Escort protection with surface combatant ships

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Abstract

In this thesis the optimal positioning of escort ships tasked with escorting and protecting a vessel from anti-ship missile (ASM) attacks is explored. With the Visby-class corvettes being outfitted with surface-to-air missiles (SAM), the study focuses specifically on how the escort ships, limited to using SAMs to shoot down the ASMs, best position themselves to protect the escorted vessel.

As part of the study a missile allocation problem, based on the missile kinematics and the probability of successful interception, for minimizing the expected number of SAMs fired in order to value the characteristics of an optimal position, is developed. A baseline scenario, where the escort ship is placed as close to the escorted vessel as possible, is used as a starting point of the optimization. From this baseline, different ship placements seeking to increase the number of shoot-look-shoot (SLS) opportunities, a crucial factor of the missile allocation problem, are explored.

The key contribution of the work is finding the positions that maximize the sector within which the number of SLS opportunities are increased compared to the baseline positioning. The results show that while the baseline positioning is often effective, collaboration between ships can lead to significant improvements. Additionally, the study explores how the optimal ship placements change based on varying degrees of uncertainty in the threat direction.

The developed model and results provide a means to model a combat scenario involving an escorted ship, escorting ships and attacking ASMs. The model can be implemented to improve a larger scale naval combat model or as a starting point for more exact optimization models

Sammanfattning

I denna avhandling undersöks den optimala positioneringen av eskortfartyg som har uppdraget att eskortera och skydda ett fartyg mot anfall från sjömålsrobotar. När Visby-klasskorvetterna utrustas med luftvärnsrobotar fokuserar studien särskilt på hur eskortfartygen, begränsade till att använda luftvärnsrobotar för att skjuta ner sjömålsrobotar, bäst positioneras för att skydda det eskorterade fartyget.

Studien utvecklar ett robotallokeringsproblem baserat på robotarnas kinematik och sannolikheten för framgångsrik nedskjutning, för att minimera det förväntade antalet avfyrade luftvärnsrobotar och på så sätt utvärdera egenskaperna som utgör en optimal position. Ett basscenario, där eskortfartyget placeras så nära det eskorterade fartyget som möjligt, används som utgångspunkt för optimeringen. Från detta basscenario utforskas olika fartygsplaceringar som syftar till att öka antalet ”shoot-look-shoot”-möjligheter, en avgörande faktor i robotallokeringsproblemet.

Det viktigaste bidraget i arbetet är att identifiera de positioner som maximerar sektorn inom vilken antalet SLS-möjligheter ökar jämfört med basscenariot. Resultaten visar att även om basscenariot ofta är effektivt, kan samarbete mellan fartygen leda till betydande förbättringar. Dessutom undersöks i studien hur de optimala fartygsplaceringarna förändras beroende på olika grader av osäkerhet kring hotets riktning.

Den utvecklade modellen och resultaten ger ett sätt att modellera ett stridsscenario som involverar ett eskorterat fartyg, eskorterande fartyg och angripande sjömålsrobotar. Modellen kan implementeras för att förbättra en större sjöstridsmodell eller som en utgångspunkt för mer exakta optimeringsmodeller.

Contents

List of abbreviations

SLS	–	Shoot-look-shoot
ASM	–	Anti-ship missile
SAM	–	Surface-to-air missile
MAP	–	Missile allocation problem
SAP	–	Section allocation problem

List of symbols

x	–	x-coordinate of escort ship position
y	–	y-coordinate of escort ship position
u_{ASM}	–	ASM velocity
u_{SAM}	–	SAM velocity
t_f	–	Time it takes detect an incoming ASM after it enters radar reach and to evaluate the outcome of an engagement
N	–	SAMs available fot the entire engagement
T	–	SLS opportunities
u_t	–	SAMs fired in SLS slot t
p_{kill}	–	Probability of shooting down ASM with a single SAM
θ	–	Angle between line of sight between escort ship and ASM when a salvo of SAMs are fired and the point where the SAM(s) will intercept the ASM
p_t	–	p_{kill} value in SLS slot t
y_t	–	State of ASM in SLS slot t
r	–	Distance between ASM and escorted vessel
R	–	Radar reach of escort ship
V	–	Cost-to-go function
y	–	y-coordinate of escort ship position
β	–	Angle of ASM trajectory
λ	–	Angle of line of sight between escort ship and ASM
r_{min}	–	Radius around escorted vessel within which the SAMs are not allowed to chase the ASM

1 Introduction

1.1 Background

The Swedish Defense Research Agency (FOI) conducts research regarding technologies with military application mainly for the Swedish Civil and Military defense. The objective of this research is to provide support to the civil and military defence in their work to develop new tactics and acquire new equipment.

One research area is how to accurately simulate combat scenarios for example in the naval domain where one possible task for the navy is to escort transport ships and protect the escorted ships from anti-ship missiles (ASM). Simulating such a scenario requires an accurate model in order to capture the effects of the interactions between the escorted ship, the escorting ships and the enemy ASMs. One part of that model is the positioning of the escorted ships and their effect of the probability of survival of the escorted ships. This is done specifically in the context of the Visby-class corvettes being upgraded and outfitted with surface-to-air missiles.

The aim of this thesis is to define a simplified model of these interactions and study the optimal positioning of surface combatant ships tasked with escorting and protecting a certain vessel. The optimization is centered specifically around the problem of shooting down incoming anti-ship missiles with surface-to-air missiles [1].

1.2 Literature Review

As a defense research agency FOI has previously compiled information and conducted research regarding naval escort missions, ASM and defensive measures against ASMs. In [2] possible requirements for a future naval air defense is presented for a number of combat assignments and threats, among them escort missions. In describing the general tactic of this type of mission the authors write that normally multiple ships are deployed in order to defend against threats coming from multiple directions. Regarding ASMs the authors write that they will likely remain the greatest threat to naval platforms in the foreseeable future and that the ASMs in use at the present generally fly at subsonic speeds, are designed for elaborate end phase maneuvers and fly at low altitudes to avoid detection. Also they use a late stage "pop-up" maneuver to orient themselves towards the intended target.

Regarding the anti-aircraft robots that can be used to shoot-down incoming ASMs and which in this paper is referred to as SAMs, the authors write that although it is possible to deploy these towards targets at a long range they are limited by their ability to detect incoming threats. They also write that within a certain area around the ship SAMs can not be deployed without risk to the safety of the ship itself and that when deciding the number of SAMs to be deployed against a threat a measure of the probability of disabling the threat with

one SAM, referred to as p_{kill} is used. This probability depends on the type of threat and the actions of the threat and include both the probability of hitting the target and the probability of this hit being damaging enough to disable the target.
In describing the duel between ASMs and SAMs the authors write that this can be seen as a battle of time. For the ASMs to be successful at destroying the escorted vessel the enemy would like to delay their detection as much as possible using low signature, high speeds and low altitude. Meanwhile the defense, which benefits from creating as much time as possible to act on the incoming ASMs, aim for early detection, short reaction time, high SAM speed, high p_{kill} and the ability to fire a lot of SAMs in a short period of time.

In researching the problem of defending against ASMs the focus usually lies on the missile allocation problem (MAP) as in [3] and [4]. MAP is an NP-complete problem which means that there doesn't exist an polynomial-time algorithm for solving it [5]. It is therefore a complex problem to solve that may require significant computation power or approximations in order to reduce the complexity of the problem [6]. Although the problem of missile allocation is well studied there are few studies which include the positions of the defending ships.
One author that have, to a certain extent, combined MAP with the positioning of ships is [7] who uses MAP to solve a sector allocation problem (SAP). By solving MAP the author calculates the expected value of the coverage that a ship in one sector provides to a ship in another sector and uses this value to formulate an optimization problem of assigning the ships to sectors. One drawback of [7], besides focusing mainly on MAP, is that it is assumed, without any real motivation, that it is more effective to maximize the protection of the individual ships instead of trying to protect a large area.
There are other studies in which the positioning or formation of the ships are partly incorporated. In [8] the optimal navigation of a naval task group going through the strait of Hormuz is studied where the positions of the threats are assumed to be known. The results show that the optimal placement of the escorting ships is usually in front of the carrier, which is the vessel being escorted, and always between the carrier and the closest ground-based threat.
The authors in [9] formulate an optimization problem for selecting which ships to include in a defensive formation based on having a base of certain number of ships to choose from which all have different equipment onboard with varying reliability. The optimized deployment of the fleet is based on the cooperative quality of the ship-borne equipment and is calculated using reliability theory. To simulate the anti-saturation capability of a group of ships coming under attack by anti-ship missiles [10] uses Monte-Carlo methods. The simulations are based on the spatial characteristic of the ship formation, numerical characteristics of the ASM shooting stream and the dynamic combat process. Also, cooperative models including detection, interception region and confrontation of the ship formation to anti-ship missiles were included in the simulations as well. Through these the authors were able to calculate the number of ASMs needed in order

to penetrate the defense, but only in one defensive formation using three ships. The authors in [11] similarly also look at only one specific set-up. Here queuing theory is used to look at multi-ship cooperation and compares the defensive capability with and without coordination between the ships.

There are few studies that look specifically at investigating the best defensive positioning of ships. In [12] a static, probabilistic 2-dimensional optimization model is presented for maximizing the probability of survival of a carrier being protected and escorted by surface combatant ships. The probability is modeled as a function of the distance between a ship and the carrier and the minimum distance between a ship and the trajectory of an incoming ASM. However the model is limited to finding the optimal positioning within a 90 degree sector.

There have also been attempts to create analytical tools which can evaluate the defensive effectiveness of a naval task group coming under attack by anti-ship missiles. In [13] a tool called the Anti-Ship Missile Defense (ASMD) model is created which models the process by which the escorts assign defensive fire and which can be used to simulate combat scenarios. The authors in [14] create an analytical tool with a graphical interface for evaluating defensive effectiveness against shore-based threats and use regression analysis to find out the effect of sector allocation and what the distance from the escorting ships to the protected vessel has on it's probability for survival.

1.3 Problem description

As the object of the optimization is to increase the survival probability of the escorted vessel one might assume that the most efficient way to do this would be to minimize the risk of an ASM surviving all the SAMs fired at it. This would mean placing the ships in a position where the probability of shooting down the ASM, p_{kill}, is the highest and where they have time to fire as many SAMs as possible before the ASM reaches the escorted vessel. However, based on the radar reach of the ships and their firing rate, firing a sufficient number of SAMs won't be the limiting factor, instead it will be having to set limitations on how many SAMs that can be afforded to be spent per ASM, both to reduce cost and reduce the risk of running out of ammunition, while still maintaining a high survival probability in every engagement.

Based on this reasoning the aim of increasing the escorted vessel's survival probability is translated, firstly, into formulating an optimization problem for minimizing the expected number of fired SAMs given one attacking ASM and one defending ship where the position of the ship and the attack direction of the ASM are simply input parameters, not variables. In other words a missile allocation problem.

Then, reasoning that when a ship moves away from the baseline scenario of positioning itself as close to the escorted vessel as possible, the span of attack directions will be divided into two sectors. One in which the survival probability has been improved and one in which it has become worse. An optimization model is set up to maximize the angle of this improved sector, given a set of constraints on the improvements one would like to achieve and using the missile

allocation problem to compare the improved position to the baseline scenario. This model is also designed to incorporate more ships and cooperation between ships.
Lastly, in order to complement this sector defense scenario model, uncertainty in the threat direction is introduced which by using a probability distribution as input is able to weigh the improvements within the improved sector against the losses outside it.

1.4 Delimitations

To keep the thesis within a reasonable scope a number of assumptions were made. First, for both the ASMs and the SAMs it was assumed that there were no difference between individual missiles and that all missiles fly in straight 2-dimensional trajectories with constant speed. Furthermore it was assumed that the ASMs only target the escorted vessel, not the defending ships, and that the speed of the ships in relation to the missiles is negligible. Also, it was assumed that the only available defensive measure on the ship were SAMs. Lastly, it was assumed that the ships are able to fire several SAMs at once.

1.5 Outline

The thesis is divided into sections where Section 1 introduces the subject and purpose of the thesis. Section 2 describes how the kinematics and probabilities involved in the battle of the ASMs and SAMs were modeled. This part is followed by Section 3 where the method behind formulating the optimization problem of finding the best position is described. In Section 4 the solutions of the optimization problems are presented. These solutions are then discussed in Section 5 where future work is also discussed. The thesis is concluded in Section 6 with a summery.

2 Model

In the following chapter Section 2.1 describes how the escort mission is described mathematically. Section 2.2 describes the process and limitations of the SAMs flight and interception of the attacking ASM. Finally, Section 2.3 describes the p_{kill} function and what factors decide whether the SAMs are successful in destroying the ASM.

2.1 Dynamic model

In the event of an attack the defending ships will try to defend the escorted vessel by shooting down incoming ASMs with SAMs. The escorted vessel is placed at the origin and as modeled the ASMs always fly in a straight line towards it. The position of the ASMs at any given time can be described in radial coordinates (r, v), where r is the radial distance from the escorted vessel and v is the angle as defined in Figure 1.

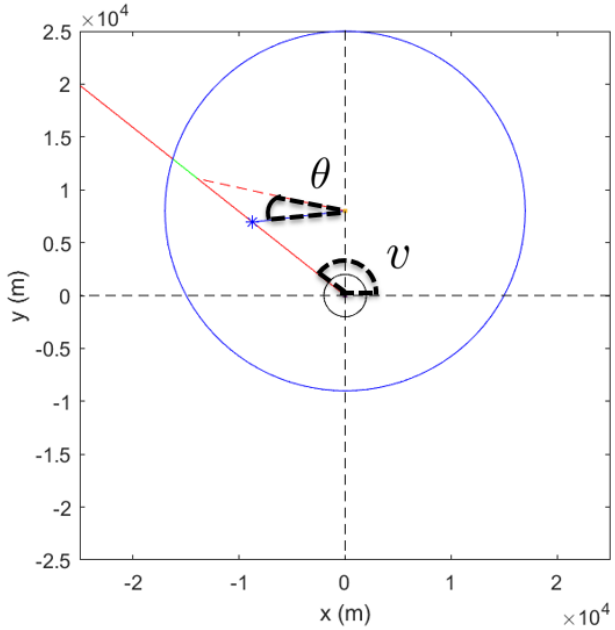


Figure 1: Illustration of the detection, flight and engagement between SAMs and ASMs. The escort ship is located at (0, 8km). The blue circle is the radar range of the ship. The red line outside the circle is the trajectory of the ASM before it is detected. As the ASM is detected it travels a certain distance before the ship is able to react and fire a salvo of SAMs at it. This distance is specified by the green line. θ is the angle between the vector between the ship and the ASM's position when the salvo is fired and the trajectory of the SAMs to the the point of interception. v is the angle of the ASM trajectory within the coordinate system.

Each ship has a position in Cartesian coordinates (x, y) and has a radar reach that extends an equal distance R in every direction within which it is assumed that the ship will immediately detect incoming ASMs. When the ASM is detected it will take a certain time for the ships to react, decide how many SAMs should be fired and launch those SAMs. The estimation of the time that it would take to finish this procedure is defined as the variable t_f which is constant.

This means that in the case when the ASM has just been detected or when a previous engagement has just finished t_f is the time it will take before the ship can fire a new salvo. The black circle in the figure represents r_{min} which is the distance from the escorted vessel within which it is no longer possible to intercept the ASMs. After being fired the SAM or SAMs fly towards the ASM until intercepting it at which point they blow up and destroy the ASM with a certain probability p_{kill} which is dependent on the angle, θ. The bigger the angle is the more the SAMs have to turn into and chase the ASM, thereby lowering their chance of destroying it.

2.2 Missile interception

The interception point of the SAM and the ASM depends on the respective positions of the missiles at the time the SAM is fired, their respective velocities as well as the relative angle between the missiles. When the problem is limited to two dimensions as shown in Figure 2 the trajectory to the intercept point is described by the angle θ [15]. This angle is described by the equation

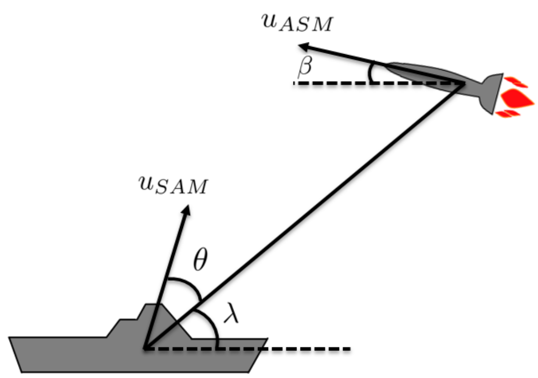


Figure 2: Illustration of the angles describing the linear interception process between ASMs and SAMs.

$$\theta = \sin^{-1} \frac{u_{ASM} \sin(\beta + \lambda)}{u_{SAM}}, \tag{1}$$

where u_{ASM} and u_{SAM} are the velocities of ASMs and the SAMs respectively. Translating this into the models coordinate system where the escorted vessel is placed at the origin, the interception point is calculated accordingly. The ship and ASM Cartesian coordinates is described by (x, y) and $(r\cos(v), r\sin(v))$ respectively and β in Figure 2 corresponds to $-v$. In order to calculate the angle θ, λ must first be calculated. This is done using the following formula:

$$\lambda = \arctan\left(\frac{r\sin(v) - y}{r\cos(v) - x}\right). \tag{2}$$

Using θ, β and the fact that the ASM and SAM have to reach the same x coordinate for the interception to occur it is then possible to calculate the time

t until interception.

$$(u_{ASM}\cos\beta - u_{SAM}\cos(\theta+\lambda))t = r\cos v - x \tag{3}$$

$$\implies t = \frac{r\cos(v) - x}{u_{ASM}\cos(\beta) - u_{SAM}(\theta+\lambda)}. \tag{4}$$

From this the position at which it occurs $(\bar{x}, \bar{y})$ and the distance s that the ASM has traveled from the moment the SAM was fired until interception is calculated:

$$(\hat{x}, \hat{y}) = (x, y) + u_{SAM}(\cos(\theta+\lambda), \sin(\theta+\lambda))t \tag{5}$$

$$s = ||(r\cos(v), r\sin(v)) - (\hat{x}, \hat{y})||_2. \tag{6}$$

As the velocities u_{ASM} and u_{SAM} is treated as constant s is a function of the radial coordinates (r, v) to the position of the ASM when the SAM is launched. This means that the position at interception can be written as $(r - s(r, v), v)$. Using this to create an iterative process by which the successive interception points are calculated it can be written that r_{k+1} from detection/engagement to interception is:

$$r_{k+1} = r_k - u_{ASM}t_f - s(r_k, v). \tag{7}$$

The maximum number of shoot-down opportunities possible should all the SAMs fail to shoot down the ASM then depends on how many iterations can be performed before $r_{k+1} \leq r_{min}$.

2.3 Probability of successful interception

The probability of a successful interception p_{kill} is modeled as a function of the angle θ so that it achieves its maximum value when $\theta = 0$ and then decrease exponentially as the angle increases, see Figure 3. This is an approximation describing the decreasing probability of a successful hit if the SAM has to "turn into" the ASM. The true value of p_{kill} depends on a large number of factors and the performance of SAMs are generally classified. Therefore the function defined by equation 8 is assumed to describe the p_{kill} value without further investigation. As can be seen in equation (8) the maximum value is set to 0.8.

$$p_{kill}(\theta) = 0.8e^{-\theta^2}. \tag{8}$$

In reality the distance the SAM has to travel before reaching its target also affects its performance. Based on the range estimates for a typical SAM this has been assumed to not have any effect within the rather short distances in the model.

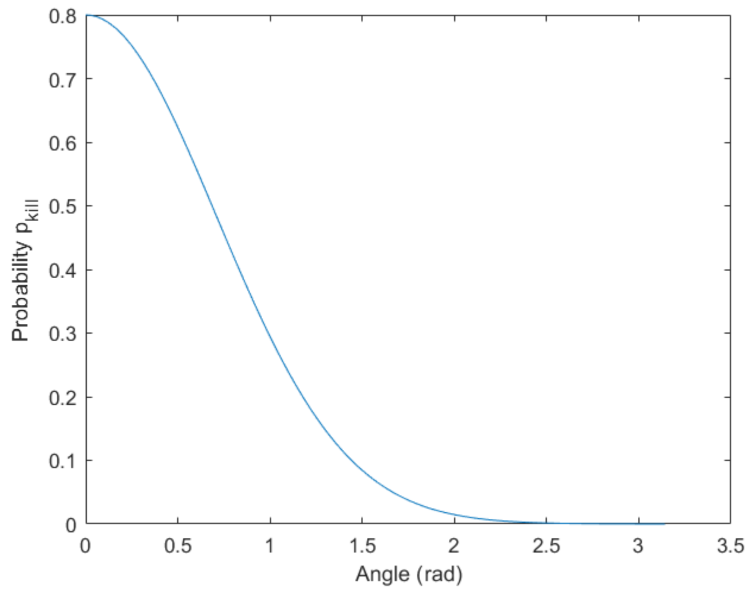


Figure 3: Probability of shooting down ASM with SAM as a function of the angle θ.

3 Method

In the following chapter Section 3.1 describes the baseline which the solutions of the optimization problems are compared to. In section 3.2 the method for minimizing the expected number of fired SAMs is explained. The optimization problem for finding the best position is defined in section 3.3 and a model for dealing with an uncertain threat distribution is described in section 3.4. Finally, the optimization methods used to calculate the results are described, first, in section 3.5, the dynamic programming recursion formula and in section 3.6 the Interior point optimization method.

3.1 Baseline

In order to get a sense of the problem dynamics and to establish a baseline from which to optimize from, the first positioning that is analyzed is that of placing the ships as close to the escorted vessel as possible. In a real world application this positioning would depend on limitations like safety, not wanting to position the ship so that there is a risk of collision, or radar blockage, assuming that the escorted vessel is a rather large commercial ship which could block the escorting ships radar. However, to simplify the optimization algorithm these limitations have been disregarded. The purpose of the baseline scenario is to create a relevant comparison for the optimized solutions. Therefore it is assumed that the defending ships can be placed right on top of the vessel.

3.2 Minimization of expected number of fired SAMs

Although the aim of this thesis is to study the effect of ship positioning, a method for calculating missiles allocation has to be included in order to evaluate the efficiency of different positions. The object of the optimization is to find the position that minimizes the expected number of SAMs spent per shot down ASM while still maintaining a low enough ASM penetration probability. Looking at the baseline position, where $p_{kill}(\theta)$ attains it's highest value $p_{kill,b}$ and is constant regardless of attack direction, assume that the $(1 - p_{kill,b})^N$ is considered a sufficiently low probability that an ASM hits the escorted vessel for some N number of SAMs. The optimal policy to minimize the expected value of fired SAMs would be to fire the SAMs one at a time and evaluate if the ASM is destroyed before firing anew. This doctrine is known as the shoot-look-shoot doctrine (SLS) which means that a shoot-down attempt has to be concluded before another SAM is fired. However, in the context of this thesis the term takes on a slightly different meaning since the ships are able to fire a salvo of SAMs, look to see if they are successful, then fire a new salvo. In order to strictly follow the SLS doctrine for the maximum number N of allotted SAMs, there has to be enough time to be able to both fire up to N SAMs as well as evaluate every single attempt, which is often not the case. In the case where the number of possible shoot-down attempts according to the SLS doctrine, which will be denoted by T, is less than the number of allotted SAMs, N, the question becomes how to portion out the SAMs optimally within the T slots. This is equivalent to the following optimization problem:

$$
\begin{aligned}
\min \quad & \mathbb{E}\{\sum_{t=0}^{T-1} u_t\} \\
\text{s.t} \quad & x_{t+1} = x_t + u_t \\
& y_{t+1} = \begin{cases} 1 \text{ with probability } 1 - (1-p_t)^{u_t} \text{ if } y_t = 0 \\ 0 \text{ with probability } (1-p_t)^{u_t} \text{ if } y_t = 0 \\ 1 \text{ with probability } 1 \text{ if } y_t = 1 \end{cases} \\
& u_{T-1} = (N - x_{T-1})(1 - y_{T-1}) \\
& x_0 = 0 \\
& u_t \in \mathbb{Z}^+
\end{aligned} \tag{9}
$$

where the variables x_t describes the total number of SAMs spent at time t, or simply the sum of the decision variables u_t up to that point. As u_0 describes the allocated SAMs in the first time slot, u_{T-1} describes the allocation in the last step. The constraint on this last decision variable ensures that if the ASM has survived to this time slot all remaining SAMs will be fired at it. The y_t variables describe if the ASM has been shot down or not in which case it attains the value 1, otherwise it is 0. The probability of shooting down the ASM with one SAM is described by the parameter p_t. The last constraint have to be included since the u_t variables can only take on integer values.

For the defined optimization problem N is a parameter value that has to be set before solving the problem. However, when the ships move away from the baseline position there is a possibility that their p_{kill}, or p_t, values will differ between engagements as the angle θ is no longer constant but can take on a variety of values $\hat{\theta}_t$. This will not only affect the expected value of spent SAMs but also the ASM penetration probability which for certain values N will no longer be sufficiently low. This is solved by simply trying different N until a sufficiently low penetration probability is attained. As this formulation only deal tangentially with positioning, in that it uses SLS opportunities and p_{kill} values as parameters, it is not sufficient for investigating this core issue. Therefore a larger scenario has to be modeled in which the above optimization problem is used as a tool to solve the optimal positioning problem.

3.3 Sector defense scenario

In evaluating the optimal position of the ships the center line of the threat is assumed to be in the positive y direction, which means that the ships will only be moved along this axis. Their position can therefore always be described by the coordinates $(0, y)$ for some value y.

As stated in the problem description when a ship moves away from the baseline position it will divide the span of possible threat directions into two sectors. One in which the chances for survival are improved and one in which they have become worse. This is because, should the ASM arrive in the improved sector, the ship has created more time for itself to act on the ASM, while the opposite is true for the deteriorated sector.

The sector defense scenario model assumes that for whatever improvement one would like to achieve it is always desirable to do so for as broad a span of threat directions as possible. This can be translated into an optimization problem where the constraints are the improvements one would like to achieve and the cost is the threat sector within which these improvements are possible. In other words this model attempts to find the position that maximizes the size of the improved sector while disregarding the deterioration outside it.

Given the influence of SLS opportunities on the missile allocation problem, the boundary between these two sectors should reasonably be placed along the threat direction where there is cutoff between the number of SLS opportunities. This is why the constraints are constructed as inequality constraints based on the desired number of SLS opportunities.

As increases in SLS opportunities will always come at a cost of decreasing the sector within which they can be achieved the optimization will focus on maximizing the sector angle of the smallest improvement possible, namely gaining at least one more SLS opportunity than the number possible in the baseline positioning. As shown by [16] this approach is reasonable due to the diminishing margin of return of an increased number of SLS slots. The optimization problem can therefore be states as:

$$\begin{aligned} \max \quad & v \\ \text{s.t} \quad & \text{There is at least one more SLS} \\ & \text{opportunity within the whole sector} \\ & \text{than in the baseline positioning.} \end{aligned} \tag{10}$$

The general model incorporates one attacking ASM and one defending ship. The variables being optimized is the angle v of the improved sector, and the position of the ship along the y axis, as illustrated in Figure 4.

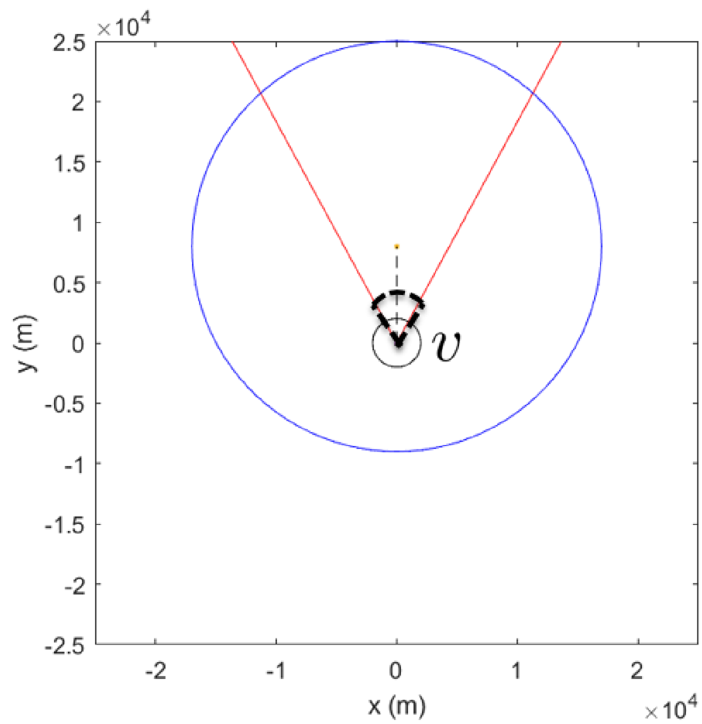


Figure 4: Sector defense scenario. Optimization model aimed at finding the y coordinate that maximizes the angle v within which the chances for survival have improved compared to the baseline scenario.

The constraints are constructed using the series of equations leading up to equation 7 where n is the minimum number of SLS opportunities within the sector and r_0 is the radial distance from the origin at which the ASM is detected.

$$
\begin{aligned}
\max \quad & v \\
\text{s.t} \quad & r_0 = y\sin(v) + \sqrt{y^2(\sin^2(v) - 1) + R^2} \\
& r_1 = r_0 - u_{ASM}t_r - s(r_0, y, v) \\
& \ldots \\
& r_n = r_{n-1} - u_{ASM}t_r - s(r_{n-1}, y, v) \\
& r_n \geq r_{min}.
\end{aligned}
\tag{11}
$$

However, it is not necessarily true the SLS boundary sought in the optimization actually is the boundary of the improved sector.

Consider the scenario in which a single ship placed in the baseline position has 2 SLS opportunities in every direction. If this ship was moved a certain distance in the positive y direction it could encounter the situation described in Figure 5, which is that it has created a sector with 3 SLS opportunities centered on the y axis while still maintaining 2 SLS opportunities in a limited area outside it, although with poorer p_{kill} values and also a sector with 1 and 0 SLS opportunities in the opposite direction. Since 0 and 1 SLS opportunities is worse than 2 and since the ship is only able to attain lower p_{kill} values than the maximum, 0.8, within the 2 SLS sector, it has made conditions worse for all directions outside the 3 SLS sector but because of the deterioration of p_{kill} values away from the centerline it might have made conditions inside the 3 SLS sector worse as well.

Therefore in order to validate that the boundary sought is actually the boundary of the improved sector, the missile allocation formulation with the integer constraint is used to calculate the expected value of fired SAMs in the baseline position and recalculate it in the new position. This is done by simply trying different N values until the penetration probability constraint is satisfied.
Also, in order to incorporate more ships into the defense both a simple strategy of adding sectors together is used as well as a modified formulation of the general one ship problem.

This modification entails placing another ship at the baseline position with which the ship being moved along the y axis can cooperate, thereby creating a two ship defense as seen in Figure 6. This further entails replacing the y variable by 0 in the r_i equations where the ship placed at baseline is shooting.

3.4 Threat distribution uncertainty

The threat distribution uncertainty is introduced to complement the sector defense scenario by using a probability distribution to weigh the improvements

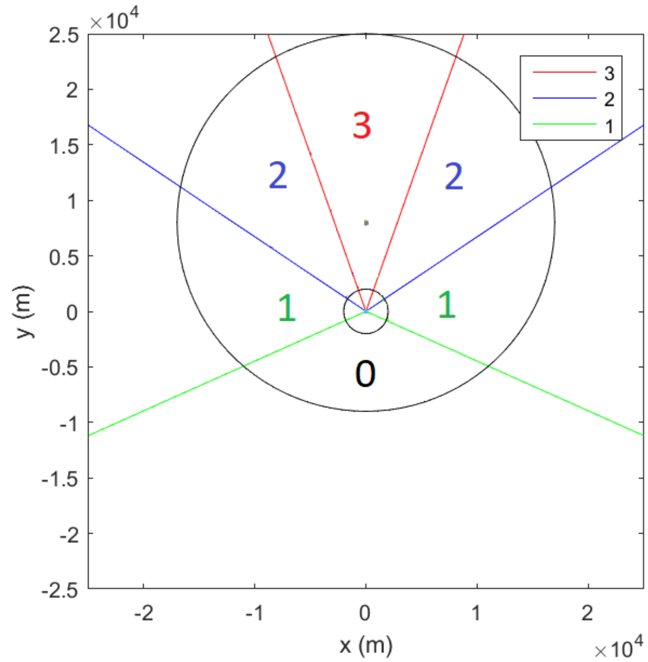


Figure 5: Illustration of the possibility that a ship may encounter different SLS sectors as it moves in the positive y direction. The ship is placed at $y = 8$ km

within the improved sector against the losses outside it.

The two sectors of improvements and deterioration might themselves be divided into smaller sectors of differing SLS opportunities, as seen in the previous section, this means that it is no longer possible to just look at one boundary direction as in the original sector defense scenario model. Instead, all the gains and losses inside the different SLS sectors have to weighed against each other in conjunction with the given threat distribution.

This creates a discontinuous problem that is difficult to implement in standard optimization solvers which is why a search method was implemented instead.

In setting up the search, using the formulation in (9) and trying different values of N was considered too time consuming when there is a desire to evaluate the expected value for many different attack directions. Therefore the optimization problem was modified. By removing the integer constraint on the allocated missiles and using a deterministic formula for calculating the expected value, it was possible to create a continuous optimization problem where the penetration probability limit could be introduced as a constraint.

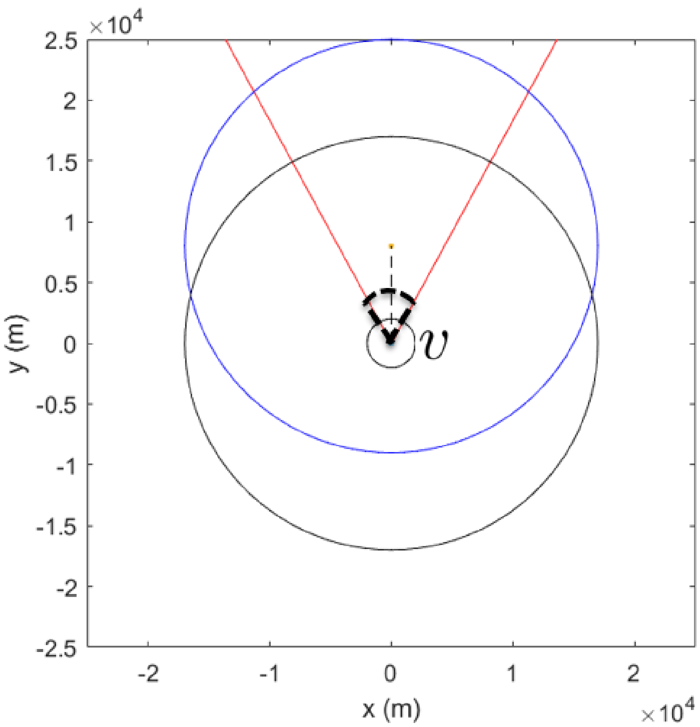


Figure 6: Two ship sector defense scenario with one of the ship's placed at the origin. Optimization model aimed at finding the y coordinate that maximizes the angle v within which the chances for survival have improved compared to the baseline scenario.

$$\begin{aligned} \min_{u} \quad & u_0 + \sum_{t=1}^{T-1} u_t \prod_{i=0}^{t-1} (1-p_i)^{u_i} \\ \text{s.t} \quad & \prod_{t=0}^{T-1} (1-p_t)^{u_t} \leq P_{\text{sat, limit}}. \end{aligned} \tag{12}$$

In (12) the objective function is the deterministic formula for calculating the expected value and $P_{\text{sat, limit}}$ is the limit for the lowest tolerated penetration probability. As in the original problem the u_t variables are the allocated SAMs in time slot t. Although this is quite a severe relaxation of the original problem it is considered a good enough approximation in order to allow the use of standard solvers for optimization problems.

As both the number of SLS opportunities T and the p_{kill} values $\{p_0, p_1, ...p_{T-1}\}$ are functions of the ASM attack direction v and the ship position y, the solution to (12) could be described as a function $f(v, y)$ such that:

$$\begin{aligned} f(v,y) = \min_{u} \quad & u_0 + \sum_{t=1}^{T(v,y)-1} u_t \prod_{i=0}^{t-1} (1 - p_i(v,y))^{u_i} \\ \text{s.t} \quad & \prod_{t=0}^{T(v,y)-1} (1 - p_t(v,y))^{u_t} \leq P_{\text{sat, limit}}. \end{aligned} \tag{13}$$

Thus, given a certain discrete probability distribution $\phi(v)$ the search method could be formulated as the following optimization problem:

$$\min \quad \sum_{v=0}^{2\pi-h} \phi(v) f(v,y), \tag{14}$$

where h is the discretization step of a certain set of discrete angles between 0 and $2\pi - h$. The procedure for calculating T and $\{p_0, p_1, ...p_{T-1}\}$ was presented in section 2.2 and 2.3. As $f(v, y)$ provides the expected value of the individual discrete attack directions, the solution to (14) provides the expected value of fired SAMs given the probability distribution.

The search method was conducting by moving the ship forward along the y axis using 100 meter increments . As the ship was allowed to move forward to the point where it would have 0 SLS opportunities for certain directions centered around the negative y direction, this was handled by setting the expected values of spent SAMs in this case to 100. The distributions that were used to describe the threat were uniform distributions within a certain sector and normal distributions.

In both cases three arbitrary distributions were used to illustrate how the expected value of spent SAMs changed as the ship moved away from the baseline. Then, expecting that, for distributions that were too spread out, the optimal positioning would invariably be the baseline position this cutoff point or "widest" possible distribution for which the optimal position was not the baseline was also investigated.

3.5 Dynamic programming

The optimization problem in (9) was solved using dynamic programming. Denoting the value function as $V(x, y, l)$ where l is the time step, the boundary condition of having to fire all remaining SAMs at time T can be translated into $V(x, y, T) = (N - x)(1 - y)$. As for the cost-to-go function that is to be minimized, this can be written as:

$$u + (1 - p_l)^u V(x + u, y = 0, l) + (1 - (1 - p_l)^u) V(x + u, y = 1, l).$$

Since if the ASM is shot down the optimal allocation is to not shoot any more SAMs regardless of time step l and allocated missiles x, $V(x, y = 1, l) = 0 \ \forall x, l$. This means that the recursion can be formulated as follows:

$$\begin{cases} V(x,T) = N - x \\ V(x,l-1) = \min\{u + (1-p_l)^u V(x+u,l)\}. \end{cases}$$

.

$$\left(1 - \prod_{t=0}^{T-1} (1-p_t)^{u_t} \geq P_{limit}\right) \tag{15}$$

3.6 Interior point optimization

For solving the relaxed problem in (12), used in the search method described in the previous section, as well as the sector defense scenario optimization problem in (11) fmincon was used. This built-in Matlab function is a nonlinear programming solver that can find the minimum of a problem specified by:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t} \quad & c(x) \leq 0 \end{aligned} \tag{16}$$

where in the case of solving the relaxed formulation (12):

$$f(x) = u_0 + \sum_{t=1}^{T-1} u_t \prod_{i=0}^{t-1} (1-p_i)^{u_i}, \quad c(x) = \prod_{t=0}^{T-1} (1-p_t)^{u_t} - P_{\text{sat, limit}} \tag{17}$$

and in the case of the sector defense scenario optimization problem in (11):

$$f(x) = -v, \quad c(x) = r_{min} - r_n \tag{18}$$

where r_n is the result of the recursive calculation in the constraints in (11).

The fmincon function is a gradient-based method designed to work on problems where the objective and constraint functions are both continuous and have continuous first derivatives. In the sector defense scenario model the angle of the improved sector is put into f(x), while the SLS improvements are contained in c(x). The default algorithm, which is the one that is used in this paper, is the Interior-Point Optimization algorithm. The basic framework of this algorithm is the following [17]:

Consider an inequality constrained problem on the form:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t} \quad & g(x) \leq 0 \end{aligned} \tag{19}$$

Using the strategy of interior point methods this problem is reformulated into a barrier problem

$$\begin{aligned} \min_x \quad & f(x) - \mu \sum_{1=1}^{m} lns^{(i)} \\ \text{s.t} \quad & g(x) + s = 0 \end{aligned} \tag{20}$$

The first order optimality condition of this problem is formulated from the Lagrangian:

$$L(x, z, \lambda) = f(x) - \mu \sum_{1=1}^{m} lns^{(i)} + \lambda^T (g(x) + s) \tag{21}$$

Where an optimal solution (x, s) has to fulfill:

$$\Delta_x L(x, s, \lambda) = \Delta f(x) + A(x)\lambda = 0 \tag{22}$$

$$\Delta_s L(x, s, \lambda) = -\mu S^{-1} e + \lambda = 0, \tag{23}$$

where $A(x) = (\Delta g^{(1)}(x), ..., \Delta g^{(m)}(x))$ is the matrix of constraint gradients and where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} s^{(1)} & & \\ & \ddots & \\ & & s^{(m)} \end{pmatrix}. \tag{24}$$

Using the fmincon function the solution that fulfills these optimality conditions can be found.

4 Results

In the following chapter Section 4.1 describes the results from the sector defense model and Section 4.2 describes the solutions for the uncertain threat distributions. Section 4.3 describes the results of the expended missile allocation model. The parameter values used in the calculations is listed in Table 1.

R	17 km
u_{ASM}	300 m/s
u_{SAM}	400 m/s
t_f	10 s
r_{min}	2 km

Table 1: Parameter values

4.1 Sector defense scenario

In order to validate that an increase in SLS opportunities actually resulted in a lower expected value of spent SAMs per ASM, the baseline positioning was investigated first. The maximum number of allocated missiles N was set to five which, because $p_{kill} = 0.8$ for all directions, meant that the probability of a single ASM reaching the escorted vessel was $(1 - 0.8)^5 = 3.2 \times 10^{-4}$. With the given parameter values the SLS opportunities in this position was 2 in every direction, as seen in Figure 7. For $T = 2$ the dynamic programming recursion resulted in:

$$V(x, 2) = N - x$$

$$V(x_0, 1) = \min\{u + (1 - p)^u (N - x_0 - u)\}.$$

Since $x_0 = 0$

$$\implies V(0, 1) = \min\{u + (1 - p)^u (N - u)\}.$$

Solving this for $N = 5$, the resulting allocation was given by $\bar{u} = \{1, 4\}$ which resulted in an expected value of 1.8 SAMs per ASM. Given that the number of SLS opportunities in the baseline positioning was two the constraint for the optimization problem of finding an improved position was that there would be at least three SLS-opportunities. The optimization problem was set

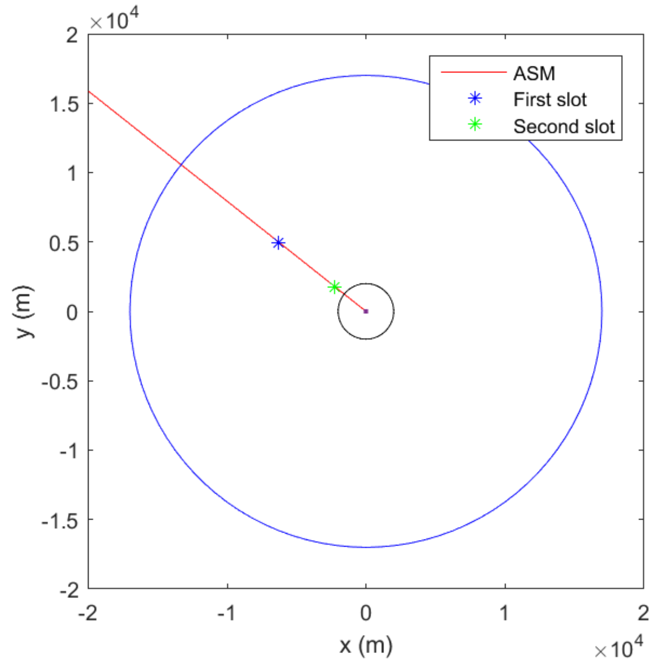


Figure 7: Illustration of the baseline scenario given the parameter values in Table 1 and an arbitrary attack direction.

up accordingly, based on equation 11:

$$\begin{aligned} \max \quad & v \\ \text{s.t} \quad & r_0 = y\sin(v) + \sqrt{y^2(\sin^2(v)-1)+R^2} \\ & r_1 = r_0 - u_{ASM}t_r - s(r_0, y, v) \\ & r_2 = r_1 - u_{ASM}t_r - s(r_1, y, v) \\ & r_3 = r_2 - u_{ASM}t_r - s(r_2, y, v) \\ & r_3 \geq r_{min}. \end{aligned} \tag{25}$$

The dynamic programming recursion for $T = 3$ resulted in:

$V(x, 3) = N - x$

$V(x, 2) = \min\{u + (1-p)^u(N - x - u)\}$

$V(x_0, 1) = \min\{u + (1-p)^u V(x_0 + u, 2)\}$

Since $x_0 = 0$

$$\implies V(0,1) = \min\{u + (1-p)^u V(u,2)\}$$

4.1.1 One ship model

In the model where only one ship is included the resulting solution, visualized in Figure 8, was $y = 7.980$ km and $v = 0.6948$ rad.

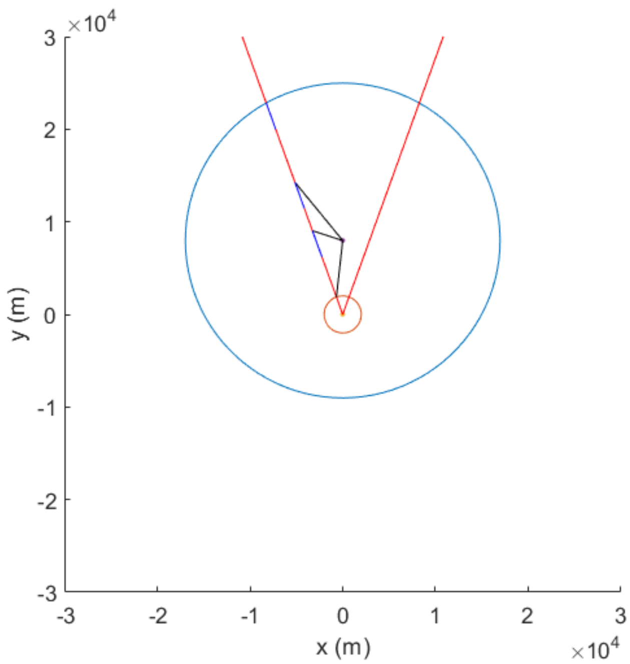


Figure 8: Illustration of 3 SLS improvement sector for one ship defense.

Recalculating the expected value at the boundary it was found that, in order for the probability of an ASM striking the target to be equal to or lower than the value of the baseline positioning, the number of allotted missiles had to be increased to $N = 11$. Given the p_{kill} values, $\{0.78, 0.69, 0.43\}$ of the three different attempts this resulted in an $\bar{u} = \{1, 2, 8\}$ allocation, a penetration probability of 2.4×10^{-4} and an expected value of 1.6 SAMs per ASM. This further meant that the expected value of spent SAMs was guaranteed to be lower in the whole sector for this positioning than it was when the ship was placed at the origin.

The evolution of the size of the improved sector from the baseline position to the optimal solution of $y = 7.89$km can be seen in Figure 9. Extrapolating a multi-ship sector defense from these results it can be calculated that with two ships the maximum sector angle is $2v = 1.3896$ rad and with three $3v = 2.0844$ rad, visualized in Figure 10.

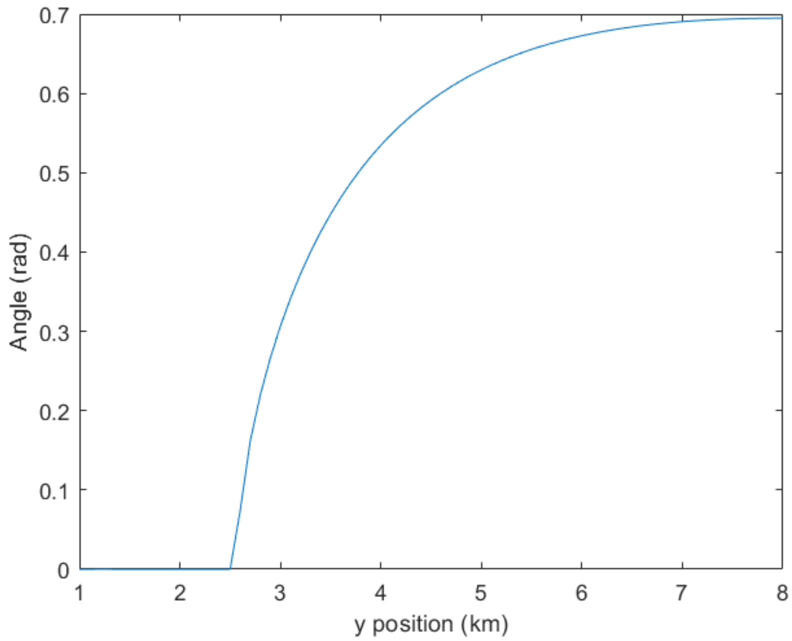
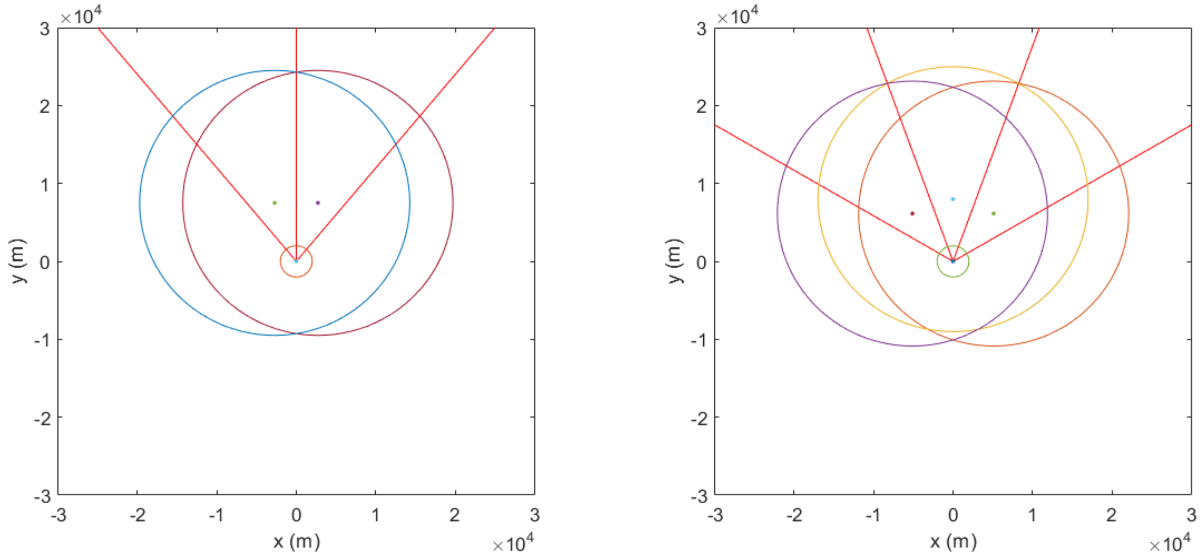


Figure 9: The angle of the improved sector for all the positions between the baseline position and the optimal solution. The angle reaches a maximum as the y axis approaches $y = 7.980$ km.

4.1.2 Two ship model - case 1

In the two-ship cooperative model where there were two ships that could alternate missile launches and where one of the ships remained in the baseline positioning the optimization problem was formulated in three different cases. Firstly, in case 1, with $y = 0$ in the r_3 equation which meant that the forward ship would take the first two SLS opportunities, while the ship placed at baseline would take the third. This resulted in the solution $y = 8.487$ km and $v = 1.3312$ rad, visualized in Figure 11a.

When recalculating the expected value at the boundary of this sector, given the p_{kill} values $\{0.74, 0.52, 0.8\}$ this resulted in the number of allotted missiles having to be increased to $N = 7$, which resulted in an $\bar{u} = \{1, 2, 4\}$ allocation, an ASM penetration probability of 9.5×10^{-5} and an expected value of 1.76 SAMs per ASM at the boundary.



(a) Two ship formation

(b) Three ship formation

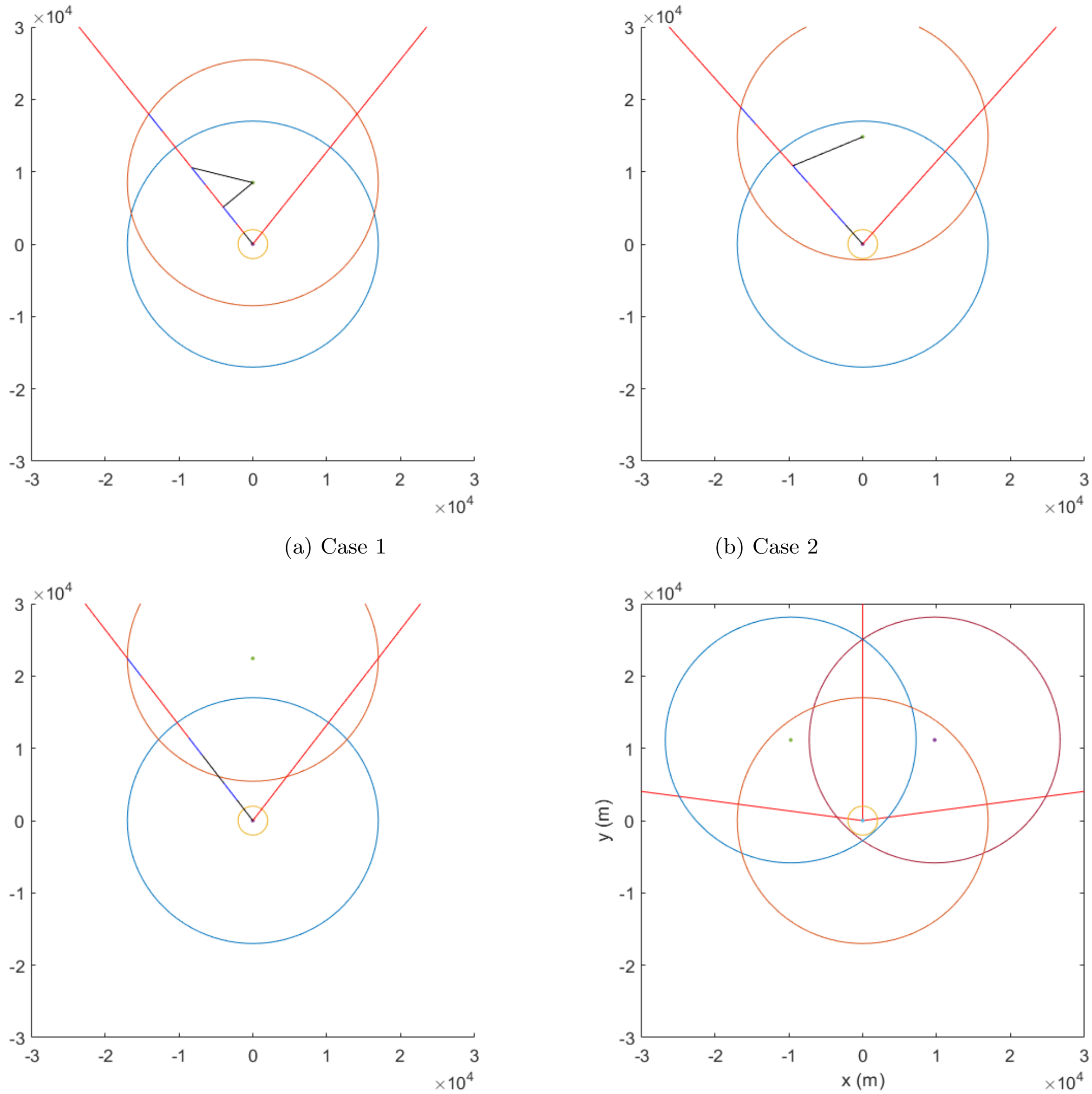
Figure 10: One ship defense extrapolated

4.1.3 Two ship model - case 2

In case 2, where the ship at the baseline position would take both the second and third slots, $y = 0$ in both the r_2 and r_3 equations, which resulted in the solution $y = 14.817$km and $v = 1.4370$ rad, visualized in Figure 11b. Given the p_{kill} values $\{0.61, 0.8, 0.8\}$, this resulted in the number of allotted missiles having to be increased to $N = 6$, which resulted in an $\bar{u} = \{1, 1, 4\}$ allocation, an ASM penetration probability of 1.25×10^{-4} and an expected value of 1.72 SAMs per ASM at the boundary.

4.1.4 Two ship model - case 3

In case 3, the forward ship only acted as a radar detector for the ship at the baseline position, which meant that $y = 0$ in all the r_i equation except the detection radius r_0. This resulted in the solution $y = 22.445$ km and $v = 1.2964$ rad, visualized in Figure 11c. For this example the alloted missiles can still be $N = 5$ and give the same ASM penetration probability as in the zero point positioning, resulting in an expected value of spent SAMs of 1.32 in the entire sector. Extrapolating this last example into a three ship defense it was found that they could increase their SLS opportunites to 3 or more for a sector of 1.2964 rad, as seen in figure 11d, while still maintaining 2 SLS opportunites for all directions outside the sector. All the results from the sector defense scenario model is summarized in Table 2.



(a) Case 1

(b) Case 2

(c) Case 3

(d) Case 3 extrapolation

Figure 11: Optimal positioning of two ships given different collaboration schemes. One ship placed at the origin at the center of the blue radar circle and one ship placed in a forward position at the center of the red circle. The black lines are shoot-down attempts.

	y (km)	v (rad)	$\mathbb{E}\{\sum u\}$	P_{sat}	N
Baseline	0	-	1.8	3.2×10^{-4}	5
One ship	7.98	0.6848	1.61	2.4×10^{-4}	11
- Two ship extrapolation	-	1.3896	-	-	-
- Three ship extrapolation	-	2.0744	-	-	-
Two ship Case 1	8.49	1.3312	1.71	9.6×10^{-5}	7
Two ship Case 2	14.817	1.4370	1.70	1.25×10^{-4}	6
Two ship Case 3	22.445	1.2964	1.32	3.2×10^{-4}	5
- Three ship extrapolation	-	2.5928	-	-	-

Table 2: Results from the sector defense scenario model

	y (km)	v (rad)	$\mathbb{E}\{\sum u\}$	P_{sat}	N
Baseline	0	-	1.8	3.2×10^{-4}	5
Two ship Case 1	8.49	1.3312	1.71	9.6×10^{-5}	7
Two ship Case 2	14.817	1.4370	1.70	1.25×10^{-4}	6
Two ship Case 3	22.445	1.2964	1.32	3.2×10^{-4}	5

Table 3: Results from the sector defense scenario model

	y (km)	v (rad)	$\mathbb{E}\{\sum u\}$	P_{sat}	N
Baseline	0	-	1.8	3.2×10^{-4}	5
One ship	7.98	0.6848	1.61	2.4×10^{-4}	11

Table 4: Results from the sector defense scenario model

4.2 Threat uncertainty distribution

4.2.1 Uniform distribution

In other words, in these two scenarios, the estimation of the threat is narrow enough to warrant a more aggressive positioning. The three sector angles for which a uniform distribution was examined, illustrated in Figure 12 were $\pi, \pi/2$ and $\pi/4$. The discretization step used was $\pi/80$. As seen in Figure 13, when the sector angle is π the expected value grows monotonously as the ship is moved forward along the y axis.

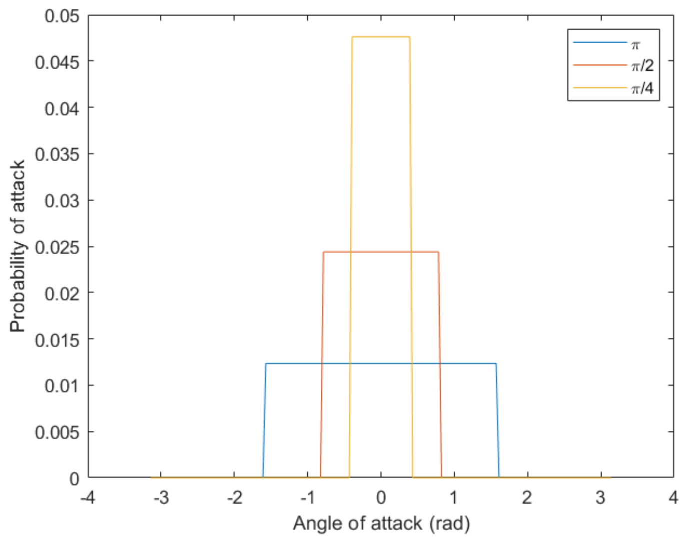


Figure 12: Threat direction probability uniform distributions in sectors $\pi, \pi/2, \pi/4$. The origin is centered on the y axis.

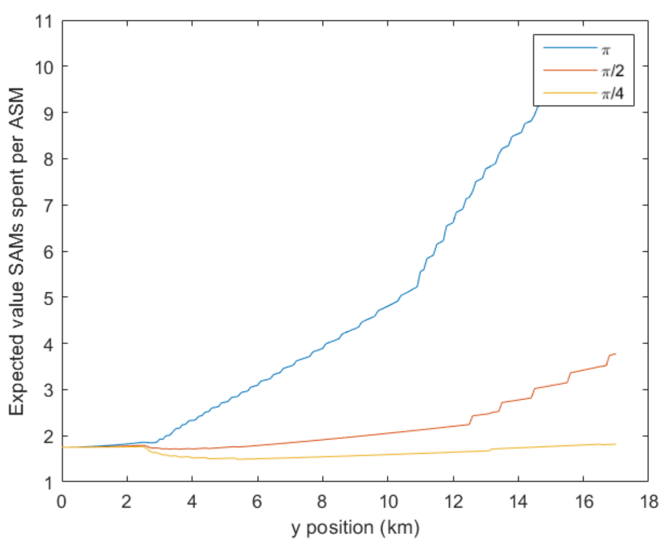


Figure 13: Expected value of fired SAMs given search along the positive y axis and given sector angle (rad). For uniform threat sectors with span $\pi, \pi/2, \pi/4$.

In other words, in these two scenarios, the estimation of the threat is narrow enough to warrant a more aggressive positioning. The results for the three different sector angles were $y = 0$ km, 3.6 km and 5.3 km.

As for the cutoff when the sector angle was small enough to warrant moving away from the baseline position this was found to be 1.72 rad, as illustrated in Figure 14.

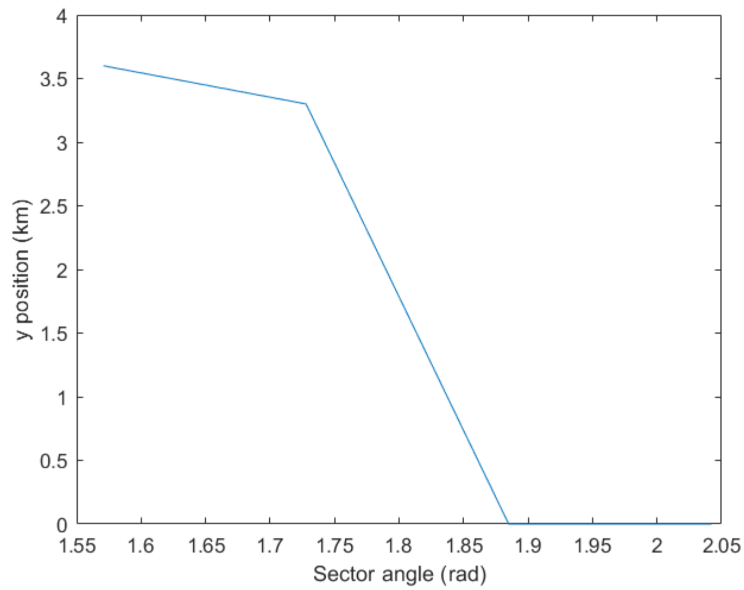


Figure 14: Optimal y position given four different sector angles $\{1.57, 1.72, 1.87, 2.02\}$ for a uniform distribution. For sector angles larger than 1.87 rad the baseline position is optimal, while the cutoff point when a more aggressive positioning better occurs somewhere between 1.72 and 1.87 rad.

4.2.2 Normal distribution

The three different standard deviations that were examined when using a normal distribution were $\sigma = 1, 0.5, 0.2$. As seen in Figure 16 given the distributions in Figure 15, when the normal distributions were not centered enough the expected value of spent SAMs would only increase as the ship moved away from the baseline, but for more centered distributions the expected number of SAMs would decrease. The optimal y position for the three different standard deviations were $y =$0 km, 3.3 km and 4.5 km.

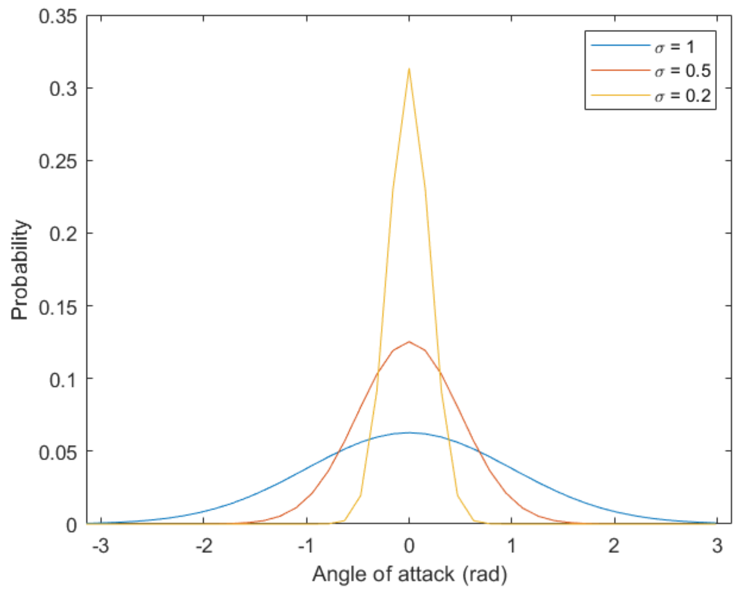


Figure 15: Threat direction probability normal distributions for $\sigma = 1, 0.5, 0.2$. Discretization step $\pi/20$. The origin is centered on the y axis

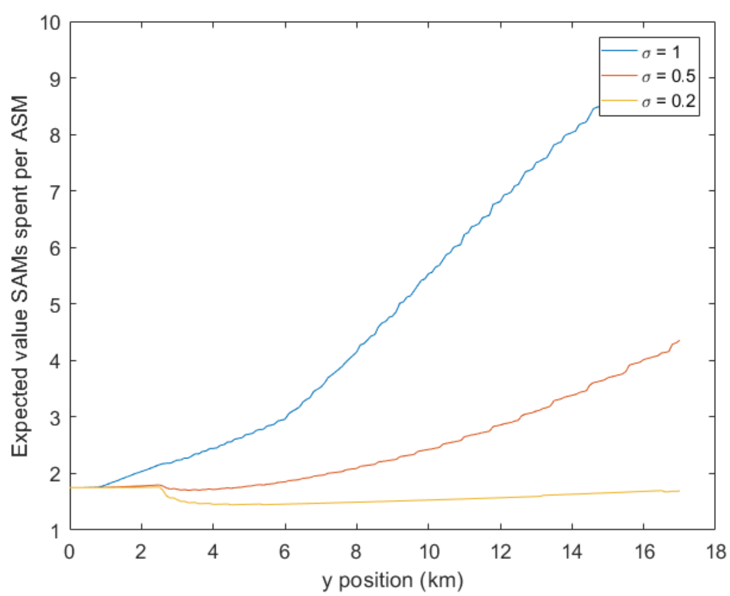


Figure 16: The expected value of spent SAMs given search along the positive y axis for $\sigma = 1, 0.5, 0.2$.

As seen in the figure a cutoff occurs around $y = 2.5$ km and that this is where the 3 SLS opportunity sector is first opened up.

As for the cutoff point when the distribution is centered enough to motivate moving away from the baseline position it can be seen in Figure 17, in which a search has been conducted for finding the optimal position for different values of σ, that the optimal position is the baseline position until sigma decreases to 0.6 at which point it jumps to $y = 3.3$ km.

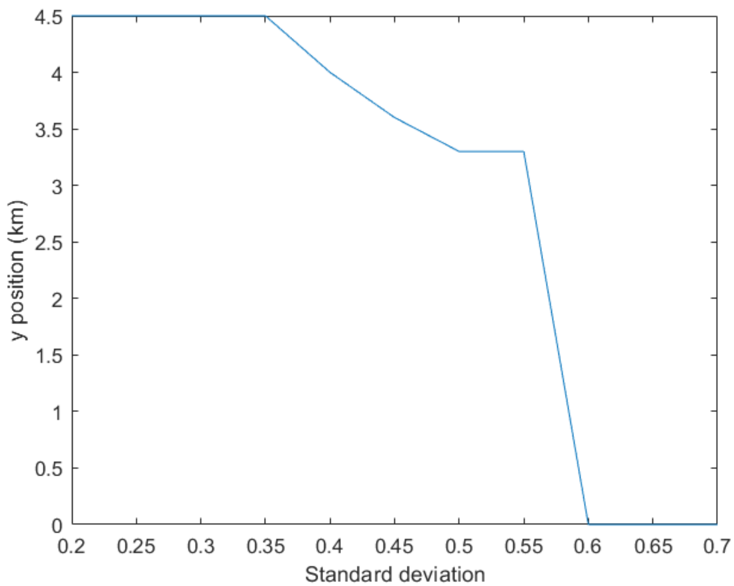


Figure 17: Optimal y position given different σ values for the normal distribution. For values larger than 0.6 the optimal position is always to remain in the baseline positioning. Only when the σ value is smaller than 0.55 can a more aggressive positioning be motivated.

4.3 Optimal shooting doctrine using dynamic programming

When calculating the optimal allocation for $T = 3$ slots, although dynamic programming is an effective way of solving the problem, it could be solved just as quickly by simply trying all the different solutions. In other words, for this limited number of slots, it is not really a necessity to limit the complexity. However, in the case of more time slots the complexity grows and dynamic programming becomes a necessary tool for solving the problem in a reasonable amount of time. In order to illustrate this and to analyze the missile allocation problem further, dynamic programming is used to solve (9) for $T = 10$, $N = 20$

and three different p_{kill} values, $\{0.9, 0.5, 0.1\}$ that remain constant for all the attempts.

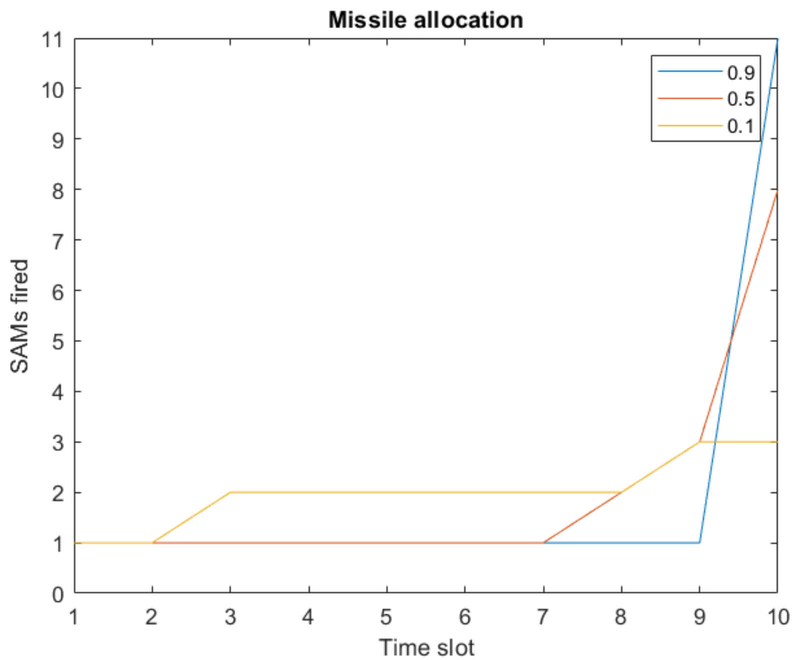


Figure 18: Optimal allocation for $T = 10$, $N = 20$ and $p_{kill} = 0.9, 0.5, 0.1$

As seen in Figure 18, with higher p_{kill} it is best to shoot the SAMs one at a time until the last slot, while for lower p_{kill} the preferred doctrine is to spend more SAMs in earlier attempts.

5 Discussion

5.1 Comparison between baseline and the optimal solutions

The baseline scenario was introduced as a comparison to the optimal solutions. At first sight there are several reasons why the baseline solution could be considered the best solution. Firstly, a ship in this position will detect incoming ASMs at the same distance from the escorted vessel in every direction. Since the SAMs are assumed to be launched vertically the probability of intercepting the ASMs are the same regardless of the direction of the threat. This is adventageous if the threat direction is unknown. Secondly, since ships in this positioning will always fire their SAMs straight at the incoming ASMs they will have the highest possible p_{kill} value in every engagement. Lastly, since if more than one ship is placed in this position all ships operate under the same conditions, it makes for easy cooperation as the ships can share the load without affecting the survival probability negatively.

For these reasons a comparison to the baseline solution is an integral part in motivating what is and what is not an optimal solution. A position is considered strictly better than the baseline if it results in both higher or equal probability of survival and a lower expected value of used SAMs.

As seen in Table 2, it is possible to find strictly better solutions. It is however, a tradeoff between improving the chances of survival in one direction while decreasing it in others. In some cases the improvement seems to be fairly limited while in other cases, especially considering multiple ships, the improvement can be substantial. In the end the challenge is to determine whether or not the improvement in one direction warrants the risk of decreasing the protection in other directions, which is not further investigated in this work.
Given a certain number of ships the sector defense model is able to provide which tactic is the best and what type of defense it is possible to achieve. It allows the tactician to explore different options and how those options are affected by what type of ASMs the enemy is using. The core question being, is it possible to motivate a positioning other than the baseline?

In terms of options, the results from the one ship model is limited. This means that if only one ship is available the baseline position is most likely the best. With two ship's available one of the collaborative models would be the preferred choice, as they guarantee that there is no SLS loss in the opposite direction. Also for the case 2 model the improved sector is the largest. As for three ships, the extrapolations from the one ship and two ship models show that with collaboration the sector can be increases by about 0.5 rad.

The results from the two ship model also show that with five ships it would almost be possible to the lower the expected number of fired SAMs for the whole span. Another to put would that if the object was to increase the survival probability for the whole sector than the optimal division of fire would be the case 2, two ship model and the optimal number of ships five. For this same object to be achieved with the one ship model defense strategy it would take nine ships.

If the roles were reversed and the object was to find weaknesses in the opposition's defense, the preferred tactic would be to avoid the most deadly sector, the boundary of which could be found using the sector defense model. However, another tactic could also be, if the enemy would use a two ship model with a forward ship and not just a radar, to fire the ASMs close to the boundary of the sector in order to get the forward ship, with worse p_{kill} than the baseline ship, to still engage and fire at the ASMs, but with the least likely probability of hitting it. This way the probability of both enemy ships having to fire SAMs on the same ASM would be maximized, which in turn would also increase the chances of them running out of ammunition all together as well.

5.2 Threat direction uncertainty

In order to create more SLS opportunities than what is possible in the baseline positioning, one or more ships have to be moved in the direction from which the ASMs are attacking from. However, with a small number of ships it is impossible to increase the number of SLS opportunities in every direction. This makes it crucial to be able to estimate the direction of the incoming threats. Accurately estimating the threat distribution might however be hard to do. Thus an increased understanding of the effects of an uncertain threat distribution is important. Any move away from the baseline positioning has to be motivated or limited by what is known about the threat direction, either in the form of a fixed sector within which the defending ships are certain that the ASMs will be arriving from, or in the form of a probability distribution as a function of the angle.

In section 4.1.1 the maximum angle of the improved sector was 0.68 rad which was achieved at $y = 7.98$km. The cost of placing the ship in this position is therefore that the ship's sensitivity to attacks in 5.58 rad of the span is increased. This constitutes 89 % of the possible attack directions. This increased sensitivity is illustrated in Figure 13 where it can be seen that if a uniform distribution over a 3.14 rad span centered on the y axis is applied to this same position the expected value of SAMs is four instead of at least 1.6. This means that in attempting to lower the expected value from 1.8 to somewhere between 1.32 and 1.6 the ship risks increasing it to 4 should the real span of the uniform distribution of the threat be 3.14 rad instead of the estimated 0.68 rad.

Another interesting aspect of the results in section 4.1.1 is that it can be seen in Figure 9 that the growth speed of the angle of the improved sector reduces as the position gets closer to the optimal. This means that if the estimation of the attack direction were more uncertain and some of the losses experienced outside the improved sector would have to be considered it is possible that the optimal position of the ship is closer to the origin than 7.89 km. This is due to the effect of an uncertain threat distribution.

When applying a uniform distribution of the threat within a $\pi/4$ rad sector the optimal position is 5.3 km. The size of this uniform sector is only about 0.1 rad smaller than the maximum achieved in the sector defense model which means that the ship is only forced to consider a small area of defense outside the improved sector. It can be seen in Figure 9 that even though there is a 2.7km difference in position the difference in improved sector angle is only about 0.05 rad. This means that by introducing a bit of uncertainty into the model the ship is able to maintain just about the same improved sector while considering a wider threat. Returning to the example of the real span of the uniform distribution of the threat being 3.14 rad the expected number of fired SAMs in the 5.3 km position is about 2.8 as compared to the 7.89 position in which it is about 4.

5.3 Future work

The gain of being able to lower the expected value of fired SAMs per shot down ASM is increased considerably if the attack consists of multiple ASMs arriving either in a stream or in a salvo or a combination of both. Although the optimization is founded in this idea that there will be multiple ASMs attacking this has not really been modeled. Which means that saturation effects are not considered. The introduction of multiple ASMs could motivate why, as opposed to in the current model, a ship that is moved in the direction that the ASMs are arriving from, will increase the survival probability of the escorted vessel even though it hasn't gained another SLS opportunity. This is because it might have still gained more time for itself to act on the arriving ASMs and thereby have increased the chances that it will not lose a SLS opportunity because of being overrun by a stream or salvo of ASMs.

In calculating the number of SLS opportunities several simplifications and approximations have been made, as listed in section 1.4. Easing of these constraints and a more realistic modeling of the approximations would increase the fidelity of the model. However, the dynamics of the model is considered to be realistic enough for the purpose of this thesis. The most severe limitations of the model is considered to be the approximation of the p_{kill} value. Including countermeasures, evasive maneuvers and a more exact description of the dependency of the intercept angle and distance would make the model more realistic. The current model is however, deemed sufficient to capture the dynamics of the studied problem. Furthermore, no matter how elaborate or exact one could make the underlying estimations there will always be an element of randomness and uncertainty in the estimations. Which is why there always has to be safety margin on the estimated number of SLS opportunities that decide the allocation.

Another expansion that could be made is to include multiple escorted ships. This would mean having to deal with questions like, how far apart should the escorted ships be placed from each other and how does it affect th optimization having to defends a larger area. Also, how does one deal with ASMs begin routed toward different targets and what is the division of work between the defensing ships. Are they assigned to different ships or do they collaborate?

One interesting aspect of the results in section 4.1.4 which could also be a candidate for future work is the idea of using one ship only has a radar sensor, as seen in Figure 11 c). This ship could then be equipped only for self defense and the bulk of its SAMs moved to the ship placed in the baseline position instead. Another interesting idea could be to replace this forward ship by an unmanned drone. Given that it is equipped with a similar radar capability as a ship, this could reduce costs quite severely and also reduce the risk of injury or loss of life while still achieving the same result.

6 Conclusion

The objective of this thesis was to optimize the positioning of surface combatant ships tasked with escorting a vessel and protecting it from ASM threats. Reasoning that it is essential to the mission's success to save as much ammunition as possible, the study was aimed at minimizing the number SAMs fired while still maintaining a high survival probability.

The results demonstrate that while the baseline positioning, of placing ships as close to the escorted vessel as possible, is often the best choice, depending on the information available about the threat, in certain scenarios more aggressive poisonings could theoretically be better. The sector defense scenario model revealed that it was possible to reduce the number of SAMs fired by positioning the ship(s) in non-trivial positions. It was also able to provide the position that maximized the sector within which these reductions were possible.

Building upon the reasoning behind the single ship sector defence scenario, the two-ship collaboration, with one ship always remaining in the origin, showed significant promise. Without risking any deterioration in the negative y direction this model was able find a position that lowered the expected value of fired SAMs for large sectors of attack directions. Also, by evaluating all the different collaboration schemes it could be seen which collaboration had the largest improvement sector and which had the lowest value of fired SAMs. Also, by using extrapolation, the gain of using collaboration against the not using it, could clearly be seen.

In scenarios with uncertain threat directions, the results showed that the positioning strategy depends heavily on the degree of uncertainty. As the threat distribution became more spread out, the baseline positioning became increasingly favorable.

In conclusion, the results provide valuable insight for military commanders and strategists seeking to optimize their naval formations and positions. Future work could focus on incorporating additional factors such as more complexity in the missile kinematics and engagements and streams or salvos of attacking ASMs.

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