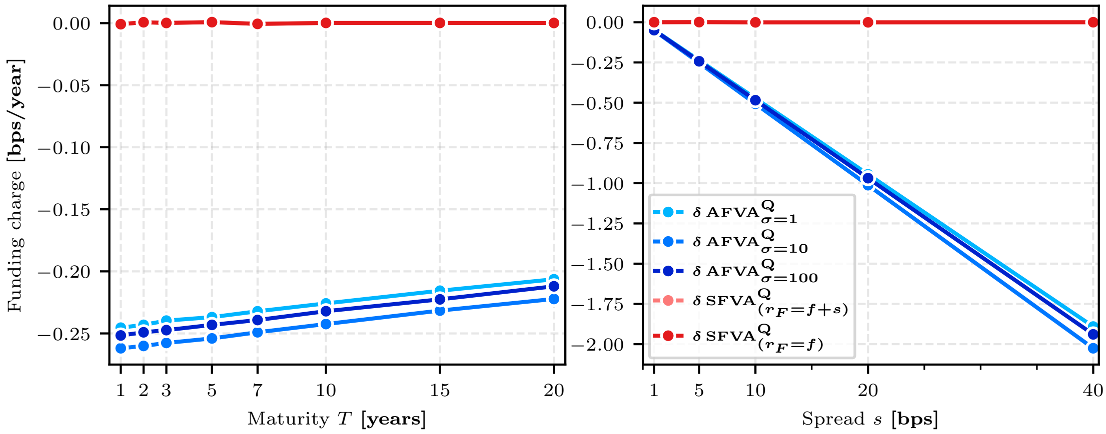

Figure 6: MC-estimated funding charges for **Portfolio 5-7** and **Derivative 1**

Recall that an increase in portfolio number (i.e. from 5 to 7) corresponds to an increase in yearly volatility σ, as determined by the construction of portfolios 5 to 7. Referring to Figure 6, we observe that an increase in portfolio number, and therefore an increase in yearly volatility, does not seem to have a significant impact on the asymmetric funding charge quote for any of the tested maturities T. Additionally, the asymmetric funding charge quotes do not appear to converge to the symmetric funding charge quotes (using $r_F = f$ or $r_F = f + s$) as the yearly volatility increases. Further analysis of Figure 6 reveals that the asymmetric quotes do not seem to converge to the symmetric quotes for any of the tested spreads s as well.

Moreover, if we again use a difference of 0.01 bps as the threshold for considering two values meaningfully different. Then looking at Table 20 we find that it would it not be possible to substitute a symmetric funding charge for an asymmetric one for any of the portfolios 5 to 7, for all spreads s tested.

5.4.3 Derivative 2

The resulting funding charges for the second derivative is now presented.

Table 21: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 1-4** and **Derivative 2**

Maturity	δ **AFVA**$^{\mathbb{Q}}_{0\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{1\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{2\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{3\sigma}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
T = 1	-0.187202	-0.07394	-0.010541	-0.001294	-0.001039	-0.00099
T = 2	-0.184601	-0.07245	-0.009854	-0.001382	0.000174	0.000166
T = 3	-0.182538	-0.071784	-0.010364	-0.000995	0.000463	0.000441
T = 5	-0.179407	-0.07099	-0.009733	-0.000612	0.000184	0.000176
T = 7	-0.176012	-0.069111	-0.010021	-0.001109	0.000673	0.000641
T = 10	-0.171948	-0.066625	-0.008444	-0.000249	-0.000273	-0.00026
T = 15	-0.164587	-0.063691	-0.008956	-0.001058	-0.000192	-0.000183
T = 20	-0.157542	-0.059559	-0.007946	-0.001425	0.00022	0.00021

Table 22: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 1-4** and **Derivative 2**

Spread	δ **AFVA**$^{\mathbb{Q}}_{0\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{1\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{2\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{3\sigma}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
s = 0.0001	-0.036062	-0.014448	-0.002043	-0.000168	0.000161	0.00016
s = 0.0005	-0.17991	-0.070671	-0.009311	-0.000166	0.000184	0.000176
s = 0.001	-0.359084	-0.141683	-0.019372	-0.001178	-0.000763	-0.000693
s = 0.002	-0.719633	-0.281918	-0.039125	-0.002499	-0.000556	-0.000463
s = 0.004	-1.436783	-0.566019	-0.076706	-0.005444	0.000143	0.000102

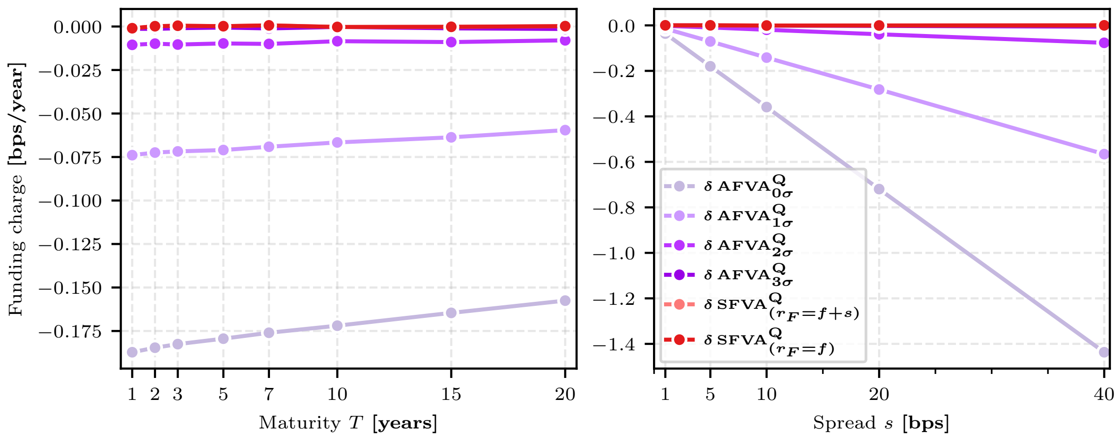

Figure 7: MC-estimated funding charges for **Portfolio 1-4** and **Derivative 2**

Examining Figure 7, we observe a similar trend for the FX forward between EUR and GBP (derivative 2) as seen with the ATM FX forward (derivative 1). As yearly volatility increases (as the portfolio moves from 1 to 4), the asymmetric funding charge quotes again seem converge to the symmetric funding charge quotes for all maturities T and spreads s tested. However, since the symmetric quotes using the funding rates $r_F = f$ and $r_F = f + s$ are very close to each other, it is difficult to determine to which one the asymmetric quotes are converging (or in some sence, the convergence is to both of them).

Furthermore, looking at Table 22, using the practical threshold of 0.01 bps, we see that for $s = 0.0001$ and $s = 0.0005$, a symmetric charge could replace an asymmetric charge when the expected portfolio value is twice the yearly volatility. For the remaining spreads $s = 0.001, 0.002, 0.004$, the substitution appears feasible when the portfolio's expected value is equal to or greater than thrice the yearly volatility.

Once again, note that from the two Tables 21 and 22, we cannot distinguish whether the funding rate $r_F = f$ or $r_F = f + s$ should be used when substituting for the asymmetric funding charge quote, as the symmetric charges are very close. This is still expected, since derivative two has an expected value that is zero., so in some sense the convergence is to both symmetric quotes.

Table 23: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 5-7** and **Derivative 2**

Maturity	δ **AFVA**$^{\mathbb{Q}}_{\sigma=1}$	δ **AFVA**$^{\mathbb{Q}}_{\sigma=10}$	δ **AFVA**$^{\mathbb{Q}}_{\sigma=100}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
T = 1	-0.071301	-0.079927	-0.010263	-0.001039	-0.00099
T = 2	-0.070118	-0.079182	-0.010505	0.000174	0.000166
T = 3	-0.071261	-0.077512	-0.011108	0.000463	0.000441
T = 5	-0.070226	-0.076735	-0.009712	0.000184	0.000176
T = 7	-0.067069	-0.075613	-0.009159	0.000673	0.000641
T = 10	-0.06587	-0.07323	-0.009786	-0.000273	-0.00026
T = 15	-0.06295	-0.069762	-0.008917	-0.000192	-0.000183
T = 20	-0.060589	-0.067089	-0.009609	0.00022	0.00021

Table 24: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 5-7** and **Derivative 2**

Spread	δ **AFVA**$^{\mathbb{Q}}_{\sigma=1}$	δ **AFVA**$^{\mathbb{Q}}_{\sigma=10}$	δ **AFVA**$^{\mathbb{Q}}_{\sigma=100}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
s = 0.0001	-0.013675	-0.015444	-0.00163	0.000161	0.00016
s = 0.0005	-0.068864	-0.076247	-0.010023	0.000184	0.000176
s = 0.001	-0.138737	-0.152885	-0.021456	-0.000763	-0.000693
s = 0.002	-0.276857	-0.307551	-0.038645	-0.000556	-0.000463
s = 0.004	-0.554771	-0.617687	-0.079877	0.000143	0.000102

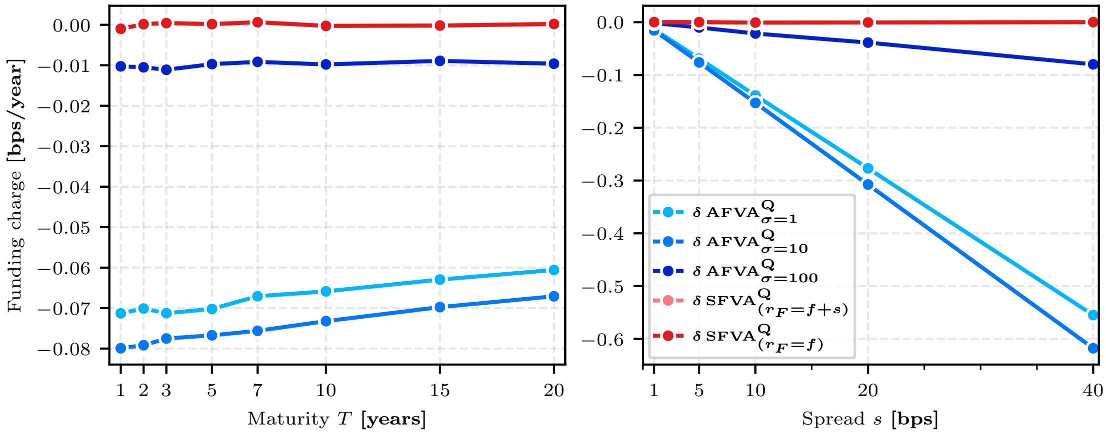

Figure 8: MC-estimated funding charges for **Portfolio 5-7** and **Derivative 2**

If we examine Tables 23 and 24, we observe that the asymmetric quotes are similar across all maturities T and spreads s for portfolios 5 and 6, respectively. However, for portfolio 7, the asymmetric funding quotes are closer to the symmetric ones compared to the asymmetric charges for portfolios 5 and 6. This observation suggests that increasing the yearly volatility could potentially cause the asymmetric funding charge quote to converge to a symmetric funding charge quote for an FX forward between EUR and GBP (derivative 2). Looking at Figure 8, this observation is graphically visible as well.

5.4.4 Derivative 3

The resulting funding charges for the third derivative is now presented.

Table 25: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 1-4** and **Derivative 3**

Maturity	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{0\sigma}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{1\sigma}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{2\sigma}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{3\sigma}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f+s)}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f)}$
T = 1	-11.107797	-11.012797	-11.080313	-11.086764	-11.086714	-10.558775
T = 2	-11.024936	-10.930577	-10.997984	-11.004034	-11.003802	-10.479812
T = 3	-10.942641	-10.850177	-10.915425	-10.921812	-10.922351	-10.402239
T = 5	-10.781422	-10.690452	-10.754795	-10.76095	-10.761431	-10.248982
T = 7	-10.622774	-10.53354	-10.597121	-10.603504	-10.603978	-10.099026
T = 10	-10.391222	-10.304963	-10.366104	-10.37212	-10.372882	-9.878935
T = 15	-10.020658	-9.938282	-9.99706	-10.002456	-10.002994	-9.526661
T = 20	-9.667167	-9.590303	-9.644787	-9.650048	-9.649968	-9.190446

Table 26: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 1-4** and **Derivative 3**

Spread	δ **AFVA**$^{\mathbb{Q}}_{0\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{1\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{2\sigma}$	δ **AFVA**$^{\mathbb{Q}}_{3\sigma}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f+s)}$	δ **SFVA**$^{\mathbb{Q}}_{(r_F=f)}$
s = 0.0001	-10.356076	-10.337115	-10.350557	-10.351474	-10.351551	-10.249061
s = 0.0005	-10.781509	-10.690679	-10.75424	-10.761158	-10.761431	-10.248982
s = 0.001	-11.313099	-11.132267	-11.261291	-11.273653	-11.274405	-10.249459
s = 0.002	-12.378664	-12.014976	-12.271946	-12.297023	-12.298529	-10.248775
s = 0.004	-14.505917	-13.779029	-14.296788	-14.345769	-14.348613	-10.249009

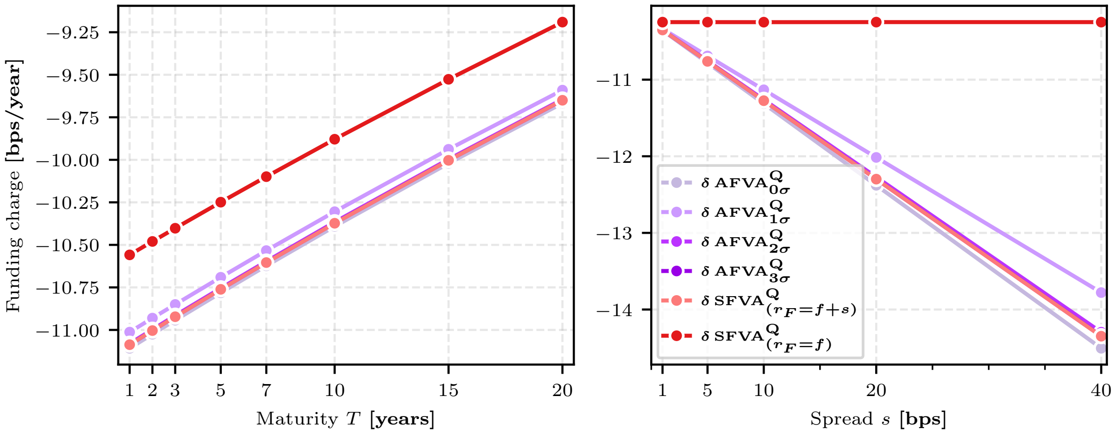

Figure 9: MC-estimated funding charges for **Portfolio 1-4** and **Derivative 3**

Looking at Figure 9, we observe that the asymmetric quotes seem to converge to the symmetric quotes as the mean portfolio value increases relative to the yearly volatility, similar to the behavior observed with the other two derivatives. However, one noticeable difference is that we seem to be able to distinguish which symmetric quote the asymmetric quotes converge to, specifically, the one using the funding rate $r_F = f + s$. This aligns with the asymptotic result in theorem 4.1, because the expected portfolio value is positive.

Table 27: MC-estimated funding charges for different maturities T and fixed spread $s = 0.0005$, using **Portfolio 5-7** and **Derivative 3**

Maturity	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=1}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=10}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=100}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f+s)}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f)}$
T = 1	-10.961695	-10.796877	-10.764462	-11.086714	-10.558775
T = 2	-10.880848	-10.716052	-10.68304	-11.003802	-10.479812
T = 3	-10.79802	-10.634989	-10.603866	-10.922351	-10.402239
T = 5	-10.640096	-10.478656	-10.447111	-10.761431	-10.248982
T = 7	-10.484309	-10.325409	-10.294054	-10.603978	-10.099026
T = 10	-10.256384	-10.099256	-10.068893	-10.372882	-9.878935
T = 15	-9.889725	-9.737519	-9.708635	-10.002994	-9.526661
T = 20	-9.542271	-9.394082	-9.364154	-9.649968	-9.190446

Table 28: MC-estimated funding charges for different spreads s using forward contract maturity $T = 5$ with **Portfolio 5-7** and **Derivative 3**

Spread	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=1}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=10}$	$\delta\,\mathbf{AFVA}^{\mathbb{Q}}_{\sigma=100}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f+s)}$	$\delta\,\mathbf{SFVA}^{\mathbb{Q}}_{(r_F=f)}$
s = 0.0001	-10.328159	-10.295409	-10.288578	-10.351434	-10.248945
s = 0.0005	-10.640967	-10.477855	-10.447602	-10.761431	-10.248982
s = 0.001	-11.032929	-10.708001	-10.645849	-11.274039	-10.249126
s = 0.002	-11.815112	-11.165608	-11.042959	-12.299539	-10.249616
s = 0.004	-13.378226	-12.081444	-11.839281	-14.349646	-10.249747

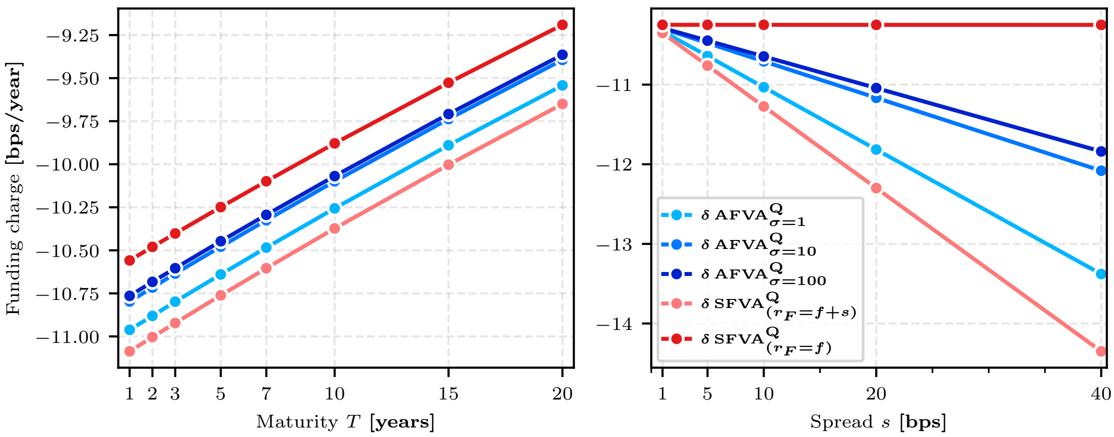

Figure 10: MC-estimated funding charges for **Portfolio 5-7** and **Derivative 3**

From Table 27, we observe a slight decrease in the asymmetric funding charge quotes as the yearly volatility increases (i.e., as the portfolio number increases). The same trend is observed in Table 28. The Figure 10, gives a graphical view of this. We also see that the asymmetric charges seem to be between the symmetric ones for all maturity times T and spreads s.

6 Discussion

6.1 Analysis of results

We begin the discussion by noting that we observe that the symmetric charge seems to be quicker to compute compared to the asymmetric charge, which is expected (see Table 16).

for derivative 1, when the spread s is varied, the symmetric funding charge quote using $r_F = f$ ($\delta\,\mathrm{SFVA}^{\mathbb{Q}}_{r_F=f}$) represents estimates of the same "true" value, as it does not depend on s. The reason for the slight differences in these estimates (see Tables 18 and 20, rightmost column) is that a different seed was used for each estimate. This variation in estimates using $r_F = f$ is also observed in the other two derivatives when varying s, for the same reason. Nevertheless, it would not be considered an issue, for the reason that all the estimates are consistent up to two decimal places.

Furthermore, when examining the results from portfolios 1 to 4, it is observed that for all three derivatives, the asymmetric funding charge quotes converge to a symmetric funding charge quote, when the expected portfolio value increases relative to the yearly volatility of the portfolio ($\sigma = \sqrt{\mathrm{Var}\left(\mathcal{V}_1\right)}$). This observed convergence suggests that for portfolios with a sufficiently large expected value compared to the yearly volatility, an asymmetric charge can be substituted with a symmetric charge.

Additionally, the two symmetric funding quotes using $r_F = f$ or $r_F = f + s$ are very similar in value for derivatives 1 and 2. This similarity is expected, as these derivatives are chosen to be at-the-money, i.e., $\mathbb{E}\left[\mathcal{E}^1\right] = \mathbb{E}\left[\mathcal{E}^2\right] = 0$. Analytically, this means that the symmetric charge $\delta\,\mathrm{SFVA}$ in 32 is zero for both r_F values. Therefore, derivatives 1 and 2 do not provide sufficient information to determine whether $r_F = f$ or $r_F = f + s$ should be used in the substitution.

However, for derivative 3, we observe that the asymmetric quotes seem to converge to the symmetric quotes using $r_F = f + s$ as the expected value increases relative to the yearly volatility. This could indicate that $r_F = f + s$ is the appropriate rate to use when substituting a symmetric charge for an asymmetric one. The results for derivative 3 are also consistent with the asymptotic argument presented in theorem 4.1, where it was shown that the asymmetric charge converges to a symmetric charge as the expected portfolio value approaches infinity.

The first portfolio where the asymmetric quotes are sufficiently close (within 0.1 bps) to the symmetric quote with $r_F = f + s$ across all tested maturities T and spreads s is portfolio 3, where the expected portfolio value equals twice the yearly volatility. Therefore, we suggest that a symmetric funding charge with $r_F = f + s$ can be substituted for an asymmetric funding charge when the portfolio's expected value is greater than or equal to three times the portfolio yearly volatility. This finding addresses the research question and will be used as a guideline to SEB.

We highlight that in cases where the expected value is equal to zero or equal to the yearly volatility, the results do not support substituting SFVA for AFVA. This is because the AFVA quote remains too far from the SFVA quote, as seen, for instance, with $\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{0\sigma}$ in Table 17 or $\delta\,\mathrm{AFVA}^{\mathbb{Q}}_{1\sigma}$ for $s = 0.001$ in Table 22. Considering different spreads s is needed, because, in practice, s might not be constant and could increase, thereby widening the gap between $\mathrm{AFVA}^{\mathbb{Q}}$ and $\mathrm{SFVA}^{\mathbb{Q}}$. This effect is observed, for example, in Table 22.

We now move the discussion to the results from portfolios 5 to 7. When we examine the effect of increasing the portfolio's yearly volatility while keeping the expected value at zero (as the portfolio number increases), we observe no significant impact on the asymmetric quotes for derivatives 1 and 2, and only a small decrease in the asymmetric quotes for derivative 3. The minimal impact of increasing volatility in the case of zero expected value suggests that the primary factor to consider when determining whether SFVA can substitute for AFVA is the relative size of the expected portfolio value compared to its volatility.

6.2 Limitations

One limitation of the numerical analysis in section 5 is that it is conducted using a single model, and the results might differ when another model is considered. But as we mentioned earlier, the model we chose is widely used in reality, so we expect the results to be practically useful. Additionally, since the asymptotic argument in subsection 4.2 demonstrated that the AFVA charge converges to the SFVA charge, we have both theoretical evidence and numerical results from two different models suggesting that AFVA converges to SFVA. This suggests that the result could hold in other models as well.

Furthermore, forward pricing is only considered in the numerical analysis. This is something that one might want to consider if one wants to apply the results of this thesis to other derivatives.

Another limitation is the Gaussian assumption in the theoretical result in theorem 4.1. Thus, it is important to note that the asymptotic result is not a proof that AFVA converges to SFVA in general.

6.3 Future work

The findings of this thesis opens up for interesting future research.

Stochastic interest rate model: One research direction is executing the numerical investigation of section 5 to a more advanced model, such as calculating funding charges for FX forwards in a stochastic rate model, using the Hull-White model [19], i.e.

$$\begin{aligned} dS_t^i &= r_t^i S_t^i \, dt + \sigma_i S_t^i \, dW_t^i, \qquad i = 1, \dots, 5, \\ dr_t^i &= (\theta_t^i - \alpha^i r_t^i) \, dt + \varsigma^i \, dZ_t^i, \end{aligned}$$

where for $i = 1, 2, \dots, 5$ we have that θ_t^i is a deterministic function, α^i and ς^i are constants and Z^i is a Wiener process, independent of W and Z^j for $i \neq j$. This investigation could be done to further validate that the observed results is not only a property of the chosen model in this thesis. One could also try to extend the analysis by looking at more types of derivatives, such as options, to investigate that the results does not solely for forwards.

Extend simplified model: Another direction is to extend the simplified model in section 4 by considering general Itô diffusions,

$$\begin{aligned}
d\mathcal{V} &= \mu_t^{\mathcal{V}}\, dt + \sigma_t^{\mathcal{V}}\, dW^1, \\
d\mathcal{E} &= \mu_t^{\mathcal{E}}\, dt + \sigma_t^{\mathcal{E}}\, dW^2, \\
\langle W^1, W^2 \rangle_t &= \rho t,
\end{aligned}$$

and investigate if the asymptotic result can be established under some assumptions on $\mu_t^{\mathcal{V}}, \sigma_t^{\mathcal{V}}, \mu_t^{\mathcal{E}}$ and $\sigma_t^{\mathcal{E}}$. The Theorem 4.2 (Tanaka's formula) in [20] might be of use here. If such an extension is possible, this could theoretically show that the asymmetric funding charge tends to the symmetric funding charge as the expected portfolio value increases for a wide range of models.

Taylor approximation: Lastly, one could also study the Taylor approximation in the simplified model from 4 to evaluate how well it performs compared to the regular AFVA. This comparison can be conducted in the FX model from section 5, by examining the relative difference between $\delta\overline{\text{AFVA}}$ and $\delta\,\text{AFVA}$. Since we observed in this thesis that the SFVA does not always approximate AFVA well, this research direction is valuable because the Taylor approximation might be useful in cases where SFVA fails to adequately approximate AFVA.

6.4 Social and Ethical aspects

This thesis improves the knowledge of funding charges and may therefore contribute to more accurate and well-informed funding charge models. Accurate models are important for the financial market, and can be motivated by the contrary. Inaccurate financial models can contribute to systemic risks, such as those seen in the 2008 financial crisis. Such risks can have severe social consequences, such as loss of savings and financial distress.

Ethically, it is important that FVA models accurately reflect true market values. Misrepresenting asset values can mislead stakeholders and lead to poor decision-making that can harm investors or clients. We believe that the results of this thesis can be used to aid an informed decision on how to accurately calculate an FVA charge.

7 Conclusions

The main research question of this thesis centers around investigating the differences between the symmetric funding value adjustment (SFVA) and the asymmetric funding value adjustment (AFVA), and whether the SFVA could approximate the AFVA. The need for approximating AFVA with SFVA arises from the computationally intensive nature of a Monte Carlo estimation of the AFVA. Through a simplified Gaussian model, we successfully derive analytical expressions for the positive and negative exposures and use these to derive an asymptotic result. This finding shows that, in the Gaussian model, AFVA converges to SFVA as the portfolio's expected value goes to infinity. The result provides insight that AFVA seems to converge to SFVA.

Furthermore, we used a simple yet practically useful FX model to numerically investigate how large AFVA is compared to SFVA. In this model, we conclude that the AFVA seems to converge to SFVA as the expected portfolio value increases. This finding, although in another model, aligns with the asymptotic result. Additionally, we conclude that an SFVA, with a funding rate equal to the mid-rate plus the funding spread, can be used instead of an AFVA for the FX model when the portfolio's expected value is three times the portfolio's yearly volatility.

The thesis contributes by adding more explanations and details to the derivation of the value adjustments by Burgard and Kjaer (see [5] and [1]). This might help a novice in value adjustments to understand the derivation more clearly and make some of the assumptions more transparent.

The findings of this thesis open up for several future research directions, which may improve our understanding of funding charges.

References

[1] C. Burgard and M. Kjaer. "Funding costs, funding strategies". In: *Risk* (2013).

[2] J. Hull. *Options, Futures, and Other Derivatives.* Pearson, 2012.

[3] J. Gregory. *The xVA Challenge: Counterparty Risk, Funding, Collateral, Capital and Initial Margin.* Wiley, 2020.

[4] Bank for International Settlements. *Basel III: A global regulatory framework for more resilient banks and banking systems.* BIS, 2011.

[5] C. Burgard and M. Kjaer. "PDE representations of options with bilateral counterparty risk and funding costs". In: *The Journal of Credit Risk* (2011).

[6] J. Hull and A. White. "The FVA debate". In: *Risk* (2012).

[7] C. Albanese and L. BG Andersen. "Accounting for OTC derivatives: Funding adjustments and the re-hypothecation option". In: *SSRN* (2014).

[8] I. Ruiz. *FVA Calculation and Management.* 2014. URL: https://www.mocaxintelligence.com/wp-content/uploads/2016/02/iRuiz_FVA_calc_and_mngmnt.pdf (visited on 07/20/2024).

[9] European Central Bank. *What is collateral?* 2016. URL: https://www.ecb.europa.eu/ecb-and-you/explainers/tell-me/html/collateral.en.html (visited on 07/16/2024).

[10] T. Björk. *Arbitrage Theory in Continuous Time.* 4th edn. Oxford University Press, 2020.

[11] T. Björk. *An Introduction to Point Processes from a Martingale Point of View.* 2011. URL: https://www.math.kth.se/matstat/fofu/PointProc3.pdf (visited on 07/31/2024).

[12] D. Brigo et al. "Illustrating a problem in the self-financing condition in two 2010-2011 papers on funding, collateral and discounting". In: *arXiv preprint arXiv:1207.2316* (2012).

[13] Nicolas Privault. *Stochastic Calculus for Jump Processes.* 2024. URL: https://personal.ntu.edu.sg/nprivault/MA5182/stochastic-calculus-jump-processes.pdf (visited on 07/10/2024).

[14] B. Øksendal. *Stochastic differential equations.* Springer, 2003.

[15] London FX Ltd. *Quotation Styles.* URL: http://www.londonfx.co.uk/quoting.html#top (visited on 07/28/2024).

[16] P. Glasserman. *Monte Carlo Methods in Financial Engineering.* Springer, 2004.

[17] T. Sauer. *Numerical analysis.* 2nd edn. Pearson Education, 2012.

[18] G. Casella and R. L. Berger. *Statistical inference.* Cengage Learning, 2002.

[19] D. Brigo and F. Mercurio. *Interest Rate Models Theory and Practice.* 2nd edn. Springer, 2007.

[20] T. Björk. "The Pedestrian's Guide to Local Time". In: *arXiv preprint arXiv:1512.08912* (2015).

A Financial mathematics

In this section of the appendix, we go through some terminology in the theory of financial mathematics from Björk [10]. This terminology might be helpful in understanding the thesis.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, $\mathbb{P}$ is a probability measure and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration with $\mathcal{F}_t \subset \mathcal{F}$ sub-σ-algebras.

Definition A.1. *A continuous time financial market model is a collection of N $\mathbb{F}$-adapted random variables $\{S_t^i\}_{i=1}^N$, each representing the price/value of one unit of an asset at time t. These are known as asset value processes or just asset values. S_t denotes the N-dimensional vector with S_t^i as components, i.e. $S_t = (S_t^1, S_t^2, \ldots, S_t^N)$.*

Definition A.2. *A portfolio strategy, also known as just portfolio, is an $\mathbb{F}$-adapted N-dimensional random vector process $\beta_t = \begin{bmatrix}\beta_t^1 & \beta_t^2 & \ldots & \beta_t^N\end{bmatrix}$.*

Definition A.3. *Given a portfolio strategy β, the value process of the portfolio strategy β at time t is defined by*

$$V_t^\beta = \sum_{i=1}^N \beta_t^i S_t^i \tag{80}$$

Definition A.4. *Let β be a portfolio strategy. If dV_t^β and dS_t^i represents the incremental change of a portfolio value process V_t^β and an incremental change of an asset value process S_t from time t to $t + dt$, respectively, then β is said to be a self-financing portfolio if*

$$dV_t^\beta = \sum_{i=1}^N \beta_i dS_t^i \tag{81}$$

for all $0 \leq t < \infty$.

Definition A.5. *A financial derivative, also known as a contingent claim or just derivative, with exercise date[2] T, is a $\mathcal{F}_T$-measurable random variable $\mathcal{X}$.*

Definition A.6. *A financial derivative $\mathcal{X}$ is called simple if $\mathcal{X} = \phi(S_T)$ for some function $\phi : \mathbb{R}^N \to \mathbb{R}$.*

Definition A.7. *Let $\mathcal{X}$ be a financial derivative with exercise date T and β a self-financing portfolio strategy. Then β is said to be a replicating[3] portfolio strategy of $\mathcal{X}$ if*

$$V_T^\beta = \mathcal{X}, \quad \mathbb{P} - a.s. \tag{82}$$

The derivative $\mathcal{X}$ can be replicated (is replicable) if there exists a corresponding replicating portfolio.

Definition A.8. *A self-financing portfolio strategy β is said to be an arbitrage portfolio strategy if*

$$V_0^\beta = 0, \tag{83}$$

$$\mathbb{P}\left(V_T^\beta \geq 0\right) = 1, \tag{84}$$

$$\mathbb{P}\left(V_T^\beta > 0\right) > 0. \tag{85}$$

We say that the market model is free of arbitrage (arbitrage free) if no arbitrage portfolios exist.

[2]Also known as maturity date/time or time of delivery

[3]or hedging

Theorem A.1. *Let Π_t denote the price at time $t \leq T$ of a replicable derivative $\mathcal{X}$ with exercise time T. Since $\mathcal{X}$ is assumed to be replicable, let β be a corresponding replicating portfolio strategy. Then the only derivative price process not admitting arbitrage is*

$$\Pi_t = V_t^{\beta}, \quad \forall t \in [0, T] \tag{86}$$

Definition A.9. *Let $\mathcal{X}$ be a financial derivative with maturity time T. A forward contract on $\mathcal{X}$, contracted at t, with time of delivery T and forward amount K follows the payment scheme (for the point of view of the contract owner):*

- *At time t, the amount K is determined*
- *At time T, the owner of the forward contract gets the amount $\mathcal{X}$*
- *At time T, the owner of the forward contract pays K*

Let $\Pi_s[\mathcal{X}]$ be the price of $\mathcal{X}$ at time s. Then the price of a forward $F_{\mathcal{X}}$ on $\mathcal{X}$ is given by $\Pi_s[F_{\mathcal{X}}] = \Pi_s[\mathcal{X}] - p(s, T)K$.

B Supporting computations

Claim B.1. *Let $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$. If $X \sim \mathcal{N}(a, b^2)$ is a Gaussian random variable, then*

$$\mathbb{E}\left[X^+\right] = \frac{1}{2}\left(\sqrt{\frac{2b^2}{\pi}} \exp\left(\frac{-a^2}{2b^2}\right) + a \cdot \operatorname{erf}\left(\frac{a}{\sqrt{2b^2}}\right) + a\right). \tag{87}$$

Proof. First compute

$$\mathbb{E}\left[X^+\right] = \int_{-\infty}^{\infty} \frac{x^+}{\sqrt{2\pi b^2}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx \tag{88}$$

$$= \int_{0}^{\infty} \frac{x}{\sqrt{2\pi b^2}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx \tag{89}$$

$$= \int_{-a}^{\infty} \frac{t+a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt \tag{90}$$

$$= \underbrace{\int_{-a}^{\infty} \frac{t}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt}_{(*)} + \underbrace{\int_{-a}^{\infty} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt}_{(**)}. \tag{91}$$

$$\tag{92}$$

Then, looking at (*) and (**) separately, we have

$$(*) = \int_{\frac{a^2}{2b^2}}^{\infty} \frac{b^2}{\sqrt{2\pi b^2}} \exp(-u)\, du \tag{93}$$

$$= \sqrt{\frac{b^2}{2\pi}} \exp\left(-\frac{a^2}{2b^2}\right) \tag{94}$$

and

$$(**) = \int_{-a}^{\infty} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt \tag{95}$$

$$= \int_{-a}^{0} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt + \int_{0}^{\infty} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt \tag{96}$$

$$= \int_{-a}^{0} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt + \frac{1}{2}a \tag{97}$$

$$= \int_{0}^{a} \frac{a}{\sqrt{2\pi b^2}} \exp\left(-\frac{t^2}{2b^2}\right) dt + \frac{1}{2}a \tag{98}$$

$$= \frac{a}{\sqrt{\pi}} \int_{0}^{\frac{a}{\sqrt{2b^2}}} \exp\left(-u^2\right) du + \frac{1}{2}a \tag{99}$$

$$= \frac{a}{2} \operatorname{erf}\left(\frac{a}{\sqrt{2b^2}}\right) + \frac{1}{2}a. \tag{100}$$

Combining (*) and (**) yields the claim. □

Claim B.2. *Let $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$. If $X \sim \mathcal{N}(a, b^2)$ is a Gaussian random variable, then*

$$\mathbb{E}\left[X^-\right] = -\frac{1}{2}\left(\sqrt{\frac{2b^2}{\pi}} \exp\left(\frac{-a^2}{2b^2}\right) + a \cdot \operatorname{erf}\left(\frac{a}{\sqrt{2b^2}}\right) - a\right). \tag{101}$$

Proof. First compute

$$\mathbb{E}\left[X^-\right] = \int_{-\infty}^{\infty} \frac{x^-}{\sqrt{2\pi b^2}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx \tag{102}$$

$$= \int_{-\infty}^{0} \frac{x}{\sqrt{2\pi b^2}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx \tag{103}$$

$$= -\int_{\infty}^{0} \frac{-t}{\sqrt{2\pi b^2}} \exp\left(-\frac{(-t-a)^2}{2b^2}\right) (-1)dt \tag{104}$$

$$= -\int_{0}^{\infty} \frac{t}{\sqrt{2\pi b^2}} \exp\left(-\frac{(t+a)^2}{2b^2}\right) dt \tag{105}$$

$$= -\mathbb{E}\left[(-X)^+\right]. \tag{106}$$

But it is known that $-X \sim \mathcal{N}(-a, b^2)$. By claim B.1 we obtain

$$\mathbb{E}\left[X^-\right] = -\frac{1}{2}\left(\sqrt{\frac{2b^2}{\pi}} \exp\left(\frac{-a^2}{2b^2}\right) + (-a) \cdot \operatorname{erf}\left(\frac{-a}{\sqrt{2b^2}}\right) - a\right). \tag{107}$$

Using the fact that the error function is odd gives

$$\mathbb{E}\left[X^-\right] = -\frac{1}{2}\left(\sqrt{\frac{2b^2}{\pi}} \exp\left(\frac{-a^2}{2b^2}\right) + a \cdot \operatorname{erf}\left(\frac{a}{\sqrt{2b^2}}\right) - a\right), \tag{108}$$

which concludes the claim. □

C Analysis of approximation formula

We apply our approximation formula from subsection 4.3 across various model values to assess the magnitude of the $\delta\overline{\text{AFVA}}$ charge. Additionally, using the same model parameters, we compute the asymmetric correction from (68) to evaluate its influence on the approximation formula $\delta\overline{\text{AFVA}}$.

We choose the following parameters, $T = 5$, $\rho = 0.5$, $f = 0.01$, $r = 0$, $s = 0.0005$, $\lambda_A = 0.005$, and $\lambda_B = 0.01$. We vary $\sigma_{\mathcal{E}}$ with values of 1, 5, 10, 50, 100, and 200. Furthermore, we vary $\alpha = -10, -5, 0, 5$ and 10. We let $\sigma_{\mathcal{V}} = 10\sigma_{\mathcal{E}}$, since in practice we would assume that the portfolio has a greater standard deviation. Additionally, we define μ as $k\sigma_{\mathcal{V}}$, where the scale factor k ranges from -4 to 4. The choice of parameters are both for simplicity and to get a good overview of different values.

We get the following Figure 11 and 12. On the x-axis we show the scale factor k and on the y-axis we show either $\delta\overline{\text{AFVA}}$ or the asymmetric correction.

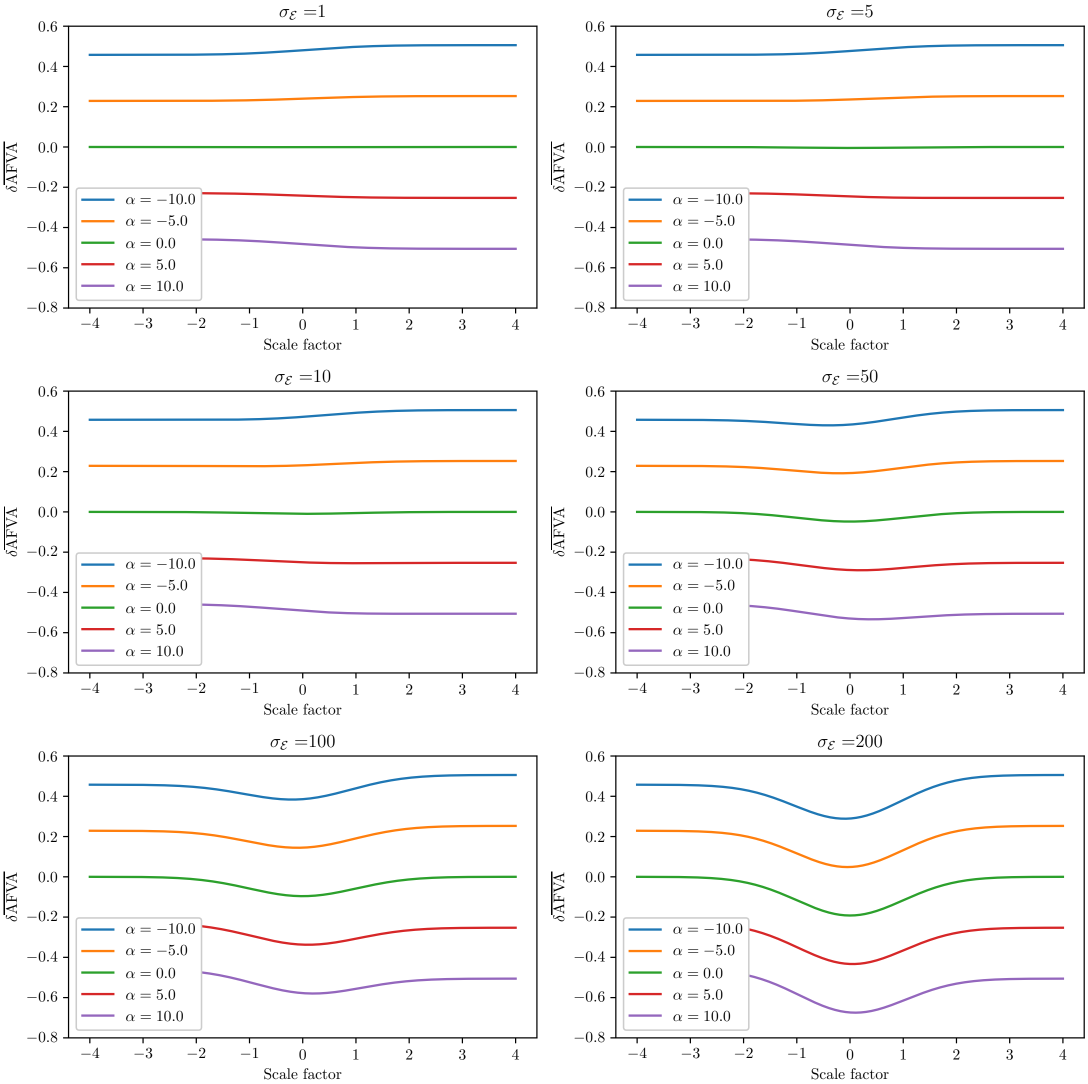

Figure 11: Approximated funding charges using $\delta\overline{\text{AFVA}}$.

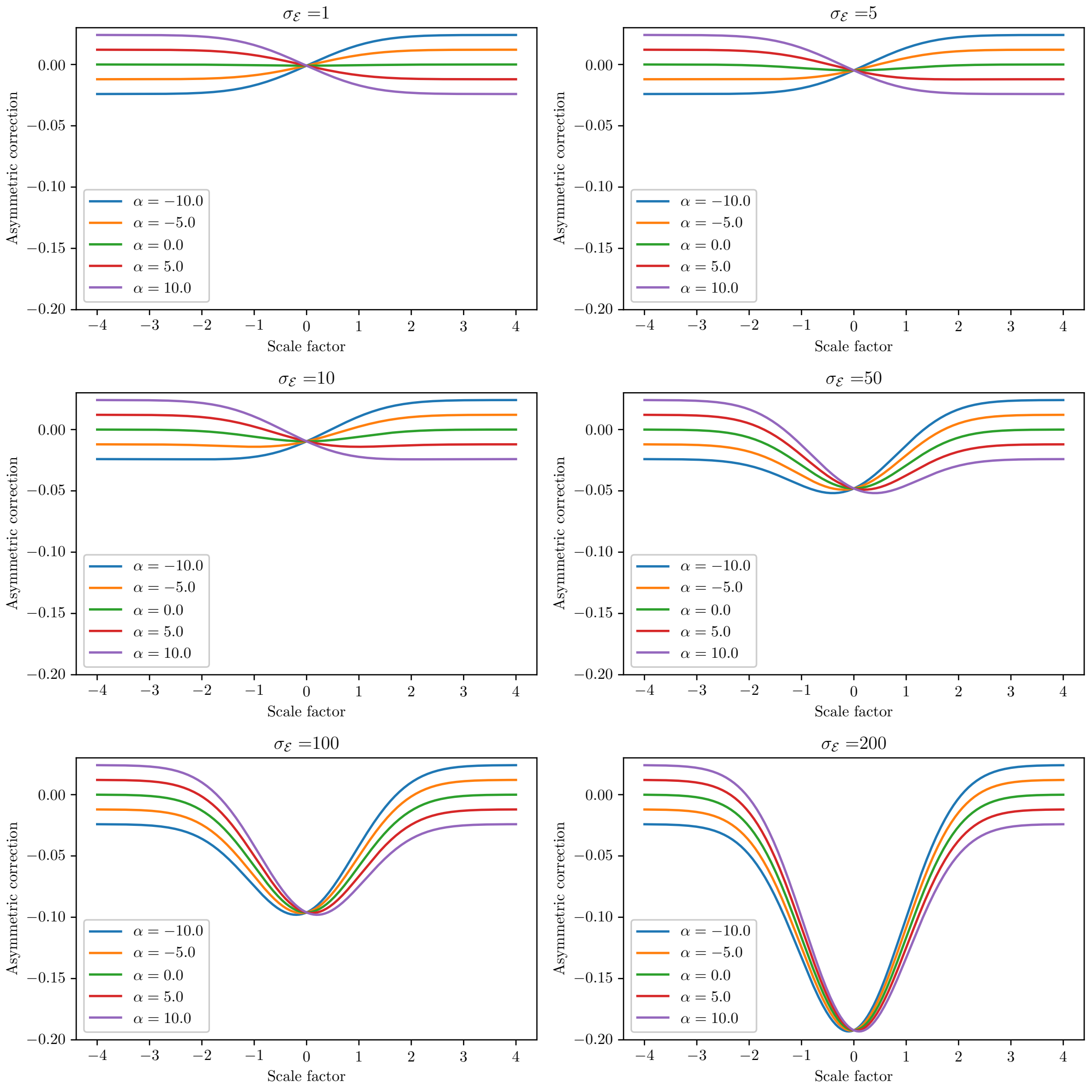

Figure 12: Asymmetric corrections.

Notice that a negative funding charge value means that the issuer will receive money, whilst a positive funding charge means that the issuer has to pay money. Thus, when we say that the charge increase, we mean that the actual value becomes more negative. Looking at Figure 11, it is observed that increasing the derivative expected value increases the approximated funding charge as expected. Furthermore, we see that the $\delta\overline{\text{AFVA}}$ becomes, in some sense, more and more convex around zero expected portfolio value as volatility of the derivative increase. We believe that this can be explained by Figure 12, as $\sigma_{\mathcal{E}}$ increase, the asymmetric correction increase (becomes more negative), with the largest value around zero. Thus, it seems like the asymmetric correction impact is largest around a zero expected portfolio value when the volatility of the derivative increases.

D Antithetic sampling for GBM

We now cover a method for variance reduction in Monte Carlo sampling. Assume that the stochastic process $(S_t)_{t\geq 0}$ has the following geometric Brownian motion (GBM) dynamics

$$dS_t = aS_t\,dt + bS_t\,dW_t, \tag{109}$$

$$S_0 = s_0, \tag{110}$$

where W_t is a Wiener process and a and b are real constants. Then the solution [14] is

$$S_t = s_0 \exp\left((a - \frac{b^2}{2})t + bW_t\right). \tag{111}$$

Thus S_t can be seen as a function of W_t, $S_t = S(W_t)$. Let $\psi : \mathbb{R} \to \mathbb{R}$ and assume that we want to estimate $\mathbb{E}\left[\psi(S_T)\right]$ using the N samples $\{W_T^i\}_{i=1}^N$. Then the Monte Carlo estimate with antithetic sampling τ_N is given by

$$\tau_N = \frac{1}{N}\sum_{i=1}^{N} \frac{\psi\left(S(W_T^i)\right) + \psi\left(S(-W_T^i)\right)}{2}. \tag{112}$$

It can be shown that this estimator has a variance less than or equal to that of the usual Monte Carlo estimate using $2N$ samples

$$\frac{1}{2N}\sum_{i=1}^{2N} \psi\left(S(W_T^i)\right), \tag{113}$$

since W and $-W$ are negatively correlated [16].