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Analyzing Antagonism in Hypernetworks

Measuring degree of balance in oriented hypergraphs

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Abstract

Competition and cooperation in natural systems, such as biological and social networks, have traditionally been modeled using signed graphs. The effect of antagonistic interactions in otherwise cooperative social networks, modeled as signed graphs, is captured by the notion of structural balance, a measure of social tension. Likewise, measures of the degree of structural balance, i.e., how far a social system is from structural balance, have been introduced in the literature. However, signed graphs fail to adequately capture the complexity of group interactions. This motivates the need for a framework that captures antagonistic and cooperative group interactions, while also retaining the pairwise interactions modeled by signed graphs. Existing models of higher-order social interactions typically assume that group interactions are fully cooperative. Although structural balance has been extended beyond signed graphs, the degree of structural balance in models with both antagonism and group interactions remains underexplored.

This thesis reviews signed hypergraphs, which represent cooperative and antagonistic interactions at both the pairwise and higher-order levels. The work also proposes a novel measure of the degree of structural balance in signed hypergraphs. The validity of the measure is supported by both theoretical grounds and numerical simulations. Furthermore, the novel measure shows a strong correlation with an existing NP-complete measure of the degree of structural balance in signed hypergraphs.

Keywords

signed graphs, structural balance, signed hypergraphs, oriented hypergraphs

Sammanfattning

Konkurrens och kooperation i naturliga system, såsom biologiska och sociala nätverk, har traditionellt modellerats med hjälp av teckengrafer, grafer som kan ha både positiva och negativa bågvikter. Effekten av antagonistiska interaktioner i annars kooperativa sociala nätverk, återigen modellerade med teckengrafer, fångas effektivt av begreppet strukturell balans, ett mått på social spänning. Likaså har mått på graden av strukturell balans, dvs. hur långt ett socialt system är från strukturell balans, introducerats i litteraturen. Teckengrafer lyckas dock inte fånga komplexiteten i gruppinteraktioner. Detta motiverar behovet av ett ramverk som fångar antagonistiska och kooperativa gruppinteraktioner, samtidigt som de parvisa interaktioner som modelleras av teckengrafer bibehålls. Befintliga modeller av högre ordningens sociala interaktioner antar vanligtvis att gruppinteraktioner är helt kooperativa. Även om strukturell balans har utvidgats bortom teckengrafer, är graden av strukturell balans i modeller med både antagonism och gruppinteraktioner fortfarande underutforskad.

Denna avhandling granskar hyperteckengrafer, som representerar kooperativa och antagonistiska interaktioner på både parvisa och högre ordningens nivåer. Arbetet föreslår också ett nytt mått på graden av strukturell balans i hyperteckengrafer. Måttets giltighet stöds av både teoretiska grunder och numeriska simuleringar. Dessutom visar det nya måttet en stark korrelation med ett befintligt NP-komplett mått på graden av strukturell balans i hyperteckengrafer.

Nyckelord

teckengrafer, strukturell balans, hyperteckengrafer, orienterade hypergrafer

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Contents

1 Introduction **1**
1.1 Background . . . 2
1.2 Problem . . . 2
1.3 Purpose . . . 3
1.4 Goals . . . 3
1.5 Research Methodology . . . 3
1.6 Outline of the thesis . . . 3

2 Background **5**
2.1 Notations and Definitions . . . 5
2.2 Pairwise Graphs . . . 6
2.3 Signed Pairwise Graphs . . . 11

3 Review of Hypergraphs **19**
3.1 Hypergraphs . . . 19
3.2 Signed Hypergraphs . . . 23

4 Research Method, Definitions, and Implementations **29**
4.1 Research Process . . . 29
4.2 Proposed Measurements of Degree of Balance of Signed Hypergraphs . . . 30
4.3 Data Generation and Manipulation . . . 37
4.4 Experimental Design . . . 41

5 Results and Discussion **43**
5.1 Fundamental Results . . . 43
5.2 Data Analysis Results and Discussion . . . 43

6 Conclusions and Future Work **53**
6.1 Conclusions . . . 53

6.2 Ethical and Sustainability Aspects 53
6.3 Limitations 54
6.4 Future Work 54

References **55**

A Hardware and Software Specifications **59**

List of Figures

2.1 An acyclic graph with labeled vertices. 8
2.2 Two cyclic graphs, one periodic and one aperiodic. 10
2.3 On the left, a balanced signed graph. Red dashed lines indicate antagonistic relationships, while green solid lines indicate cooperative relationships. On the right, a bipartite graph. . . . 12
2.4 A signed graph with a negative cycle. Dotted lines are negatively signed edges, and solid lines are positive. 14
2.5 A structurally unbalanced signed graph. The green solid lines are cooperative; the red dotted ones are antagonistic. 16

3.1 A 2-uniform hypergraph with three hyperedges, each containing three nodes. 20
3.2 2-order hypergraph with three hyperedges. 25
3.3 An unbalanced s-hypergraph. The smaller, black circles represent the hyperedges; the larger, white circles represent the nodes. Red dotted lines represent antagonistic incidences; green solid lines represent cooperative incidences. 25

4.1 Three orientations of the same hypergraph, the all positive orientation H^{+}, a balanced orientation H^{σ}, and an unbalanced orientation H^{τ}. 34

5.1 Measurements on synthetic pairwise graphs. Scatter plots of $\widetilde{AC}$ and NER; $\widetilde{FI}$ and NER. 44
5.2 Histograms of distribution of $\widetilde{AC}$ and $\widetilde{FI}$ on synthetic pairwise graphs. 46
5.3 Distribution of average hyperedge cardinality of the hypergraphs; Distribution of $|E|$. 47
5.4 Measurements on synthetic hypergraphs. (a) Scatter plots of $\widetilde{MB}$ and NIR, (b) $\widetilde{SP}$ and NIR. 49

5.5 Measurements on synthetic hypergraphs. (a) Scatter plots of $\widetilde{MB}$ and NIPR, (b) $\widetilde{SP}$ and NIPR. 50
5.6 Facet representation of a subgraph of the Senate Committees hypergraph. Red, dotted lines represent antagonistic incidences; green, solid lines represent cooperative ones. 52

List of Tables

5.1 Correlation table of pairwise measurements and properties. . . 45
5.2 Correlation table of hypergraph measurements and properties. 48
5.3 Properties of US data. $|R|$ is the number of Republicans, likewise $|D|$ for Democrats. AHC stands for Average Hyperedge Cardinality, i.e., the average number of elected officials per committee/bill. 52

A.1 Hardware specifications and software versions used for generation, measurements, and data analysis 59

Chapter 1

Introduction

The existing literature has not fully addressed the analysis of group interactions in social networks characterized by both antagonistic and cooperative interactions. The problem is two-fold. Firstly, real-world networked systems are too complex to be represented with only dyadic (pairwise) relationships. Social systems have been studied using more general objects such as hypergraphs, see [1] for a review, but the systems have traditionally been assumed to be cooperative only (see e.g., [2]). Secondly, while antagonistic/cooperative social networks have been analyzed in the literature (see, e.g., [3]), these systems have traditionally been modeled using pairwise graphs, which are not sufficiently general to capture group interactions.

As explained in [1], we see a growing interest in modeling systems with the more complex object classes: hypergraphs, a generalization of graphs, and simplicial complexes, a structure similar to graphs that highlights group interactions. This thesis examines hypergraphs, which are more general than the alternative. By definition, simplicial complexes cannot represent group interactions in the absence of underlying dyadic relationships. Take, for instance, a communication system that uses only multicast or broadcast; as there will be no unicast connections, the model shall not include any pairwise relationships between the agents. This would be impossible to model correctly using simplicial complexes. The shortfalls motivate the use of hypergraphs, a generalization of graphs that allows edges to connect an arbitrary number of nodes. Hence, agents with only group interactions and agents with only dyadic interactions can be represented by the same mathematical object.

In the literature, signed graphs have been used to represent networked systems with antagonistic and cooperative relationships. Most notably for this study, see the work of Harary [4], who defined the enmity/amity structure balance for

pairwise graphs. This notion of structure balance proved important in the study of social networks, but the definition is inherently binary: a system is either structurally balanced or it is not. The notion has been extended to capture degrees of structural balance using several measures; see, e.g., [5]. Indeed, the notion of structural balance has been generalized to signed hypergraphs, see [6] and [7]; however, there is a lack of measures on the degree of balance of signed hypergraphs.
This thesis generalizes the structure balance measurements from signed graphs to signed hypergraphs. More concretely, a measurement from the domain of pairwise graphs (see [5]) and ideas on signed hypergraphs from [7] were combined to form a novel measure of the degree of balance in signed hypergraphs.

1.1 Background

The idea of structural balance was coined by the social scientist Harary in [4]. Harary first defined structural balance as a decision problem: any social network is either structurally balanced or structurally unbalanced. In particular, social networks are structurally balanced if they can be bipartitioned into two disjoint sets such that antagonistic relationships (if any) occur only between the sets, not within them.
Real-world social networks are complex, and the probability of structural balance is low. This motivates the use of a measure to quantify how structurally unbalanced a social network is; see, e.g., [3]. As mentioned, determining the degree of structural balance is still limited for signed hypergraphs.

1.2 Problem

This thesis investigates the following questions. Can we measure antagonism, e.g., social tension, in signed hypergraphs? Can we generalize antagonism measurements used on signed pairwise networks to signed hypergraphs, i.e., is it possible to, while preserving their meaning, extend selected pairwise measures to the domain of hypergraphs?

1.3 Purpose

This work will contribute to the analysis of complex systems, e.g., social networks. Research in biology, economics, and other complex systems could benefit from studying more complex interactions, such as systems in which agents interact exclusively in groups and relationships can be antagonistic.

1.4 Goals

The project aims to define balance in signed hypergraphs and analyze measures of structural balance in hypernetworks. The project will lay the foundation for extending existing work on decision-making in cooperative hypernetworks [3] and frustration in signed networks [2]. The goal is to answer the research questions in Section 1.2. To answer the research question, the thesis will need to include both the theoretical definition of the new measure, in addition to the results and analysis of the measurement applied on synthetic and real-world hypernetworks.

1.5 Research Methodology

The research project was structured as follows. *Theoretical derivation*. The theoretical aspects of measurements of structural balance in signed graphs were analyzed. The ones that seemed likely to generalize to hypergraphs were further analyzed. We introduced all definitions, including that of the generalized novel measurement, in Chapter 4.2. *Implementations* The novel measurement and an existing measurement defined on signed hypergraphs were implemented and compared; likewise, two measurements defined on signed graphs were implemented. *Synthetic validation*. The four measurements were applied to synthetic signed graphs and signed hypergraphs; the procedure of generating random (hyper)graphs was given in Chapter 4.3 and Chapter 4.4. *Real-world validation*. Finally, the novel measurement was applied to real-world data.

1.6 Outline of the thesis

This thesis is structured as follows. Chapter 2 presents relevant background information. A review of hypergraphs and signed hypergraphs is provided in Chapter 3. Chapter 4 presents the method, definitions, and software

implementations used to achieve the results. The results and discussion are presented in Chapter 5. Chapter 6 contains the conclusions, limitations, and future work.

Chapter 2

Background

For clarity and completeness, we begin this chapter with some notation from matrix and graph theory before moving on to structural balance. This chapter is structured as follows. Section 2.1 introduces the notation and definitions used in this thesis. An overview of graph theory, linear algebra, and matrix theory is given in Section 2.2. Finally, Section 2.3 introduces signed graphs, the notion of structural balance for signed graphs, and measurements of degrees of structural balance.

2.1 Notations and Definitions

Regarding sets and compositions, denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Z}$ the integers, $\mathbb{N} = \{1, 2, ...\}$. In this thesis, the positive scalar n is always in $\mathbb{N}$. Let $[n] = \{1, 2, ..., n\}$. Let (a, b), where $a, b \in \mathbb{N}$, define an ordered pair. A closed interval is defined $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. The all-ones column vector is defined as $\mathbf{1}_n$. A sequence is a function $s : \{j, j + 1, ..., k\} \to \mathbb{S}$, where $j \leq k,\ j, k \in \mathbb{N}$ and $\mathbb{S}$ is some set. The cardinality of a set $\mathbb{S}$ is denoted $|\mathbb{S}|$.
For random variables, the notation $[X = x]$ denotes the event that the random variable X takes the value x; $\sum[X = x]$ is the number of times the variable takes that value. $\mathbb{P}[X = x]$ defines the probability of X taking the value x.
Regarding matrices, let $(\mathbf{A})_{ij}$ be the element at the i:th row and the j:th column of $\mathbf{A} \in \mathbb{R}^{n \times m}$, also defined a_{ij}. The matrix $\mathbf{A}$ is non-negative if all its elements are larger or equal to zero, denoted $\mathbf{A} \geq 0$, inequalities between matrices are understood component-wise. $\mathbf{A}$ is positive if all its elements are larger than zero, $\mathbf{A} > 0$. $\mathbf{A}$ is positive-semidefinite if all its eigenvalues are non-negative, $\mathbf{A} \succeq 0$; positive-definite if all of its eigenvalues are positive, $\mathbf{A} \succ 0$. The

spectrum of $\mathbf{A}$ is written $\lambda(\mathbf{A})$, the eigenvalues are ordered from smallest to largest $\lambda_1(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \leq ... \leq \lambda_n(\mathbf{A})$. Let $\rho(\mathbf{A})$ be the spectral radius of the matrix $\mathbf{A}$. $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{A}$ is orthogonal if $\mathbf{A}^T = \mathbf{A}^{-1}$. The diagonal of a matrix is a set of elements $\{a_{11}, a_{22}, ..., a_{nn}\}$. A diagonal matrix

$$(\mathbf{A})_{ij} \in \begin{cases} d & i = j \\ 0 & \text{otherwise} \end{cases} \qquad d \in \mathbb{R},$$

is defined as $\mathbf{A} = \text{diag}\{a_1, ..., a_n\}$, $a_i \in \mathbb{R}$ $\forall i$. A signature matrix $\mathbf{S}$ is a diagonal matrix where $\mathbf{S} = \text{diag}\{s_1, ..., s_n\}, s_i \in \{-1, 1\}$ $\forall i$.
A row sum is the sum of all elements in a row of a matrix. The i:th row of $\mathbf{A}$ is defined $(\mathbf{A})_i$. The i:th row sum is written $(\mathbf{A}\mathbf{1}_n)_i$. The sum of a matrix is,

$$\mathbf{1}_n^T \mathbf{A} \mathbf{1}_n = \sum_{i,j \in [n]} (\mathbf{A})_{ij}.$$

2.2 Pairwise Graphs

A graph, denoted $G = (V, E)$, is an ordered pair of two sets (V, E). The elements V are the nodes in the graph* while the elements in E are unordered pairs of elements in V, i.e., $E \subseteq V \times V$. The elements in E are called edges. Vertices that are directly connected by an edge are called adjacent. For undirected graphs, edges do not have a direction, i.e., $(v_i, v_j) = (v_j, v_i)$, $v_i, v_j \in V$. A graph is complete if all nodes are connected. A multigraph is a graph where two nodes can be adjacent through more than one edge. An edge, $e = (v_i, v_i)$, is called a self-loop. A graph is weighted if each edge is assigned a real-valued weight, represented by a function $f : E \to \mathbb{R} \cap \{0\}$. A graph is signed if its edges are signed, i.e., $f : E \to \{-1, 1\}$. Finally, a graph is simple if it is not weighted, not directional, not signed, does not contain self-loops, and is not a multigraph.

For simple graphs, we define the following. A walk is an ordered sequence of nodes: $\{v_1, v_2, \cdots, v_q\} = \{v_k\}_1^q$, where every subsequent node, except the first one, have to share an edge with the preceding node in the sequence. A path is a walk where no vertex is repeated. A cycle is a walk where no vertex is repeated except for the first one, $v_1 = v_q$.

For signed and unsigned graphs, we define the following. A graph is acyclic if it contains no cycles. A graph is connected if there exists a walk

*Nodes, agents, and vertices are synonymous in this thesis.

between any two nodes. A connected component of a graph is a set of nodes such that none of the nodes in the set are adjacent to the rest of the graph, i.e., the other connected components. The following definition introduces a special type of graph.

Definition 2.2.1 *A graph is bipartite if its vertices can be partitioned into two disjoint groups such that no two vertices in the same group are adjacent.*

Furthermore, the following definition is central in this thesis.

Definition 2.2.2 *The degree of a vertex is the number of directly adjacent vertices.*

The following two concepts, graph homomorphism and graph isomorphism, are important in graph theory. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. A graph homomorphism from G to H is a function $f : V_G \to V_H$ such that adjacency is preserved: if $\{u, v\} \in E_G$, then $\{f(u), f(v)\} \in E_H$. The function f need not be bijective. In particular, several vertices of G may be mapped to the same vertex of H, provided that all edges of G are mapped to edges of H.
A graph isomorphism is a stronger notion. It is a bijective map $f : V_G \to V_H$ that preserves adjacency in both directions: $\{u, v\} \in E_G \iff \{f(u), f(v)\} \in E_H$. Thus, an isomorphism can be viewed as a renaming of the vertices that leaves the graph structure unchanged. If such a map exists, then G and H are said to be isomorphic.

Algebraic Graph Theory

All definitions mentioned in this section are taken from [8] unless otherwise specified. The adjacency matrix $\mathbf{A}_G = [a_{ij}]$ of a graph $G = (V, E, \mathbf{A})$ is defined as

$$(\mathbf{A})_{ij} = \begin{cases} 1 & (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The subscript G is omitted when unambiguous. The adjacency matrix is a unique representation of the graph's structure. However, note that the adjacency matrices of isomorphic graphs are unique but permutation similar. If the graphs G and H are isomorphic, then $\mathbf{A}_G = P^T \mathbf{A}_H P$, for some permutation matrix P, a matrix that differs from the identity matrix only by a sequence of row interchanges. In this work, we consider undirected graphs without self-loops, which means that the corresponding adjacency matrices

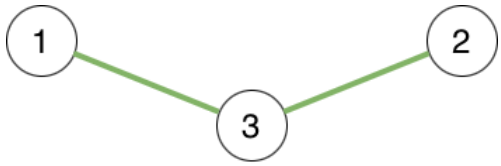


Figure 2.1: An acyclic graph with labeled vertices.

are symmetric, i.e, $\mathbf{A} = \mathbf{A}^T$, and have zero diagonals. The adjacency matrix of the graph in Figure 2.1 is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Other important matrices in algebraic graph theory include the degree matrix Δ, incidence matrix $\mathbf{I}$, and Laplacian matrix $\mathbf{L}$. The degree matrix $\Delta \in \mathbb{R}^{|V|\times|V|}$ is a diagonal matrix

$$(\Delta)_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij} & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The incidence matrix is defined $\mathbf{I} \in \{0,1\}^{|V|\times|E|}$, with rows representing the vertices, columns representing the edges. If $(\mathbf{I})_{ij}$ is non-zero then we say that v_i and e_j are incident. The incidence of the graph in Figure is written 2.1,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Furthermore, the adjacency can be expressed as $\mathbf{A} = \mathbf{I}\mathbf{I}^T - \Delta$.
The Laplacian* matrix (called Laplacian interchangeably in the thesis) is defined as $\mathbf{L} = \Delta - \mathbf{A}$,

$$(\mathbf{L})_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} & i = j, \\ -a_{ij} & \text{otherwise.} \end{cases}$$

*This thesis only considers the *combinatorial* Laplacian.

The Laplacian of the graph in Figure 2.1 is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \Delta - \mathbf{A}.$$

By definition, the row sums of the Laplacian are always zero, i.e., $\mathbf{L}\mathbf{1}_n = \mathbf{0}_n$, meaning that 0 is always an eigenvalue of $\mathbf{L}$. Finally, the normalized adjacency matrix and normalized Laplacian matrix are given by $\mathbf{H} = \Delta^{-1}\mathbf{A}$ and $\mathcal{L} = \Delta - \mathbf{H}$, respectively. The normalization gives $\mathbf{H}\mathbf{1}_n = 1$ and $\rho(\mathbf{H}) = 1$. The adjacency can be used to detect acyclicity and connectivity.

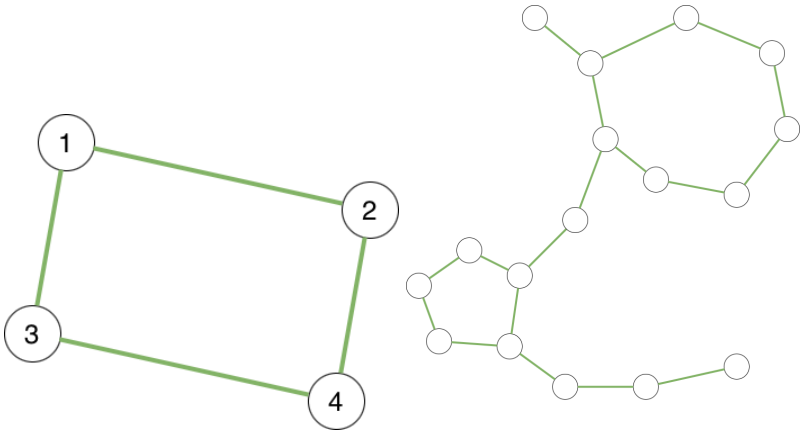
Lemma 2.2.1 *Let $G = (V, E, \mathbf{A})$ be a graph with n nodes. $\forall\ i, j \in [n]$, $k \in \mathbb{N}$: The entry a_{ij} in $\mathbf{A}^k$ contains the number of walks of length k from node i to node j; it is zero if no walk of length k exists between the nodes.*

Definition 2.2.3 *The adjacency matrix $\mathbf{A}$ of a connected graph is irreducible if $\exists\ k \in \mathbb{N}$ such that $\sum_{i=0}^{k} \mathbf{A}^i > 0$.*

Definition 2.2.4 *A graph $G = (V, E, \mathbf{A})$ is primitive if there exists a $k \in \mathbb{N}$: $\mathbf{A}^k > 0$.*

Remark 2.2.1 *If $G = (V, E, \mathbf{A})$ is primitive if there exists a $k \in \mathbf{N}$ such that any node pair in G is connected by a walk of length k.*

While all connected graphs are irreducible, not all of them are primitive. If G is primitive, then G is irreducible and acyclic or irreducible and *aperiodic*, i.e., the lengths of its cycles are coprime to 2, see Example 2.2.1.



(a) A periodic graph. (b) An aperiodic graph.

Figure 2.2: Two cyclic graphs, one periodic and one aperiodic.

Example 2.2.1 *The graph in Figure 2.2a is not primitive as it is periodic, i.e., there exists no k such that there exists a walk of k steps between v_1 and v_2 and a walk of k steps between v_1 and v_3. The graph in Figure 2.2b is primitive as there exists a k such that all pairs of vertices are connected by a k-length walk. The weighted sum of co-prime numbers can be used to express all numbers equal or larger than k, i.e., the Frobenius number + 1 (see Lemma 4.8 in [8]). This can be applied to walks: for graphs where the length of the loops are coprime to each other and to the number 2 there will exist a k such that there exists a walk of k steps between every two nodes**.
The graph in Figure 2.2b is primitive as the greatest common denominator between the lengths of its cycles, and 2 is 1, $\gcd(7,5,2) = 1$.

Spectral Graph Theory

Spectral graph theory is the study of how the spectrum of the matrices used in algebraic graph theory relates to the structure of the graphs they represent. The spectrum of the Laplacian has a particularly relevant role; for instance, it is used to determine the number of connected components in the graph. We begin by establishing some of its properties.

Lemma 2.2.2 *([8]) The Laplacian of an undirected graph has real eigenvalues.*

*There will in fact exist a walk of length $q, q \geq k$ between all nodes; if $A^k > 0$, then all $A^{k+j}, j \in \mathbb{N}$ will be positive, see Chapter 2.3 in [8].

Lemma 2.2.2 follows from the fact that the Laplacian of an undirected graph is symmetric and real, hence its eigenvalues are real.

Lemma 2.2.3 *([8], Lemma 6.5) The Laplacian of an undirected graph is positive-semidefinite, i.e., its eigenvalues are non-negative.*

Since $\mathbf{L}\mathbf{1}_n = \mathbf{0}_n\mathbf{L} = \mathbf{0}$, zero is always an eigenvalue of the Laplacian, and since the Laplacian is positive-semidefinite, the smallest eigenvalue of the Laplacian is zero.

The second smallest eigenvalue of the symmetrical Laplacian, called the Fiedler eigenvalue, is correlated with the level of connectivity in the graph, i.e., how tightly coupled the vertices are. Example 2.2.2 explains how the eigenvalue relates to connectivity.

Example 2.2.2 *Bipartition the graph's nodes into two disjoint non-empty groups with a vertex vector x such that $x \in [a,b]^{|V|}$, $a < 0 < b$. Using Rayleigh's Theorem for eigenvalues (see Theorem 4.2.2 in [9]), the second smallest eigenvalue of the Laplacian is*

$$\lambda_2 = \min_{x \notin span(\mathbf{1}_n)} \frac{x^T\mathbf{L}x}{x^Tx} = \min_{\substack{\|x\|_2=1, \\ x \notin span(\mathbf{1}_n)}} x^T\mathbf{L}x.$$

Note that $x \in span(\mathbf{1}_n)$ represents all nodes being in the same group; it is excluded as it is the eigenvector associated with the smallest eigenvalue, 0.
A graph that has two or more connected components, i.e., is not connected, has a Fiedler eigenvalue of 0; the complete graph has a Fiedler eigenvalue of $|V|$ (Example was inspired by example in Appendix 6.4 of [8]).

2.3 Signed Pairwise Graphs

We are now ready to introduce the concepts of antagonism and structural balance. The antagonistic and cooperative relationships between agents are usually denoted by the signs of the edges in the graphs. This translates to signed entries in the adjacency matrix. The adjacency matrix of a signed graph is defined as

$$(\mathbf{A})_{ij} = \begin{cases} \sigma(i,j) & (i,j) \in E, \\ 0 & \text{otherwise}, \end{cases}$$

where $\sigma : E \to \{-1, 1\}$. We call a graph *cooperative* if all its signs are positive, i.e., a standard unsigned graph defined in the previous section,

Section 2.2. We denote cooperative graphs as G^+.
The Laplacian of a signed graph can be defined in multiple ways [10]. Unless otherwise specified, this thesis uses the *opposing* Laplacian, where the adjacency degree is defined as the absolute sum of adjacent nodes. It is defined as

$$(\mathbf{L})_{ij} = \begin{cases} \sum_{k=1}^{n} |a_{ik}| & i = j, \\ -a_{ij} & \text{otherwise.} \end{cases}$$

Structural Balance

The idea of structural balance (SB) originates from Heider's work on social triangles from 1944 [11], e.g., if person A likes person B and person B likes object C, there is tension, or unbalance, if person A does not like object C. Cartwright and Harary extended this idea in 1956 [12] to social networks larger than triangles; the work builds on Harary's previous work on signed graphs [4], which defines the following:

Definition 2.3.1 *([4], Theorem 3) A signed graph is structurally balanced iff its vertices V can be partitioned into two disjoint subsets, one of which can be empty, such that all edges going between the subsets are negative and all edges within the subsets are positive.*

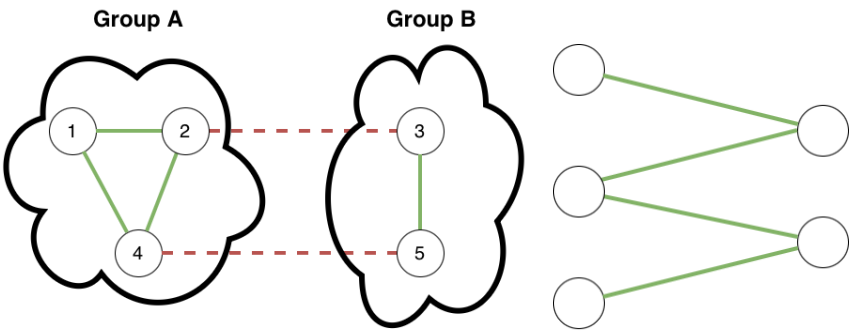


Figure 2.3: On the left, a balanced signed graph. Red dashed lines indicate antagonistic relationships, while green solid lines indicate cooperative relationships. On the right, a bipartite graph.

It is important to distinguish between a bipartition and a bipartite graph. The graph on the left in Figure 2.3 has been bipartitioned. The one on the right is a bipartite graph.

An alternative definition to Definition 2.3.1 is the following.

Definition 2.3.2 *([4]) A signed graph is SB iff it is free of negative cycles.*

Or, equivalently, by the following lemma and theorem.

Lemma 2.3.1 *([7], Lemma 2.1) Two s-graphs, $G = (V_G, E_G, \mathbf{A}_G)$ and $H = (V_H, E_G, \mathbf{A}_H)$ are switching equivalent iff there exists a signature matrix $\mathbf{A}_G = \mathbf{S}\mathbf{A}_H\mathbf{S}^{-1}$*

Theorem 2.3.2 *A s-graph $G = (V, E, \mathbf{A})$ is SB if it is switching equivalent to G^+, i.e., if there exists a signature matrix $\mathbf{S}$ such that $\mathbf{A} = \mathbf{S}|\mathbf{A}|\mathbf{S}^{-1}$, where $|\mathbf{A}|$ indicates elementwise absolute value.*

Note that switching equivalence is also called *signature similarity* in the signed graph theory.
The following theorem is another way of expressing Definition 2.3.1.

Theorem 2.3.3 *([4], Theorem 2) All paths between any two vertices in an SB graph have the same sign.*

We define the opposite of SB as follows.

Definition 2.3.3 *A signed graph is structurally unbalanced if it is not structurally balanced.*

To give the reader intuition for why the mathematical statements are related, we present the following example.

Example 2.3.1 *The graph in Figure 2.4 has two cycles: $(v_4, v_5, v_6, v_7, v_8, v_4)$ is a negative cycle, as the edge between v_5 and v_6 is negative; $(v_2, v_3, v_4, v_8, v_9, v_2)$ is a positive cycle as it contains two negative edges. There exist three paths between v_1 and v_{10}: (v_1, v_2, v_9, v_{10}) and $(v_1, v_2, v_3, v_4, v_8, v_9, v_{10})$ are positive paths while $(v_1, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ is not. The graph is not structurally balanced, as it is impossible to bipartition it into disjoint subsets such that only negative edges separate them. Removing the (v_5, v_6) edge removes all negative cycles, makes all paths share the same sign, and makes the bipartition possible.*

We are now ready to state the following.

Theorem 2.3.4 *([5], Theorem 2 and 4) The smallest eigenvalue λ_1 of the Laplacian of an SB graph, G, is always 0, i.e., it is always positive-semidefinite. Similarly, λ_1 of the Laplacian of a structurally unbalanced graph is always larger than 0, i.e., positive-definite.*

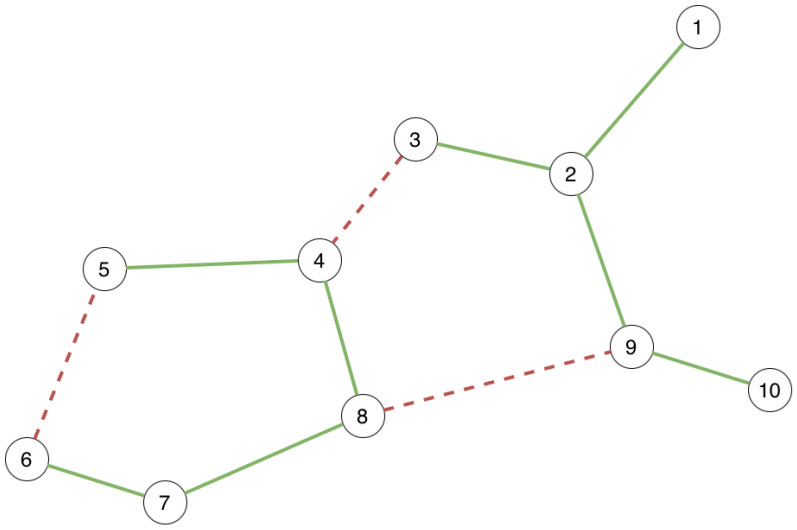


Figure 2.4: A signed graph with a negative cycle. Dotted lines are negatively signed edges, and solid lines are positive.

Analogously for the normalized adjacency matrix, $\mathbf{H}$.

Theorem 2.3.5 *If a graph $G = (V, E, \mathbf{A})$ is SB then the largest eigenvalue λ_n of $\Delta^{-1}\mathbf{A} = \mathbf{H}$ will be 1. If structurally unbalanced, $\lambda_n(\mathbf{H}) < 1$.*

Measuring the Degree of Structural Balance

In [12], Harary and Cartwright extended the idea of structural balance, which is defined as a decision problem in [4], to an optimization problem. They argued that some signed graphs are more unbalanced than others, hence their definition, which follows.

Definition 2.3.4 *([12]) For signed graphs, let the degree of structural balance be the ratio of the number of positive cycles to the total number of cycles.*

They argued that smaller structures, e.g., triangles (cycles of length three), have a larger impact on the overall structural balance in a social network than larger ones; they define a graph to be 3-balanced if all triangles are balanced, likewise for cycles larger than three, they define N-balance.

The idea was extended in Kunegis's 2014 paper [5] He defines a similar measure to that of Harary and Cartwright [12], the *signed clustering*

coefficient. Firstly, we define the clustering coefficient as the likelihood of finding triangles in a graph

$$\frac{\nabla}{\vee},$$

where ∇ is the number or triangles and $\vee$ is the number of paths of length two. The signed clustering coefficient is defined as the sum of the product of all triangles over all triangles. Let $\nabla^+(G)$ and $\nabla^-(G)$ be the number of triangles with even and odd numbers of negative edges, respectively. The signed clustering coefficient is defined

$$\frac{\nabla^+(G) - \nabla^-(G)}{\nabla +^+ (G) + \nabla^-(G)}.$$

Note that the ratio can be negative.

More importantly, Kunegis proved that the Laplacian of a signed graph is positive-semidefinite: The Laplacian can be represented as the sum of matrices

$$\mathbf{L} = \sum_{\{i,j\} \in E} \mathbf{L}^{\{i,j\}},$$

where $\mathbf{L}^{\{i,j\}} \in \mathbf{R}^{|V| \times |V|}$ with all zeros except for four elements: $l_{ii} = l_{jj} = 1$, $l_{ij} = l_{ji} = -\sigma(\{i,j\})$. $\mathbf{L}^{\{i,j\}}$ is non-negative as the quadratic from gives $x^T \mathbf{L}^{\{i,j\}} x = x_i^2 + x_j^2 - 2\sigma(\{i,j\}) = (x_i - \sigma(\{i,j\}) x_j)^2 \geq 0$. Since $\mathbf{L} = \sum_{\{i,j\} \in E} \mathbf{L}^{\{i,j\}}$, the signed Laplacian is positive-semidefinite.

Treating the vector x as a bipartition vertex vector, $x_i \in \{-1, 1\}$, gives the following proposition: the quadratic form with bipartition vector x is only zero when all vertices can be partitioned into two groups, i.e., all $x^T \mathbf{L}^{\{i,j\}} x$ terms cancel out. Kunegis defines algebraic conflict as the smallest eigenvalue of the signed Laplacian of G. The value is zero iff G is SB, and it is negatively correlated to the degree of SB.

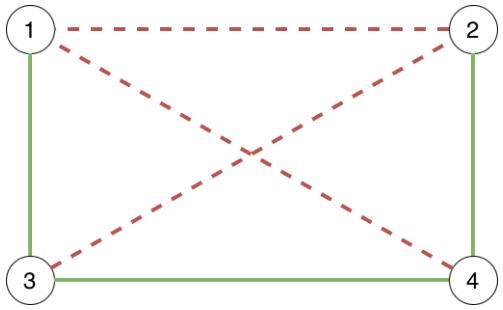


Figure 2.5: A structurally unbalanced signed graph. The green solid lines are cooperative; the red dotted ones are antagonistic.

Example 2.3.2 *Figure 2.5 shows a graph that cannot be bipartitioned as v_3 and v_4 share a positive edge: the graph is structurally unbalanced. Its Laplacian is*

$$\mathbf{L} = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}.$$

In other words, there exists no bipartition vector x such that $x^T\mathbf{L}x = 0$,

$$\nexists\, x, x \in \{-1,1\}^4 : x^T\mathbf{L}x = \sum_{\{i,j\}\in E} x^T L^{\{i,j\}} x = (x_1-x_2)^2+\cdots+(x_3+x_4)^2 = 0.$$

By Harary's Theorem (see Definition 2.3.1), the graph is structurally unbalanced. As explained by Kunegis in [5], it is possible to measure the degree of structural unbalance.
Relaxing the problem from a strictly binary partition to a continuous bipartition, $x \in \{-1,1\}^4$ to $x \in [-1,1]^4$, gives the following. The smallest value of

$$\min_{x\in[-1,1]^4} x^T\mathbf{L}x,$$

gives a measurement of how unbalanced the graph is. Without loss of generality, restricting the bipartition to $\|x\|_2 = 1$ gives,

$$\min_{x\in\mathbf{R}^4,\, \|x\|_2=1} x^T\mathbf{L}x.$$

This gives the smallest eigenvalue, $\lambda_1(\mathbf{L})$, as $\mathbf{L}$ is real and symmetric. $\lambda_1(\mathbf{L}) > 0 \Leftrightarrow \mathbf{L} \succ 0$. In this example, $\lambda_1(\mathbf{L}) \approx \frac{3}{4}$.

Kunegis gives the following theorem.

Theorem 2.3.6 *([5], Theorem 5) Let $G = (V, E)$ and $H = (V, E')$ be signed graphs such that $E = E' \backslash e$. The algebraic conflict of G, $\lambda_1(G)$ will be smaller or equal to $\lambda_1(H)$.*

The theorem states that removing an edge never decreases a graph's degree of balance; it is impossible to create a negative (or positive) cycle by removing an edge.

In 2018, Aref et al. [13] surveyed measurements of the degree of SB and classified them into three categories. **Negative cycles:** since a graph is SB iff it is free of negative cycles, a natural measurement of relative SB is the number of negative cycles. The simplest such measurement is the ratio defined by Harary and Cartwright [12], the ratio of positive cycles over all cycles

$$D(G) = \frac{\sum_{k=3}^{n} C_k^+}{\sum_{k=3}^{n} C_k}, \text{ and } D_k(G) = \frac{C_k^+}{C_k},$$

where C is a cycle and C^+ a positive cycle. As mentioned in their paper, the length of the cycles should have an inverse significance to the overall SB; a reflection on Harary and Cartwright's paper [12] where they argue that smaller social structures, e.g., triangles, have larger impacts on the SB than larger ones. Aref et al. noted that attempts have been made to implement this notion, which is why the survey also includes a weighted version; however, as Aref et al. point out, there has been no definitive best choice of weighting function. **Minimum deletion set:** (the frustration index) deleting an edge can only decrease SB, not increase it, hence it is natural to use the minimum edge deletion set as a measurement of the degree of SB. If removing two given edges* makes the graph G SB, the frustration index of G is two. **Eigenvalues:** Aref et al. found that the algebraic conflict was the most effective measurement in this class.

In the conclusion of their paper, Aref et al. state that the frustration index is the most representative measurement of the degree of SB. However, since finding the minimum deletion set is NP-complete, they suggest using algebraic conflict and $D_k(G)$ for complex graphs.

*G can have many minimum deletion sets of the same cardinality, it does not affect the score.

The frustration index from the Laplacian Fontan et al. [3] introduce a notion of the frustration index for weighted signed graphs. It corresponds to minimizing the signature similarity transformation of the Laplacian of a signed graph,

$$\text{Frustration Index}(\mathbf{L}) = \min_{\substack{S=\text{diag}\{s_1,\ldots,s_n\} \\ s_i\in\{-1,1\}\ \forall i}} \mathbf{1}_n^T(|\mathbf{L}| + \mathbf{SLS})\mathbf{1}_n,$$

which is proportional to

$$\min_{\substack{\mathbf{S}=\text{diag}\{s_1,\ldots,s_n\} \\ s_i\in\{-1,1\}\ \forall i}} \mathbf{1}_n^T(|\mathbf{A}| - \mathbf{SAS})\mathbf{1}_n.$$

Left-multiplying the adjacency matrix by the signature matrix switches the sign of all rows i where $s_i = -1$; right-multiplying does the same for the columns. Hence, it is equivalent to bipartitioning the vertices into two labeled groups, $\{-1\}$ and $\{1\}$. If the graph is SB, there exists a signature matrix that can switch all signs such that $\mathbf{SAS}$ is non-negative, hence, the matrix sum of $|\mathbf{A}| - \mathbf{SAS}$ will be zero. Minimizing for the smallest matrix sum gives a multiple of the frustration index.

Chapter 3

Review of Hypergraphs

This chapter introduces the notion of hypergraphs and gives a review of some of the literature on signed hypergraphs. Hypergraphs and their representation are defined in Section 3.1, while Section 3.2 presents signed hypergraphs and how the notion of structural balance can be extended to higher-order representation.

3.1 Hypergraphs

All definitions of hypergraphs are variations on generalizations of pairwise graphs. The notion of hypergraphs discussed in this thesis follows Jégou et al.'s definition.

Definition 3.1.1 *([14], Definition 1) A hypergraph H is a pair $H = (X, E)$ where X is a set of vertices and E is a family of subsets of X, called hyperedges.*

Hence, hyperedges are sets of nodes that can be overlapping and of arbitrary size. Furthermore, we say that a hypergraph's order is $max\{|\ e\ |: e \in E\}$ and we define a hypergraph to be k-uniform if $\forall e \in E,\ |e| - 1 = k$, where $|e|$ is the cardinality of the hyperedge e. Note that this definition renders pairwise graphs as 1-uniform hypergraphs.

Representation

As the definition of hypergraphs is lenient, hypergraphs do not require higher-order edges to be built on top of lower-order edges; a hypergraph containing the hyperedge $\{a, b, c\}$ does not require the pairwise edge $\{a, b\}$. The relaxed definition makes the adjacency matrix insufficient to represent higher-order

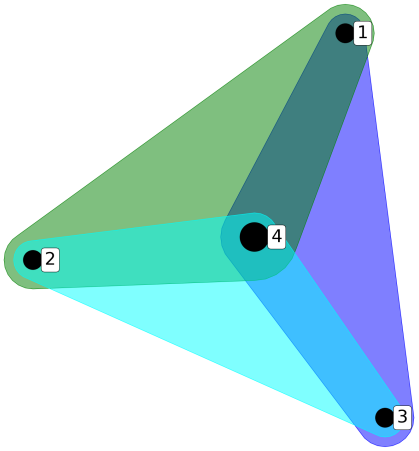


Figure 3.1: A 2-uniform hypergraph with three hyperedges, each containing three nodes.

relationships in hypergraphs. A general review of hypergraph representations can be found in [1].

A simple way to represent a hypergraph is given by its incidence matrix. The columns of the incidence matrix of a pairwise graph have exactly two non-zero elements, as each edge connects two vertices. For 2-uniform hypergraphs, there are three non-zero elements. For example, the incidence matrix of the hypergraph in Figure 3.1 is

$$\mathbf{I} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

An alternative definition is given in [7], for the hypergraph H, $\mathbf{M}(H) = \mathbf{I}^T$.

Another approach is to represent a hypergraph through the adjacency tensors. Battiston et al. suggest in [1] that order-k edges can be represented with a $(k+1)$-tensor. For instance, for second-order edges, one could use an $n \times n \times n$ tensor. As an example, the adjacency tensor for the hypergraph in Figure 3.1 is captured by the following four matrices, where each matrix $A_i^{(3)}$ corresponds to the adjacency of vertex $i, i = 1, 2, ..., |V|$ and (3) denotes the

order of the hyperedge.

$$A_1^{(3)}\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&1&1&0\end{bmatrix}, A_2^{(3)}\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&1\\1&0&1&0\end{bmatrix}, A_3^{(3)}\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&0\\1&1&0&0\end{bmatrix}, A_4^{(3)}\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&0\\0&0&0&0\end{bmatrix}.$$

Note that, similar to how undirected graphs have two entries per edge in the adjacency matrix, the adjacency tensor has six, which corresponds to the number of permutations of the nodes in the hyperedge, i.e., the factorial of the cardinality of the set.

Given the adjacency tensor, it is also possible to define a pairwise representation of a hypergraph, albeit at a loss of information. For instance, in [2], Fontan et al. define a pairwise adjacency matrix for 2-order hypergraphs as

$$\widetilde{A} = A^{(2)} + \begin{bmatrix}\mathbf{1}_n^T A_1^{(3)}\\ \vdots \\ \mathbf{1}_n^T A_n^{(3)}\end{bmatrix}, \tag{1}$$

where $A^{(2)}$ is the pairwise adjacency matrix and $A_i^{(3)}$ is the i:th matrix in the tensor. $\widetilde{A}$ for the hypergraph in Figure 3.1 is given by

$$\mathbf{0}^{4\times 4} + \begin{bmatrix}0&1&1&2\\1&0&1&2\\1&1&0&2\\2&2&2&0\end{bmatrix}.$$

Note that the matrix is not binary. Also note that this representation loses information about the higher-order relationships, since it is not possible to deduce the order of the hyperedges for $\widetilde{A}$. However, it is useful when representing dynamical systems over hypergraphs, see [2] for details.

Higher-Order Degree

In a pairwise graph $G = (V, E, \mathbf{A})$, the degree of a vertex is typically defined as the number of directly connected neighbors $d(v_i) = \sum_{j=1}^{n} (\mathbf{A})_{ij}$. Battiston et al. suggest in [1] that the degree in hypernetworks can be expressed similarly. For 2-uniform hyperedges (hypergraphs where all hyperedges contain exactly three nodes), the degree of node i considers all the higher-order interactions node i participates in. The degree matrix for 2-uniform hypergraphs is given

by

$$(\Delta)_{ij} = \begin{cases} \mathbf{1}_n^T A_i^{(3)} \mathbf{1}_n & i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Note that this is true only for 2-uniform hypergraphs. For 2-order hypergraphs, pairwise interactions may also be present. In [2], Fontan et al. extends the notion in (2) and defines the degree matrix for 2-order hypergraphs using (1),

$$\Delta = \operatorname{diag}\left(\widetilde{A}\mathbf{1}_n\right) = \operatorname{diag}\left(\sum_{j\in n}(\widetilde{A})_{1j}, \ldots, \sum_{j\in n}(\widetilde{A})_{nj}\right). \tag{3}$$

Given the notion of degree, it is also possible to introduce the normalized version of the matrix $\widetilde{H} = \Delta^{-1}\widetilde{A}$, where Δ corresponds to the representation in (3). Note that $\rho(\widetilde{H}) = 1$ if the hypergraph is cooperative, i.e., all hyperedges are positively signed.

Note that the mentioned definitions of higher-order degrees are only defined for 2-order hypergraphs. While it is possible to represent k-order hypergraphs ($k > 2$) using the sum of higher rank tensors, in this thesis, we represent them not by $\widetilde{A}$ matrices but with the incidence matrix. We define *incidence degree* as the number of hyperedges a vertex is in. The incidence degree matrix is defined as

$$(\Delta_I)_{ij} = \begin{cases} \mathbf{1}_n^T (\mathbf{I}^T)_i \mathbf{1}_n & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the incidence degree of vertex i is the i:th column sum of the incidence matrix.

Furthermore, a matrix similar to $\widetilde{A}$ can be defined for hypergraphs defined using the incidence matrix. The diagonal of $\mathbf{I}\mathbf{I}^T$ is Δ_I, while $\mathbf{I}\mathbf{I}^T - \Delta_I$ gives a matrix

$$(\widetilde{A}_I)_{ij} = \begin{cases} |\{e \in E : v_i \in e \wedge v_j \in e\}| & i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, for all i, j in $|V|$ such that $i \neq j$, the element $(\widetilde{A}_I)_{ij}$ is the number of hyperedges that v_i and v_j share.

3.2 Signed Hypergraphs

Extending the notions of signed graphs and of structural balance to signed hypergraphs, denoted as *s-hypergraphs* in this thesis, is not trivial. In the following sections, we provide a brief review of the main approaches proposed in the literature based on the so-called signed incidence matrix. Accordingly, an s-hypergraph can be defined as a pair consisting of a hypergraph and a signed incidence matrix mapping the edge-node incidences to $\{-1, 1\}$. It follows that hyperedges in s-hypergraphs are labeled with multiple edge-node signs, i.e., signed incidences, leaving the sign of a given hyperedge open to interpretation. As discussed below, note that s-hypergraphs defined in this way are also referred to as *oriented hypergraphs* in the literature, see, e.g., [7].

Shi et al.'s Work on Balance in Signed Hypergraphs

In their 1999 work [6], Shi and Brzozowski extended Harary's idea of balance in signed graphs [4] (see also Section 2.2 and Section 2.3) to s-hypergraphs*. Their definition uses the signed incidence matrix, Ψ. The columns represent hyperedges and rows represent nodes,

$$(\Psi)_{ij} = \begin{cases} 1 & v_i \text{ is in } e_j \text{ and the incident is positive,} \\ -1 & v_i \text{ is in } e_j \text{ and the incident is negative,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this form of s-hypergraphs is akin, though not equivalent, to signed bipartite graphs where the hyperedges are one subset and the vertices are the other.

Furthermore, as in pairwise graphs, paths and cycles are defined as ordered sequences of alternating vertices and edges, i.e., $(v_1, e_1, v_2, e_2, ..., v_q, e_q, v_{q+1})$. We say that a (s-)hypergraph is connected if there exists a path between all pairs of vertices. The sign of a path/cycle is the product of all incidences in the path/cycle

$$\text{sign}(v_1, e_1, ..., v_{q+1}) = \Pi_{i=1}^{q} \Psi(v_i, e_i)\Psi(v_{i+1}, e_i).$$

*While in the pairwise setting the standard terminology is *structural balance* or *structurally balanced signed graph*, in this chapter we adopt the notation used in Shi et al. [6], where *structural balance* and *structurally balanced signed graph* are simply referred to as *balance* and *balanced signed graph*.

As Harary's structure balance theorem states (see Definition 2.3.1), a graph is SB iff it can be bipartitioned into two disjoint subsets such that any edges connecting the groups are negative, and all edges within the groups are positive.
Analogously, Shi et al. [6] define an s-hypergraph to be balanced if it can be bipartitoned into two sets V_a, V_b such that $V_a \cap V_b = \emptyset$ and $V_a \cup V_b = V$. Accordingly, an s-hypergraph is balanced iff there exists a bipartition where (4) is true for all $e \in E$ and for all pairs of vertices in the hyperedge, $\forall (v_1 \in e \wedge v_2 \in e \wedge v_1 \neq v_2) \in e$.

$$\begin{aligned} \Psi(v_1, e) &= \Psi(v_2, e) \quad && \text{for } v_1, v_2 \in V_a, \\ \Psi(v_1, e) &= -\Psi(v_2, e) && \text{for } v_1 \in V_a,\ v_2 \in V_b. \end{aligned} \tag{4}$$

For example, the hypergraph in Figure 3.2 consists of three hyperedges, its incidence matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We can transform the hypergraph into an s-hypergraph by mapping the non-zero elements in $\mathbf{I}$ to $\{-1, 1\}$. For instance, the signed incidence matrix can be

$$\Psi = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The facet representation, a bipartite representation of hypergraphs, of the s-hypergraphs is depicted in Figure 3.3.

The s-hypergraph is not balanced as the edges connecting v_2 and v_3 create an inconsistency: e_3 requires the vertices to be in different partitions, while e_2 requires them to be in the same.

As with signed graphs, bipartition and negative cycles are related.

Theorem 3.2.1 *([6], Theorem 3.1) A s-hypergraph is balanced iff it is free of negative cycles.*

For instance, the s-hypergraph in Figure 3.3 is not balanced as it contains negative cycles, $(v_1, e_1, v_2, e_3, v_3, e_2, v_1)$ and $(v_2, e_3, v_3, e_2, v_2)$.

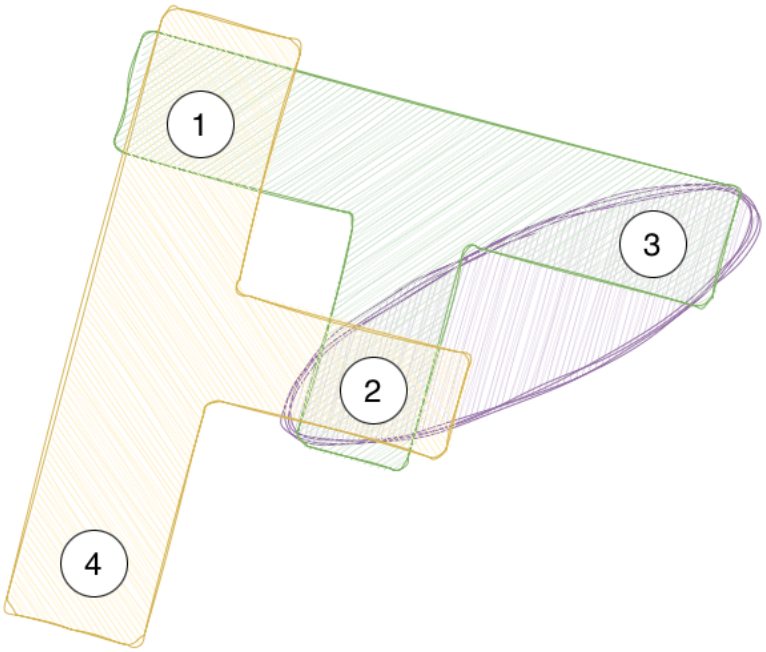


Figure 3.2: 2-order hypergraph with three hyperedges.

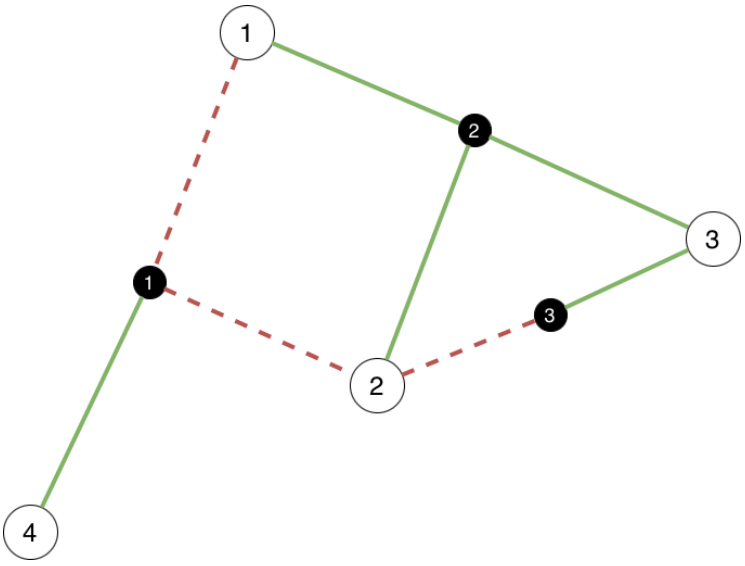


Figure 3.3: An unbalanced s-hypergraph. The smaller, black circles represent the hyperedges; the larger, white circles represent the nodes. Red dotted lines represent antagonistic incidences; green solid lines represent cooperative incidences.

Theorem 3.2.2 *([6], Corollary 3.1) Checking the balance of a signed hypergraph is in* $O(|V| + |E|)$.

Remark 3.2.1 *([6]) Checking the balance of an s-hypergraph can be done using Depth-First Search (DFS) in linear time.*

Importantly, extending notions of balance (and unbalance) to s-hypergraphs allows for the novel definition of measures of the degree of balance of an s-hypergraph. An early contribution is given by Shi and Brzozowski [6], who define the maximum balance problem, similar to the frustration index (see Section 2.3) but for incidence-represented s-hypergraphs.

Definition 3.2.1 *([6]) Given an s-hypergraph* H*, the maximum balance problem consists of finding the set* $E' \subseteq E$ *with minimum cardinality such that* $H(E - E')$ *is free of negative cycles.*

In Definition 3.2.1, $H(E - E')$ is a subhypergraph, i.e., a s-hypergraph $H = (V, E, \Psi)$ where the hyperedges in E' are removed from E. The s-hypergraph in Figure 3.3 is balanced if e_2 or e_3 is removed, hence the minimum cardinality of E' is 1. Additionally, Shi and Brzozowski proved in [6] that, as with the frustration index, the problem is NP-complete.

Wang et al.'s Study on Balance in Oriented Hypergraphs

Wang et al. extended Shi et al.'s ideas in their 2022 work [7]. They define the orientation of a hypergraph as a function from an unsigned incidence to a signed one. An oriented hypergraph is denoted H^σ, while the all positive orientation (cooperative) of the same graph is written H^+. The signed incidence matrix of a hypergraph is denoted by $\mathbf{M}(H^\sigma) \in \{-1, 0, 1\}^{|E|\times|V|}$, we write $\mathbf{M}$ instead of $\mathbf{M}(H^\sigma)$ when unambiguous. Note that this is the transpose of Ψ from Shi et al.'s work. Vertex and edge switching, denoted $\mathbf{S}_U$ and $\mathbf{S}_F$, respectively, are defined as the signature transformation on the incidence matrix,

$$\mathbf{S}_F\mathbf{M}\mathbf{S}_U,$$

where $U \subseteq |V|, F \subseteq |E|$, $(\mathbf{S}_U)_u = -1$ if $u \in U$, else 1, and similarly for $\mathbf{S}_F$.

Lemma 3.2.3 *([7], Lemma 2.1) Two oriented hypergraphs* H^σ *and* H^τ *are switching equivalent iff* $\exists \mathbf{S}_F, \mathbf{S}_U, \mathbf{S}_F, \mathbf{S}_U = diag\{s_1, \dots, s_n\}, s_i \in \{-1, 1\}\ \forall i$: $\mathbf{M}(H^\sigma) = \mathbf{S}_F\mathbf{M}(H^\tau)\mathbf{S}_U$.

In this thesis, we interpret an oriented hypergraph H^σ as an s-hypergraph, which, together with the notion of switching equivalent oriented hypergraphs (Lemma 3.2.3), allows for the definition of a balanced s-hypergraph, as shown in the following theorem. This result is analogous to Theorem 2.3.2 for (pairwise) signed graphs.

Theorem 3.2.4 *([7], Theorem 2.2) H^σ is balanced iff H^σ is switching equivalent to H^+.*

As with the signed graphs, the Laplacian of the s-hypergraphs plays a critical role in evaluating balance. Wang et al. define the Laplacian of H^σ

$$\mathbf{L}(H^\sigma) = \mathbf{M}(H^\sigma)^T\mathbf{M}(H^\sigma).$$

By construction, the Laplacian is positive-semidefinite as the eigenvalues of the matrix are the squares of the singular values of $\mathbf{M}$. Furthermore, the following properties can be derived, where we use the notation for $\mathbf{M} \in \mathbb{Z}^{n\times m}$. Let $\sigma_{max}(\mathbf{M})$ and $\rho(\mathbf{M}^T\mathbf{M})$ be the largest singular value and spectral radius of $\mathbf{M}$ and $\mathbf{M}^T\mathbf{M}$, respectively.

Lemma 3.2.5 *([7], Lemma 3.1) Let H be a connected hypergraph. H^σ is balanced iff $\sigma_{max}(\mathbf{M}(H^\sigma)) = \sigma_{max}(\mathbf{M}(H^+))$.*

Corollary 3.2.6 *([7], Corollary 3.3) Let H be a connected hypergraph. H^σ is balanced iff $\rho(\mathbf{L}(H^+))$ is in the spectrum of $\mathbf{L}(H^\sigma)$.*

Chapter 4

Research Method, Definitions, and Implementations

The goal of the thesis was to model antagonism and understand social tension in higher-order social communities. In practice, this involved extending measurements defined in the domain of signed pairwise graphs to the domain of signed hypergraphs. The contributions are twofold: first, we formally defined the measurements. Second, we evaluated the proposed notions heuristically, by applying them to synthetic and real-world hypergraphs. This chapter describes the method used to achieve these objectives. The first section, Section 4.1, presents the research process of the thesis. Section 4.2 defines the measurements used on the hypergraphs and how they were implemented. Following this, Section 4.3 presents how the graphs were synthesized and how the real-world data was used. Finally, Section 4.4 explains how the experiments were conducted.

4.1 Research Process

This was an exploratory research project. The end goal was clearly defined, while the characteristics of the end result were less defined. The research method used was both fundamental and experimental, as explained in [15], meaning that the project was curiosity-driven and the results quantitatively supported.
What follows is an explanation of how the project was executed. The project consisted of distinct, sequential, and at times overlapping, phases: Pre-study; software implementation of synthetic pairwise graph generation and measurement; explanatory literature study, i.e., evaluating research articles on

signed hypergraphs provided by the supervisors; deciding on a given approach, i.e., oriented hypergraphs; implementing synthetic hypergraph generation; finally, defining the new measurement, implementing it in software and evaluating it on synthetic and real-world signed hypergraphs.

4.2 Proposed Measurements of Degree of Balance of Signed Hypergraphs

We begin by giving a brief description of the pairwise measurements. The algebraic conflict (AC) measurement is the smallest eigenvalue of the signed Laplacian. This is proportional to the largest eigenvalue of the adjacency matrix. We compute the normalized algebraic conflict with the spectral radius of the normalized adjacency, $\mathbf{H}$ of a graph $G = (v, E, \mathbf{A})$, where $\mathbf{H} = \Delta^{-1}\mathbf{A}$. The measurement from [5] is adapted and defined in this work as

$$\widetilde{AC}(G) := 1 - \rho(\mathbf{H}).$$

As $\rho(\mathbf{H}) \leq 1, \widetilde{AC}(G) \in [0, 1]$, where $\widetilde{AC}(G) = 0$ indicating that the signed graph is balanced. The frustration index (FI), as defined in [3]

$$FI(G) := \min_{\substack{\mathbf{S}=\text{diag}\{s_1,\dots,s_n\} \\ s_i \in \{-1,1\}\ \forall i}} \mathbf{1}_n^T(|\mathbf{H}| - \mathbf{SHS})\mathbf{1}_n.$$

As shown in Chapter 2, a signed graph G is balanced iff $FI(G) = 0$. Equivalently, G is balanced iff there exists a (diagonal) signature matrix $\mathbf{S} = \text{diag}(s_1, \dots, s_n)$, $s_i \in \{+1, -1\}\ \forall i$, s.t. $\mathbf{SHS} = |\mathbf{H}|$, i.e., the graph G is switching equivalent to a cooperative graph described by the normalized adjacency matrix $|\mathbf{H}|$. We normalize the measurements to $[0, 1]$ by dividing each value by the largest value.

We are now ready to present measurements for the degree of balance of an s-hypergraph. We propose the novel measurement, spectral dissonance. Following this, we explain how we implemented the maximum balance measurement, defined by Shi et al. [6]. We also propose a novel secondary measurement: maximum switching equivalence, which remains a theoretical contribution only.

Spectral Dissonance

In this work, we use the definition of the signed Laplacian for s-hypergraphs introduced by Wang et al. [7], they define it as $\mathbf{L}(H) = \mathbf{M}(H)^T\mathbf{M}(H)$ where $\mathbf{M}$ is the signed incidence matrix of the s-hypergraph, the argument H is omitted when unambiguous.
Our intuition for a first measure of imbalance for s-hypergraphs comes from Corollary 3.2.6. Corollary 3.2.6 states that the spectral radius of $\mathbf{L}(H^+)$ belongs to the spectrum of $\mathbf{L}(H^\sigma)$ iff the s-hypergraph H^σ is balanced, i.e., H^σ is switching equivalent to H^+. We therefore propose using the difference between the largest eigenvalues of $\mathbf{L}(H^+)$ and $\mathbf{L}(H^\sigma)$ as a measure of the degree of balance. Indeed, $\lambda_n(\mathbf{L}(H^+))$ is larger than $\lambda_n(\mathbf{L}(H^\sigma))$ whenever H^σ is unbalanced, while the equality holds when H^σ is balanced. Hence, the difference $\lambda_n(\mathbf{L}(H^+)) - \lambda_n(\mathbf{L}(H^\sigma))$ is always non-negative, thus well-defined.

The all-positive Laplacian, $\mathbf{L}(H^+)$, is irreducible when H is connected. By the Perron-Frobenius Theorem (see Theorem 2.12 in [8]), the eigenvectors of an irreducible matrix have positive entries; in other words, H^+ is balanced and all vertices in the bipartition vector are in the same partition. If $\lambda_n(\mathbf{L}(H^+) = \lambda_n(\mathbf{L}(H^\sigma))$, H^σ is balanced. However, note that not all elements in the bipartition vector corresponding to $\lambda_n(\mathbf{L}(H^\sigma))$ are necessarily positive, i.e., not all vertices are in the same partition, however, definition of balance, determined by Equation (4) in Chapter 3.2 is true for all hyperedges in the s-hypergraph. By the Rayleigh Theorem for eigenvalues (see Theorem 4.2.2 in [9]), $\lambda_n(\mathbf{L}(H^+))$ and $\lambda_n(\mathbf{L}(H^\sigma))$ can be found through optimization as follows.

The largest eigenvalue of $\mathbf{L}(H^\sigma) = \mathbf{M}^T\mathbf{M}$ is given by

$$\max_{x\in\mathbb{R}^{|V|},||x||_2=1} x^T(\mathbf{M}^T\mathbf{M})x = \max_{x\in\mathbb{R}^{|V|},||x||_2=1} ||\mathbf{M}x||_2^2,$$

where,

$$||\mathbf{M}x||_2^2 = \sum_{i=1}^{|E|}(m_{i1}x_1 + m_{i2}x_1 + ... + m_{i|V|}x_{|V|})^2 =$$

$$\sum_{i=1}^{|V|}\sum_{j=1}^{|E|} m_{ij}^2x_j^2 + 2\sum_{i=1}^{|E|}\sum_{1\le j<k\le|V|} m_{ij}m_{ik}x_jx_k.$$

The first sum is invariant under the signs of M and x; the second is not. Hence, the difference in the second sum of $||\mathbf{M}(H^+)||$ and the second sum of $||\mathbf{M}(H^\sigma)||$ is enough to compute $\lambda_n(\mathbf{L}(H^+)) - \lambda_n(\mathbf{L}(H^\sigma))$.

What follows are two assumptions we make related to the novel measurement.

Assumption 4.2.1 *We assume that, for all structurally balanced, connected H^σ, the eigenvector corresponding to $\lambda_n(\mathbf{L}(H^\sigma))$ contains no zero entries, i.e., the signature of the eigenvector is a valid bipartition vector when H^σ is balanced.*

Assumption 4.2.2 *We assume that, for all connected H^σ,$\lambda_n(\mathbf{L}(H^\sigma)) \leq \lambda_n(\mathbf{L}(H^+))$. Furthermore, if H^σ is unbalanced and connected, we assume that $\lambda_n(\mathbf{L}(H^\sigma)) < \lambda_n(\mathbf{L}(H^+))$.*

We are now ready to propose the spectral dissonance measurement.

Definition 4.2.1 *Let H^σ be an oriented hypergraph. The spectral dissonance (SP) is defined as the difference between $\lambda_n(\mathbf{L}(H^+))$ and $\lambda_n(\mathbf{L}(H^\sigma))$. Moreover, let SP($H^\sigma$) and SP($H^\tau$) be the SP of the s-hypergraphs H^σ and H^τ, respectively. We say that H^σ is more balanced than H^τ if SP(H^σ) < SP(H^τ).*

Observe that $\text{SP}(H^\sigma) = 0$ implies that the s-hypergraph is balanced while $\text{SP}(H^\sigma) > 0$ means that H^σ is not balanced.

Example 4.2.1 *Let H^σ be the s-hypergraph depicted in Figure 4.1b. Its incidence matrix is*

$$\mathbf{M}(H^\sigma) = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

H^σ does not contain a negative cycle, hence it is balanced; it is switching equivalent to $\mathbf{M}(H^+)$, with H^+ illustrated in Figure 4.1a. Moreover, $\lambda_n(\mathbf{L}(H^\sigma)) = \lambda_n(\mathbf{L}(H^+)) \approx 6.12$.
To find the bipartition associated with the balance property (see Theorem 3.2.1), we can look at the signature (the signs) of the eigenvector corresponding to $\lambda_n(\mathbf{L}(H^\sigma))$. Its signs are $[-1, 1, 1, 1]$. The signs of the eigenvector corresponding to $\lambda_n(\mathbf{L}(H^+))$ are all positive, i.e., all nodes are in the same partition, as expected. Moreover,

$$||\mathbf{M}(H^+)x||^2 = (x_1 + x_2 + x_3)^2 + (x_2 + x_3 + x_4)^2 + (x_2 + x_4)^2,$$

$$||\mathbf{M}(H^\sigma)x||^2 = (-x_1 + x_2 + x_3)^2 + (x_2 + x_3 + x_4)^2 + (x_2 + x_4)^2,$$

which gives,

$$\max_{x\in\mathbb{R}^4,\|x\|_2=1} \|\mathbf{M}(H^\sigma)x\|_2^2 \;=\; \max_{x\in\mathbb{R}^4,\|x\|_2=1} \|\mathbf{M}(H^+)x\|_2^2.$$

By Raylight's Theorem for eigenvalues (see Theorem 4.2.2 in [9]),

$$\lambda_n(\mathbf{L}(H^\sigma)) = \lambda_n(\mathbf{L}(H^+)).$$

Thus, H^σ is balanced.

Example 4.2.2 *Let H^τ be the s-hypergraph shown in Figure 4.1c. Its incidence matrix is*

$$\mathbf{M}(H^\tau) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

*H^τ **does** contain a negative cycle, hence it is not balanced. $\lambda_n(\mathbf{L}(H^\tau)) \approx 4.80$, the signature of the corresponding eigenvector corresponding to its largest eigenvalue is $[1,1,-1,1]$. Moreover,*

$$\|\mathbf{M}(H^\tau)x\|^2 = (x_1+x_2+x_3)^2 + (x_2-x_3+x_4)^2 + (x_2+x_4)^2.$$

The constrained maximum of the quadratic form is lower than that of the cooperative s-hypergraph, H^+.

$$\max_{x\in\mathbb{R}^4,\|x\|_2=1} \|\mathbf{M}(H^\tau)x\|_2^2 \;<\; \max_{x\in\mathbb{R}^4,\|x\|_2=1} \|\mathbf{M}(H^+)x\|_2^2.$$

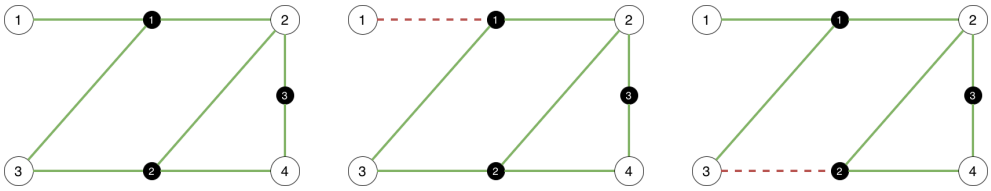
And so,

$$\lambda_n(\mathbf{L}(H^\tau)) < \lambda_n(\mathbf{L}(H^+)).$$

H^τ is not balanced.

Maximum Balance

Maximum balance (MB) is defined, and proven to be NP-complete, by Shi et al. in [6] (see Definition 3.2.1). We implement MB by removing all sets of edges, E', of size $k, k \in \{0\} \cup [|E|]$ and checking if the signed subhypergraph $H(E-E')$ is a balanced s-hypergraph. Thus, computing the MB of a s-hypergraph $H = (V,E)$ can be done by checking the balance in all signed subhypergraphs $H(E-E')$. The decision problem of checking the balance of



(a) H^+ (b) H^σ (c) H^τ

Figure 4.1: Three orientations of the same hypergraph, the all positive orientation H^+, a balanced orientation H^σ, and an unbalanced orientation H^τ.

a signed subhypergraph is linear, i.e., in $O(|V| + |E - E'|)$ (see Remark 3.2.1). However, checking the balance of all possible signed subhypergraphs is NP-complete.
What follows is a description of the algorithm we wrote for checking the balance of a given s-hypergraph. The algorithm comprises two steps. First, we compute the signed pairwise adjacencies within each hyperedge and save the results in a directory (*signed_adj*) (see Algorithm 1). Following this, we check the balance of the s-hypergraph by performing a labeled Depth First Search (DFS), an algorithm that attempts to bipartition all nodes in V into two disjoint subsets such that Equation (4) in Chapter 3.2 holds for all hyperedges. The DFS algorithm (see Algorithm 2) assigns the first vertex to be in group 0. All directly adjacent nodes are added to group 0 if they have the same sign as the first vertex; others are added to group 1. The algorithm repeats the process in the directly adjacent nodes. If an inconsistency is found, i.e., if a vertex is already assigned to one group but the current state assigns it to the opposite group, a negative cycle has been found, and the s-hypergraph cannot be bipartitioned. Hence, it is unbalanced.

It is important to emphasize that when computing MB, the signed subhypergraph $H(E - E')$ might be unconnected. As mentioned, the definition of MB (see Definition 3.2.1) allows for unconnected s-hypergraphs. Algorithm 2 does not work for unconnected hypergraphs. To make it work on unconnected s-hypergraphs, we run the algorithm with all nodes as the starting node, i.e., the algorithm is executed in a loop: $\forall\, v \in V$, start $:= v$.

Maximum Switching Equivalence

Similar to how the frustration index is computed in Fontan's work [3] (see Chapter 2.3), we propose that the degree of balance in s-hypergraphs can be found through brute-force optimization on Wang et al.'s Theorem 2.2 in [7].

Algorithm 1 Pairwise adjacencies.

Require: Signed incidence matrix $\Psi \in \mathbb{R}^{n \times m}$

return *signed_adj*
$n, m \leftarrow \text{shape}(\Psi)$
signed_adj $\leftarrow$ empty map from node to list
for $e \leftarrow 0$ to $m - 1$ **do**
    pairwise_adjacencies $\leftarrow \{v \mid \Psi[v, e] \neq 0\}$
    for u,v combinations of *pairwise_adjacencies* **do**
        sign $\leftarrow \Psi[u, e] \cdot \Psi[v, e]$
        append $(v, sign)$ to *signed_adj*$[u]$
        append $(u, sign)$ to *signed_adj*$[v]$

Algorithm 2 Balance check (DFS with binary labeling).

Require: *signed_adj* (Node to vector mapping), *start* (The starting vertex)
Ensure: **True** if the hypergraph is not balanced; otherwise **False**.

partitioning $\leftarrow$ empty map
stack $\leftarrow [(start, 0)]$
while *stack* is not empty **do**
    $(node, group) \leftarrow \text{Pop}(stack)$
    if *node* $\in$ *partitioning* **then**
        if *partitioning*$[node] \neq group$ **then**
            return True
        else
            continue
    partitioning$[node] \leftarrow group$
    for all $(neighbour, sign \in signed_adjacencies[node])$ **do**
        if $sign = 1$ **then**
            $next_group \leftarrow group$
        else
            $next_group \leftarrow 1 - group$
        $\text{Push}(stack, (neighbour, next_group))$
return False

We define the Maximum Switching Equivalence (MSE) as

$$\mathrm{MSE}(H^\sigma) = \min_{\substack{\mathbf{S}_F,\mathbf{S}_U=\mathrm{diag}\{s_1,\dots,s_n\} \\ s_i\in\{-1,1\}\ \forall i}} \frac{1_n^T(\mathbf{M}(H^+) - \mathbf{S}_F\mathbf{M}(H^\sigma)\mathbf{S}_U)1_n}{1_n^T\mathbf{M}(H^+)1_n}.$$

The algorithm for the measurement runs in $O(2^{|E|+|V|})$. As the runtime grows rapidly as we increase the size of the s-hypergraphs, we did not implement MSE in software; it remains a theoretical contribution only.

Normalization of SP and MB

To allow for comparisons, all proposed s-hypergraph measurements are normalized to be within $[0,1]$, where a score of zero means the graph is balanced. The normalized hypergraph measurements are defined as

$$\widetilde{SP}(H^\sigma) := 1 - \frac{\lambda_n(\mathbf{L}(H^\sigma))}{\lambda_n(\mathbf{L}(H^+))},$$

$$\widetilde{MB}(H^\sigma) := \frac{MB(H^\sigma)}{|E|}.$$

Remark 4.2.1 *Subtracting the quotient, $\frac{\lambda_n(\mathbf{L}(H^\sigma))}{\lambda_n(\mathbf{L}(H^+))}$ from 1 will give a value in [0,1], however, numerical simulations showed that the score is never equal to 1, as the denominator is never infinitely larger than the numerator, more probably, the numerator is never significantly smaller than the denominator. For this reason, a scaling factor for the quotient may be considered; however, determining this scaling factor is beyond the scope of this work.*

Validity and Reliability of the Measurements

The MB score is implemented by deleting a set of k, $k \in [0,|E|]$, edges and testing the balance with labeled DFS, see Algorithms 1 and 2. Deleting edges can break the hypergraph's connectivity. Note that Shi et al.'s [6] definition of balance allows for unconnected hypergraphs; the DFS algorithm works on unconnected hypergraphs.
SP is derived from Wang et al.'s work [7] (see Section 3.2). The measurements are hence only defined on connected hypergraphs. SP is not applied to unconnected hypergraphs. The reliability of SP depends on the accuracy of approximating eigenvalues, which is done with Numpy's *linalg.eigvals*. Hence, its reliability depends on the NumPy library's implementation [16].

4.3 Data Generation and Manipulation

Synthetic Graph Generation

The following paragraph outlines the constraints imposed on the graph generation. First, while no definition in the relevant literature explicitly prohibits an s-hypergraph from being a multigraph, i.e., a graph having at least two identical hyperedges, no hypergraph used in this thesis is a multigraph. We ensure that, for any given synthetic hypergraph, no two hyperedges contain exactly the same vertices. Second, the random pairwise graphs in this thesis followed the Erdos-Renyi model [17], in which the probability of any two vertices sharing an edge is dependent on a random variable. The random hypergraphs produced in this work followed a similar random model.

Pairwise graphs The pairwise signed graph generation program is listed in Algorithm 3. Note that $\mathbf{A}$ is the adjacency matrix of a random signed graph, $\mathcal{G}$ is a set of adjacency matrices. The edges in the Erdos-Renyi graphs have a given probability of existing, p_r, and of being antagonistic, α. $\odot$ symbolizes element-wise multiplication.

Algorithm 3 Signed Erdos Renyi Graphs.

$\mathcal{G} \leftarrow [\,]$
for $\alpha \in \{0, 0.05, 0.10, \ldots, 1.00\}$ **do**
  repeat
    Sample $\mathbf{A}$ s.t., for $i \in [n], j \in \{k, ..., n\}, k = 2, 3, ..., n$,

$$\mathbb{P}[(\mathbf{A})_{ij} = 1] = p_r, \qquad \mathbb{P}[(\mathbf{A})_{ij} = 0] = 1 - p_r$$

    Sample $\mathbf{B}$ s.t., for $i \in [n], j \in \{k, ..., n\}, k = 2, 3, ..., n$,

$$\mathbb{P}[(\mathbf{B})_{ij} = -1] = \alpha, \qquad \mathbb{P}[(\mathbf{B})_{ij} = 1] = 1 - \alpha$$

    $\mathbf{A} \leftarrow \mathbf{A} \odot \mathbf{B}$
    $\mathbf{A} \leftarrow \mathbf{A} + \mathbf{A}^\top$
  until $|\mathbf{A}|$ is irreducible
  Append $\mathbf{A}$ to $\mathcal{G}$
return $\mathcal{G}$

Hypergraphs The generation of synthetic s-hypergraphs is straightforward by the use of ranking/unranking* algorithms, standard algorithms from the literature of algorithmic combinatorics; see, e.g., [18]. In combinatorics, a rank corresponds to a unique sequence of ordered numbers. Given the non-negative numbers n and k such that $k \leq n$, we know that there exists $\binom{n}{k}$ sets of cardinality k of n unique elements. A rank is the unique identifier of a given set of cardinality k from n unique elements.

For instance, the ranks of combinations with 3 elements taken from 5 elements are

$$\begin{aligned}
\text{rank } 0 &\rightarrow [0,1,2],\\
\text{rank } 1 &\rightarrow [0,1,3],\\
\text{rank } 2 &\rightarrow [0,1,4],\\
\text{rank } 3 &\rightarrow [0,2,3],\\
\text{rank } 4 &\rightarrow [0,2,4],\\
\text{rank } 5 &\rightarrow [0,3,4],\\
\text{rank } 6 &\rightarrow [1,2,3],\\
\text{rank } 7 &\rightarrow [1,2,4],\\
\text{rank } 8 &\rightarrow [1,3,4],\\
\text{rank } 9 &\rightarrow [2,3,4].
\end{aligned}$$

In the context of hypergraphs, if $|V| = 5$, there exist 10 unique hyperedges with 3 nodes. Hence, generating the hyperedges for the s-hypergraphs is as simple as generating unique random numbers. What follows is a brief explanation of the unranking algorithm used (see Algorithm 4) in the thesis work.

Given rank r, the algorithm starts by checking if $v_0 = 0$ is in the unique combination mapped to r by computing q. If the rank is low enough to contain 0, it is added to the set e, and v_0 is incremented by one. If the rank is higher, i.e., 0 is not in the combination, v is continuously incremented by one until the inequality is true, and a number larger than zero is added to e. The outer loop continues until k elements have been assigned to e, and the function returns the hyperedge in the form of a vector. Naturally, the hyperedges are all cooperative at this stage; they are signed by randomly mapping the non-zero elements of the vectors to $\{-1, 1\}$; the probability of being mapped to -1 is determined by the given antagonism. To this end, the multigraph constraint mentioned at the beginning of this section is handled deterministically.

*Note that the concept *rank* in combinatorics is different from the concept *rank* in linear algebra.

Algorithm 4 Unrank

Require: Number of vertices n, edge cardinality k, rank r.

$e \leftarrow [\,]$
$v_0 \leftarrow 0$
for $s = k, k-1, \dots, 1$ **do**
    for $v = v_0, v_0+1, \dots, n-1$ **do**
        $q \leftarrow \binom{n-v-1}{s-1}$
        if $r < q$ **then**
            Append v to e
            $v_0 \leftarrow v+1$
            break
        else
            $r \leftarrow r-q$
return e

Additionally, Wang et al.'s Corollary (see Corollary 3.2.6) relies on connected s-hypergraphs (see Section 3.2), meaning that the synthetic hypergraphs must be connected. This constraint is not dealt with deterministically, but heuristically, as in the pairwise case, by checking the irreducibility of the $\widetilde{A}_I$ matrix, the matrix is defined in Chapter 3.1.

Properties of the Random Graphs

As seen in Algorithm 3 and mentioned in relation to the synthetic s-hypergraph procedure, the assigned level of antagonism is probabilistic. Increasing the level of antagonism in the generation algorithms increases the probability of negative edges/incidences; however, the programs do not guarantee a specified ratio of antagonism for any given graph. To make the analysis possible, we defined three properties, Negative Edge Ratio (NER), Negative Incidence Ratio (NIR), and Negative Incidence Product Ratio (NIPR). The properties communicate the actual ratio of antagonism in the graphs.

NER is defined for edges, while NIR and NIPR are defined for incidence; thus, the former is used for pairwise graphs, and the latter ones are used for s-hypergraphs.

NER and NIR are defined, respectively, as

$$\mathrm{NER}(\mathbf{A}) = \frac{\sum_{(i,j)\in[|V|]^2}[(\mathbf{A})_{ij} = -1]}{\sum_{(i,j)\in[|V|]^2}|(\mathbf{A})_{ij}|},$$

$$\mathrm{NIR}(\Psi) = \frac{\sum_{(i,j)\in[|V|]\times[|E|]}[(\Psi)_{ij} = -1]}{\sum_{(i,j)\in[|V|]\times[|E|]}|(\Psi)_{ij}|}.$$

Both NER and NIR counts the number of negative entries in the matrices and divides by the number of non-zero entries.

NIPR is defined

$$\mathrm{NIPR}(\Psi) = \sum_{i=1}^{|E|} \frac{n(i)p(i)}{n(i)p(i) + \binom{n(i)}{2} + \binom{p(i)}{2}},$$

where $p(i) := \sum_{j=1}^{|V|}[(\Psi)_{ji} = 1]$, $n(i) := \sum_{j=1}^{|V|}[(\Psi)_{ji} = -1]$. The numerator is the number of negative products in the edge, and the denominator is the total number of products. Indeed, the number of pairs of incidences in an edge where the products is negative is the number of negative incidences times the positive ones, $n(i)p(i)$; and the number of pairs where the product is positive is $\binom{n(i)}{2} + \binom{p(i)}{2}$.

Validity and reliability of the random graphs

Most measurements are defined only on connected graphs; all generated graphs are connected. Furthermore, all elements in the generated matrices must be in $\{-1, 0, 1\}$ and the adjacency of the pairwise graphs must be symmetrical.
Furthermore, the level of balance in the random graphs cannot be explicitly set by any argument. Both the structure and the signs of the graph determine its level of unbalance; e.g., acyclic graphs are always balanced regardless of the signs of their adjacencies/incidences, i.e., the level of antagonism. In other words, neither the level of unbalance nor the exact ratio of antagonism is explicitly set as an argument to the graph-generating functions, which is why we defined NIR, NER, and NIPR.

Real-World Hypergraph Manipulation

The hypergraph data used in this thesis is taken from the AHORN database. Three datasets were used: *House Committees* [19, 20], *Senate Committees* [19, 20], and *Senate Bills* [19, 21, 22]. As no publicly available signed hypergraphs representing real-world systems were found during this project, cooperative hypergraphs were manipulated as follows.
All vertices in the hypergraphs are either Democrats or Republicans; the

hyperedges are groups, e.g., committees, of politicians. The data is unsigned; an antagonism is introduced via a bijection from sets of numbers to vectors, $f : \{n_1, n_2, ..., n_k\} \rightarrow \{-1, 0, 1\}^{|V|}$. If the number of Democrats in the set is greater than the number of Republicans, the Democrats are cooperative in the hyperedge, and the Republicans are antagonistic, i.e., Democrats are coded 1 and Republicans -1, politicians not in the set are coded 0. The same applies in the reversed case. If the number of Democrats equals the number of Republicans, all politicians in the set are coded 1.

4.4 Experimental Design

The synthetic graphs were generated on Dardel, KTH's supercomputer. The measurements for both the synthetic and real-world graphs were also executed on the same machine. As the real-world hypergraphs were large, MB (see Definition 3.2.1) was not measured on them. The size of $|V|$ for the random graphs (both pairwise and hypergraphs) was in $[12,\ 22]$. For a connected pairwise graph, the lowest and largest number of $|E|$ was in the range of 11 to 231*, inclusive. The theoretical upper limit of $|E|$ for connected hypergraphs is large; the lower limit is 1, i.e., an edge covering all vertices. The range used was $[2, 27]$; the upper limit scales with $|V|$, and no graphs with $|E| = 1$ and $|e_1| = |V|$ were allowed as the balance measurements of such graphs are trivial. The function used to determine the upper limit was chosen heuristically based on runtime; larger values resulted in prohibitively long execution times.
The synthetic graph generation and measurement were executed on KTH's supercomputer Dardel, while the data analysis was on an M5 Apple MacBook Air. All relevant software releases and hardware used in the experiment are listed in Table A.1 in Appendix A.

Reliability of Experiments

Larger graphs and hypergraphs with $|V| > 22$, $|E| \geq 28$, are not synthesized. Hence, the sample size is constrained to smaller graphs and hypergraphs. Furthermore, to execute these measurements efficiently, all program generation or measurement was limited to a given execution time of 5 minutes.

Naturally, this can introduce bias into the data. S-hypergraphs with a large number of edges and a large degree of structural unbalance would be excluded

*$|E| \in \left[|V| - 1,\ \frac{|V|(|V|-1)}{2}\right]$.

as the MB computations exceed the allowed duration. However, in practice, there were only a few time-outs.

Chapter 5

Results and Discussion

This chapter presents the results of the thesis work. Section 5.1 presents the theoretical results, and Section 5.2 presents the measurements on synthetic graphs and real-world hypergraphs.

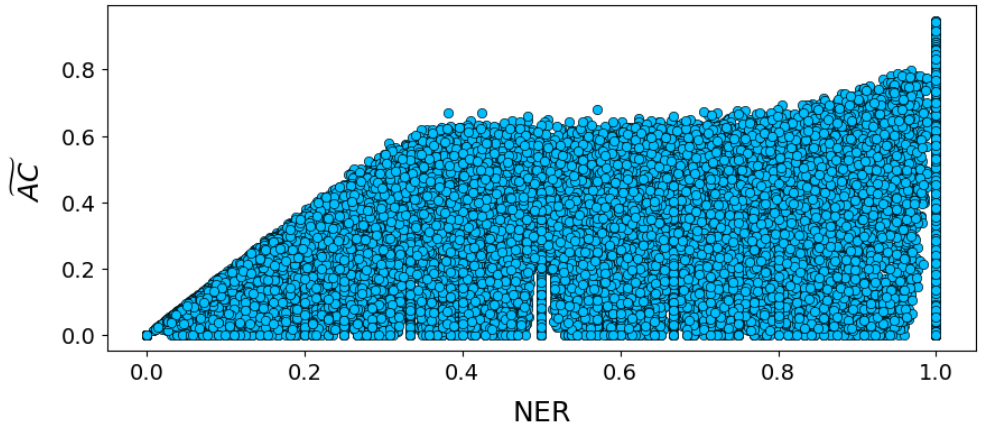
5.1 Fundamental Results

The first objective of this thesis was to adapt measures of the degree of structural balance from signed graphs to s-hypergraphs and introduce a novel measurement. The measurement, spectral dissonance (SP), see Definition 4.2.1, was adapted from algebraic conflict, defined in [5], and from Corollary 3.2.6, defined in [7].

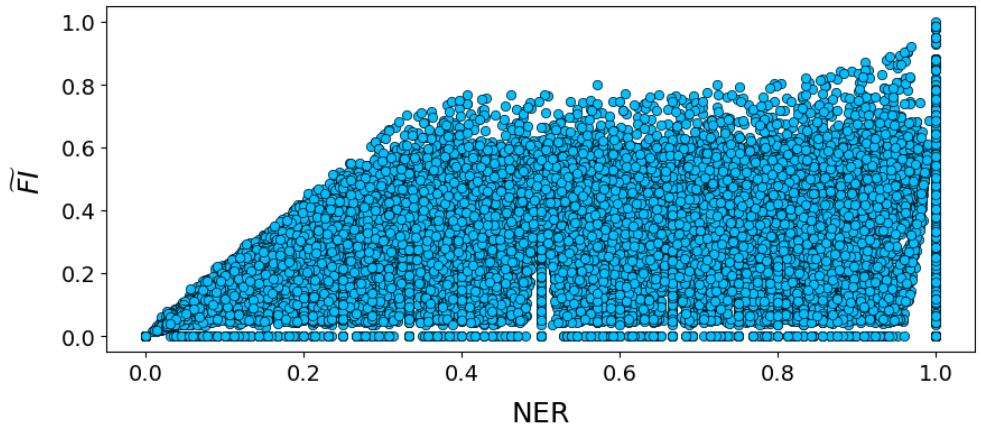
5.2 Data Analysis Results and Discussion

Synthetic Pairwise Graphs

A total of 24838 pairwise graphs were generated and measured. Figure 5.1 shows the pairwise measurements scatter plotted with NER, the ratio of negative edges, see Chapter 4.2. Each dot represents the SB degree of a graph. The horizontal axes represent the NER: NER $= 0$ means no antagonism is present; NER $= 1$ implies that all edges are antagonistic. The vertical axes are the degree of structural unbalance. In Figure 5.1a, the vertical axis is the normalized algebraic conflict ($\widetilde{AC}$); $\widetilde{AC} = 0$ implies structural balance, $\widetilde{AC} > 0$ means some degree of unbalance. Likewise, the vertical axis in the



(a) $\widetilde{AC}$



(b) $\widetilde{FI}$

Figure 5.1: Measurements on synthetic pairwise graphs. Scatter plots of $\widetilde{AC}$ and NER; $\widetilde{FI}$ and NER.

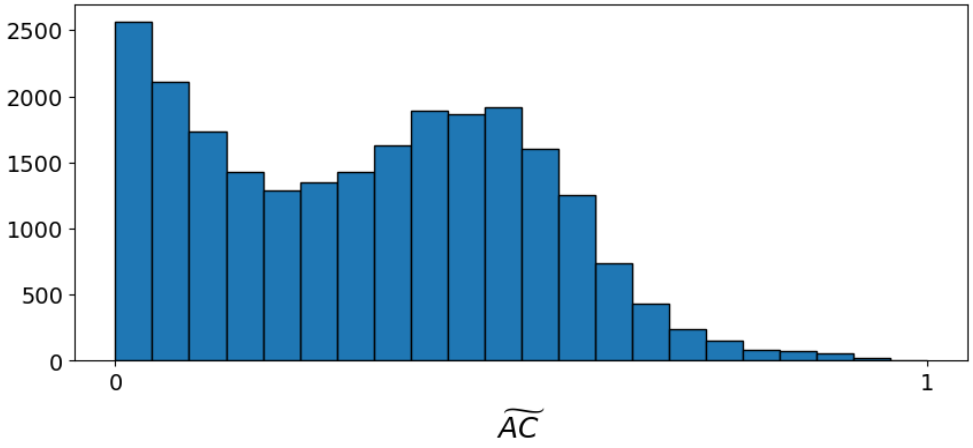
Table 5.1: Correlation table of pairwise measurements and properties.

	$\widetilde{AC}$	$\widetilde{FI}$	NER
$\widetilde{AC}$	100%	94%	47%
$\widetilde{FI}$		100%	47%
NER			100%

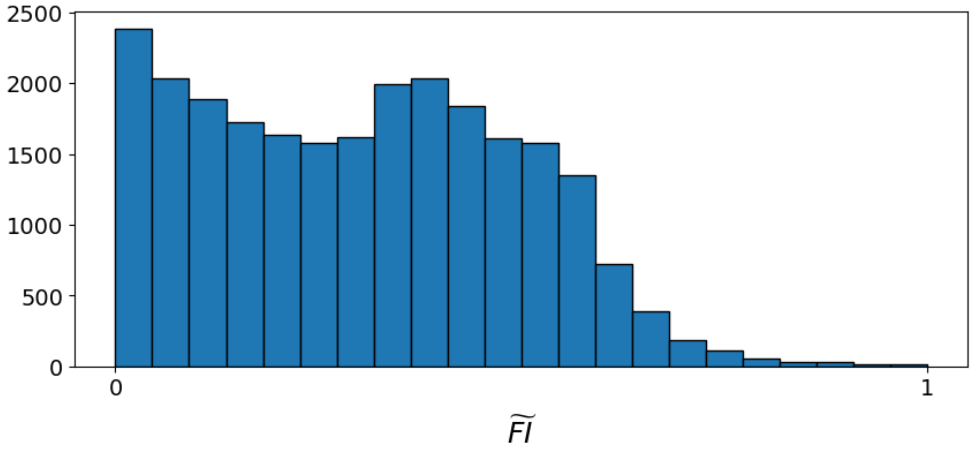
scatter plot in Figure 5.1b shows the normalized frustration index ($\widetilde{FI}$), $\widetilde{FI} = 0$ means the graph is SB, $\widetilde{FI}$ being larger than zero implies a degree of SB.
Figure 5.2 shows histograms of the measurements, both using 22 bins. Table 5.1 shows the correlations between the measurements and properties of the graphs. The strong correlation between the algebraic conflict and the frustration index observed in Table 5.1 for Erdos-Renyi graphs suggests that the former can approximate the latter. Of the two, the frustration index is the preferred measure (see [13]), whereas the algebraic conflict is easier to compute. The strong correlation suggests that the latter can be used to approximate the former.
Notice in Figure 5.1 that the shape of the scatter plots follow a similar pattern, only at different scale; the same similarity can be seen in the histograms in Figure 5.2, again suggesting high correlation between the measurements.
With a more detailed look at Figure 5.1, and essentially apparent in Figure 5.1b, the reader will find a horizontal line of datapoints at $\widetilde{FI} = 0$. This follows from the fact that certain graphs are always balanced: acyclic graphs and graphs without negative cycles. This includes all negative graphs without cycles of odd length. Likewise, note that vertical axes at NER$= 1$, i.e., samples where all edges are antagonistic; this concentration suggests that the signs of the edges are not the property affecting the degree of SB.
Notice also the absence of plots at NER $= \frac{19}{20}$ in the Figures. This indicates that no balanced graphs were observed when almost all edges were negative. However, if we again notice the vertical line at NER$= 1$, we see that the samples cover the entire spectrum; balanced graphs do exist when all edges are negative.

Synthetic Hypergraphs

A total of 23708 synthetic hypergraphs were generated and measured. Figure 5.3a shows the distribution of the average number of vertices per hyperedge, and Figure 5.3b shows the distribution of $|E|$.

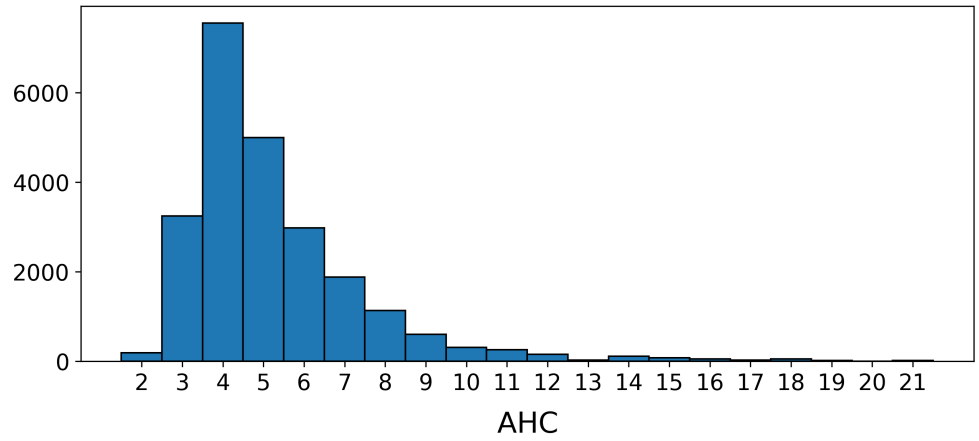


(a) $\widetilde{AC}$

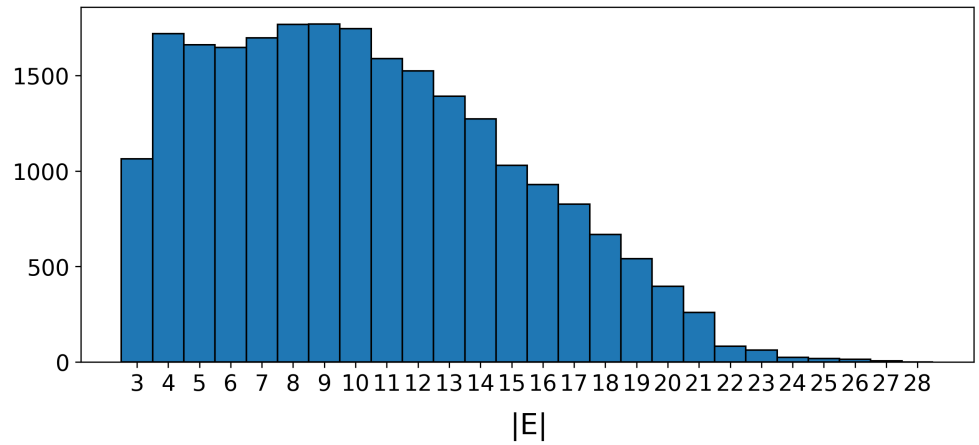


(b) $\widetilde{FI}$

Figure 5.2: Histograms of distribution of $\widetilde{AC}$ and $\widetilde{FI}$ on synthetic pairwise graphs.



(a) Ave. Hyperedge Cardinality



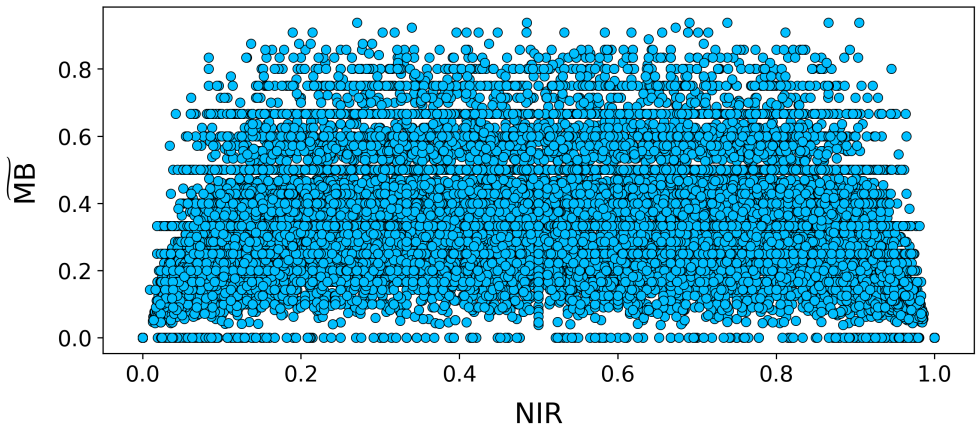
(b) $|E|$

Figure 5.3: Distribution of average hyperedge cardinality of the hypergraphs; Distribution of $|E|$.

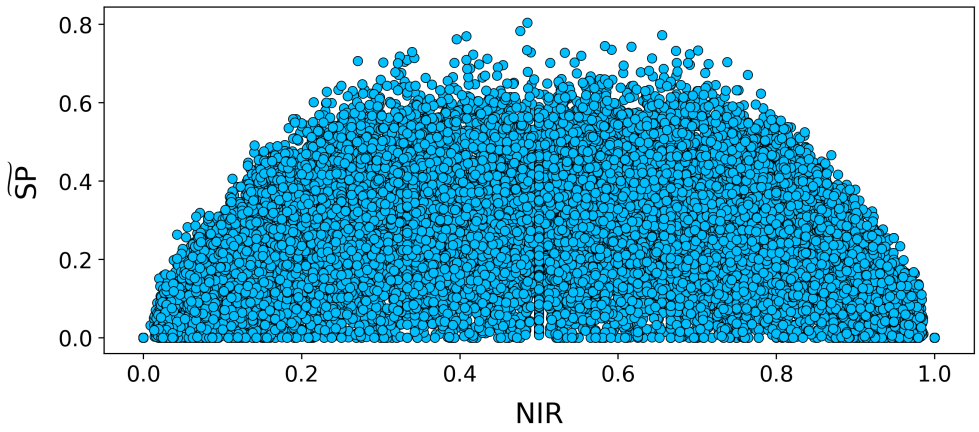
Table 5.2: Correlation table of hypergraph measurements and properties.

	$\widetilde{\text{SP}}$	$\widetilde{\text{MB}}$	MB	NIPR	NIR
$\widetilde{\text{SP}}$	100%	83%	73%	62%	0.4%
$\widetilde{\text{MB}}$		100%	67%	57%	0.03%
MB			100%	47%	0.4%
NIPR				100%	0.6%
NIR					100%

As shown in Table 5.2, the spectral measurement closely approximates the NP-complete measurement for random graphs. Furthermore, notice how in Figure 5.4, each distribution of the plots forms a concave structure with its peak in the middle. This is a reflection of Shi et al.'s definition of balance (see also Section 3.2). The horizontal axis in the scatter plots captures an increase in total antagonism; the definition of balance for oriented hypergraphs does not depend on sign alone but rather on whether two vertices have different signs. This explains why we observe the peak in the middle, as this is when most graphs have the greatest number of different signs, i.e., when the distributions of positive and negative incidences are equal.
Figure 5.5 shows the same measurements scattered with NIPR, the ratio of different signs over all signs. The levels of antagonism peak at NIPR = 0.5, when most graphs will have the greatest number of different signs.

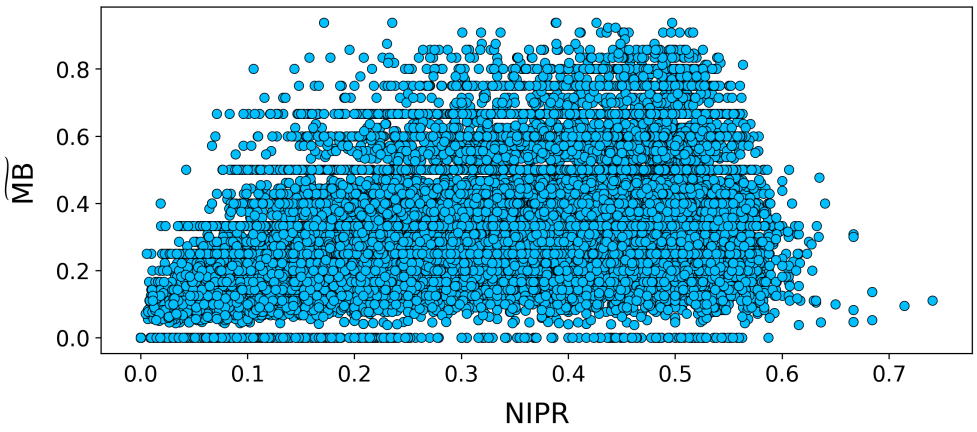


(a) $\widetilde{MB}$

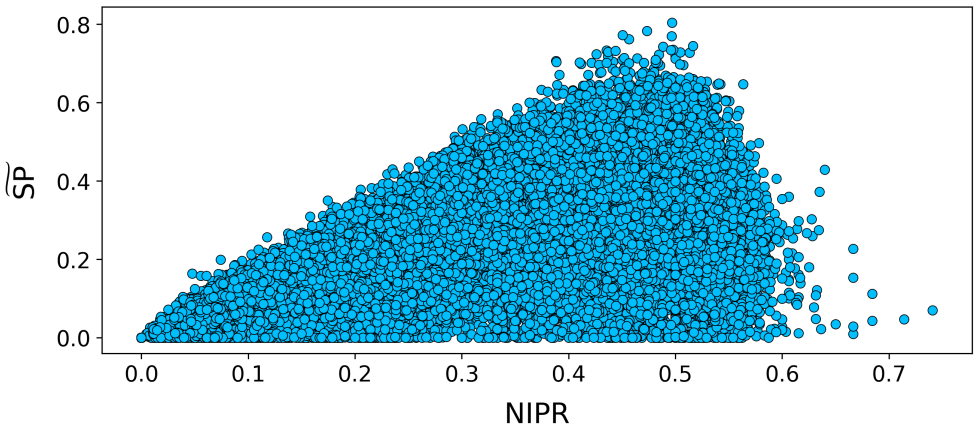


(b) $\widetilde{SP}$

Figure 5.4: Measurements on synthetic hypergraphs. (a) Scatter plots of $\widetilde{MB}$ and NIR, (b) $\widetilde{SP}$ and NIR.



(a) $\widetilde{MB}$



(b) $\widetilde{SP}$

Figure 5.5: Measurements on synthetic hypergraphs. (a) Scatter plots of $\widetilde{MB}$ and NIPR, (b) $\widetilde{SP}$ and NIPR.

Real-world Hypergraphs

The properties and measurements of the real-world hypergraphs are listed in Table 5.3; all hypergraphs are connected, and they are not structurally balanced.

Figure 5.6 shows a graphical representation of a real-world hypergraph, the bipartite representation, also called the facet representation [1]. The data represent *Senate Bills*, and only 2 of the 29157 hyperedges are shown. The larger blue nodes are hyperedges, i.e., bills, and the smaller black nodes are politicians. The large orange node amongst the politicians represents Hillary Clinton.

The reader might conclude that the low SP values in Table 5.3 indicate that the real-world s-hypergraphs are almost structurally balanced. However, the measurement is influenced by properties of the s-hypergraph that are not considered in this work (compare Remark 4.2.1), e.g., how the size of the s-hypergraph affects the measure. Thus, the percentages in Table 5.3 suggest that *Senate Committees* are more structurally unbalanced than *House Committees*; however, since the s-hypergraphs are very different in size, the measure alone is not enough to determine that this is the case.
The high correlation between SP and MB (compare Table 5.2) suggests that *Senate Committees* would need to remove more committees to achieve balance, compared to *House Committees*. However, the intuition behind SP was based on synthetic s-hypergraphs with an Erdos-Renyi-like structure that are much smaller than real-world s-hypergraphs, making it difficult to generalize the measurement to the real world.

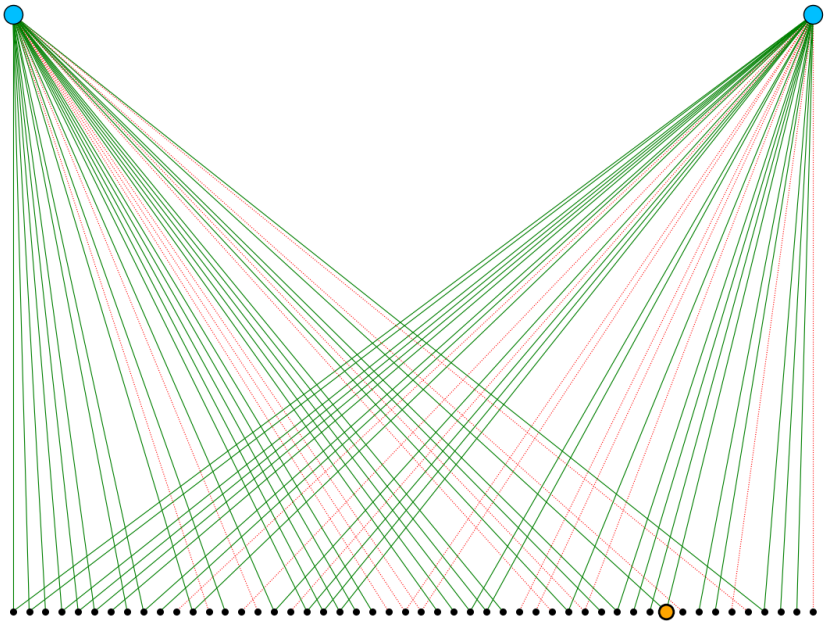


Figure 5.6: Facet representation of a subgraph of the Senate Committees hypergraph. Red, dotted lines represent antagonistic incidences; green, solid lines represent cooperative ones.

Table 5.3: Properties of US data. $|R|$ is the number of Republicans, likewise $|D|$ for Democrats. AHC stands for Average Hyperedge Cardinality, i.e., the average number of elected officials per committee/bill.

Dataset	|V|	|E|	|R|	|D|	$\widetilde{SP}$	AHC
House Committees	1290	341	670	620	1.9%	35
Senate Bills	294	29157	144	150	3.2%	8
Senate Committees	282	315	142	140	6.6%	17

Chapter 6

Conclusions and Future Work

6.1 Conclusions

The goal of the thesis work was to investigate the role of antagonism in hypergraphs and to adapt a measure of the degree of structural balance on signed pairwise graphs to signed hypergraphs. Early on, we isolated the frustration index and the algebraic conflict as measurements that could likely be generalized. We later discovered that the frustration index had a hypergraph counterpart: maximum balance, defined by Shi et al. in [6].
Building on the seminal work by Wang et al. [7], this thesis proposes a novel measure, spectral dissonance. It is a generalization of the algebraic conflict, defined by Kunegis in [5]. The results of this thesis showed that for Erdos-Renyi graphs, the frustration index and the algebraic conflict measure are highly correlated. Likewise, we found that the proposed measure is highly correlated to the frustration index's hypergraph analog, the maximum balance.

6.2 Ethical and Sustainability Aspects

The real-world data used in this study were taken from three sources. The *Senate Bills* dataset [19, 22, 21] were derived from James Fowler's work [22, 21]. The *House Committees* [19, 20] and *Senate Committees* [19, 20] datasets were derived from the work of Charles Stewart and Jonathan Woon [20]. All datasets are open access; the data in them are assumed to have been lawfully obtained from public sources.
To introduce antagonism in the hypergraphs, the data was manipulated. The introduced antagonism in the hypergraphs is in no way a reflection of how the politicians are connected to the bills and committees. Furthermore, the

manipulation did not add or remove any edges or nodes in the hypergraphs; the only change was the addition of signs to the hyperedges. The assignment of signs was explained in Chapter 4.3.
This thesis raises no sustainability concerns.

6.3 Limitations

This project used only a single graph topology, the Erdos-Renyi model. More realistic ones, such as small-world graphs [23], could be studied in future work. Furthermore, the synthetic graphs were much smaller than the real-world graphs used, which makes comparing the findings difficult. Finally, a comprehensive theoretical treatment of the proposed measure remains a topic for future work, i.e., proving or disproving the assumptions of the new measurements.

6.4 Future Work

Based on extensive numerical analysis, we assumed (see Assumption 4.2.2) the squared Euclidean norm of $\mathbf{M}(H^+)x$ is larger than the same norm of $\mathbf{M}(H^\sigma)x$ when H^σ is structurally unbalanced. We leave the proof of this as future work.
Furthermore, we assumed that the eigenvector corresponding to the largest eigenvalue of the Laplacian ($\mathbf{L}(H^\sigma)$) of a structurally unbalanced s-hypergraph contains no zero entries (see Assumption 4.2.1). This would mean that the signature of the eigenvector corresponding to $\lambda_n(\mathbf{L}(H^\sigma))$ is always a valid bipartition vector. If the signed Laplacians of all s-hypergraphs are irreducible, then the assumption is true. This can be proven by showing that all the matrices can be transformed into upper-triangle form. Also, finding a more appropriate constant in the normalization of SP, as mentioned in Remark 4.2.1, could be achieved by a thorough study of s-hypergraphs, such as the maximum or average row sum, see Gershgorin's Disk Theorem (see Theorem 2.8 in [8]).

Finally, this thesis analyzed oriented hypergraphs, i.e., hypergraphs defined on the signed incidence. Building on this work, a natural next step would be to analyze hypergraphs weighted with values in $\mathbb{R}$. Another possibility is to instead of analyzing incidence-defined graphs, work with hypergraphs defined on signed tensors, i.e., define a metric based on tensor eigenvalues.

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Appendix A

Hardware and Software Specifications

Table A.1: Hardware specifications and software versions used for generation, measurements, and data analysis

	Dardel	MacBook
CPU Model	AMD EPYC 7742 64-Core	Apple M5
L1 Data Cache Size	4 MB	64 KB
L2 Cache Size	64 MB	16 MB
Linux Release	5.14.21	6.8.0
Python Version	3.6.15	3.14.4
NumPy Version	1.17.3	2.4.3
Pandas Version	-	3.0.2

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